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DEVELOPMENT OF FOKKINK-FOKKINK-WANG'S GENERATING FUNCTION FOR $FFW(n)$

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Abstract:

In 1995, R. Fokkink, W. Fokkink and Wang defined the $FFW(n)$ in terms of $s(\pi)$, where $s(\pi)$ is the smallest part of partition π. In 2008, Andrews obtained the generating function for $FFW(n)$. In 2013, Andrews, Garvan and Liang extended the FFW-function and obtained the similar expressions for the spt-function and then defined the spt-crank generating functions. They also defined the generating function for $FFW(z,n)$ in various ways. This paper shows how to find the number of partitions of n into distinct parts with certain conditions and shows how to prove the Theorem 1 by induction method. This paper shows how to prove the Theorem 2 with the help of two generating functions.

Keywords:
Distinct parts, FFW-function, positive divisors, smallest part, spt-function, spt-crank.

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1. INTRODUCTION

In this paper we give some related definitions of P(n), $FFW(n)$, d(n), $(x)_\infty$ $(x^2;x)_\infty$, $(zx)_\infty$, $(x)_k$ $and (x^{k+1};x)_\infty$. We give two tables for FFW(5) and FFW(6) respectively and discuss the generating functions for FFW(n) and then shows a relation related to the term $d(n)$.We discuss the various generating functions for FFW(z, n) and prove the Corollary I for proving the fundamental Theorem 1 containing three parts and prove the Theorem 2

$$\text{FFW}(z,n) = \sum_{\substack{\pi\in D \\ |\pi|=n \\ k+1\le s(\pi).}} (-1)^{\#(\pi)-1}(1+z+......+z^{s(\pi)-1}) \quad \text{and then}$$

establish the Corollary 2: FFW (1, n) = FFW(n).

2. SOME RELATED DEFINITIONS

P (n) [Sabuj et al (2014b)]: The number of partitions of n like

4, 3+1, 2+2, 2+1+1, 1+1+1+1 $\therefore$ P (4) =5.

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FFW (n) [Fokkink et al (1995)]: Let D denote the set of partitions into distinct parts. We define;

$$\text{FFW (n)} = \sum_{\substack{\pi \in D \\ |\pi| = n}} (-1)^{\#(\pi)-1} s(\pi),$$

where s(π) is the smallest part of π, and # (π) is the number of parts .

d (n) : The numbers of positive divisors of n like d(1)=1, d(2)=2 ,d(3)=2,..

Product Notations [Sabuj et al (2014a)]:

$$(x;x)_\infty = (x)_\infty = (1-x)(1-x^2)(1-x^3)...$$

$$(x^2;x)_\infty = (1-x^2)(1-x^3)...$$

$$(zx)_\infty = (1-zx)(1-zx^2)....$$

$$(x)_k = (1-x)(1-x^2)...(1-x^k)$$

$$(x^{k+1};x)_\infty = (1-x^{k+1})(1-x^{k+2}).....$$

3. WE GIVE TWO TABLES FOR n= 5 AND 6 RESPECTIVELY

Table-1: Partition of 5 into distinct parts.

Partition of 5 into distinct parts	**Smallest part of (π) s(π)**
5	5
4+1	1
3+2	2

From the table we get;

FFW (5) = (-1)° 5+ $(-1)^1$.1 +$(-1)^1$.2 = 5-1 -2 =2.

Table-2: Partition of 6 into distinct parts.

Partition of 6 into distinct parts	**Smallest part of (π) s(π)**
6	6
5+1	1
4+2	2
3+2+1	1

From the table we get;

FFW (6) = 6-1-2+1 =7-3=4.

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Similarly we get;
FFW (1) =1, FFW (2) =2, FFW (3) = 2, FFW (4) =3……………
The generating Function [Andrews (2008)] for FFW (n) is given by

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{\frac{n+1}{2}}}{(x)_n(1-x^n)}$$

$$=\frac{x}{(1-x)(1-x)}+\frac{(-1).\ x^3}{(1-x)(1-x^2)(1-x^2)}+\frac{x^6}{(1-x)(1-x^2)(1-x^3)(1-x^3)}+\dots$$

$= x + 2x^2 + 2x^3 + 3x^4 + 2x^5 + 4x^6 + \dots$

$=$ FFW (1) x + FFW (2)x^2 + FFW (3)x^3 + FFW(4)x^4 +…

$$=\sum_{n=1}^{\infty}\mathrm{FFW(n)x^n}.$$

A relation related to the term d (n).

We get; FFW (1) = 1 = d (1)
FFW (2) = 2= d (2)
FFW (3) = 2= d (3)
FFW (4) = 3= d (4)
FFW (5) = 2= d (5)
--- --- ---

We can write the relation $FFW(n)$ = d (n).

4. NOW WE DESCRIBE THE VARIOUS GENERATING FUNCTIONS [Andrews et al 2001] FOR FFW (z, n)

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{\frac{n(n+1)}{2}}}{(1-zx^n)(x)_n}$$

$$=\frac{x}{(1-x)(1-zx)}-\frac{x^3}{(1-x)(1-x^2)(1-zx^2)}+\frac{x^6}{(1-x)(1-x^2)(1-x^3)(1-zx^3)}-\cdots$$

$= x + (1+z)x^2 + (z+z^2)x^3 + (z+z^2+z^3)x^4 + (-1+z^2+z^3+z^4)x^5 +$
$(-1+z^2+z^3+z^4+z^5+1)x^6 + \dots\dots$

$$=\sum_{n=1}^{\infty}FFW(z,n)x^n.$$

We get; $\sum_{k=0}^{\infty}\frac{z^k}{(x)_k}[(x)_k-(x)_\infty]$

$$=\{1-(x)_\infty\}+\frac{z}{(1-x)}\{(1-x)-(x)_\infty\}+\frac{z^2}{(1-x)(1-x^2)}\{(1-x)(1-x^2)-(x)_\infty\}+\frac{z^3}{(1-x)(1-x^2)(1-x^3)}$$
$$\{(1-x)(1-x^2)(1-x^3)-(x)_\infty\}+\dots..$$

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$= x + x^2 + z(x^2 + x^3 + x^4) + z^2(x^3 + x^4) + z^3x^4 + ...$
$= x + (1+z)x^2 + (z+z^2)x^3 + (z+z^2+z^3)x^4 + (-1+z^2+z^3+z^4)x^5 + ...$
$= \sum_{n=1}^{\infty} FFW(z,n)x^n$.

Again we get; $\sum_{k=0}^{\infty} z^k\{1-(x^{k+1};x)_\infty\}$

$= \{1-(x)_\infty\} + z\{1-(x^2;x)_\infty\} + z^2\{1-(x^3;x)_\infty\} + ...$

$= \{1+z+z^2+z^3+....\} - \{(1-x)(1-x^2)... + z(1-x)(1-x^3).... + z^2(1-x^3)(1-x^4)....\}$

$= \{1+z+z^2+z^3+....\} - \{1-x-x^2+z-zx^2-zx^3+z^2-z^2x^3+....\}$

$= x + x^2 + z(x^2 + x^3 + x^4) + z^2(x^3 + x^4) + z^3x^4 + ...$
$= x + (1+z)x^2 + (z+z^2)x^3 + (z+z^2+z^3)x^4 + (-1+z^2+z^3+z^4)x^5 + ...$
$= \sum_{n=1}^{\infty} FFW(z,n)x^n$.

Corollary1: $\frac{x}{(1-zx)(1-x)} = \sum_{k=1}^{\infty} \left(\frac{z^k-1}{(z-1)}\right) x^k$.

Proof: L.H.S = $\frac{x}{(1-zx)(1-x)}$
$= x\,(1+zx+z^2x^2+z^3x^3+.......)(1+x+x^2+x^3+\cdots)$
$= x + (1+z)x^2 + (1+z+z^2)x^3 + (1+z+z^2+z^3)x^4 + ...$
$= x + \frac{(1+z)(1-z)}{(1-z)}x^2 + \frac{(1+z+z^2)(1-z)}{(1-z)}x^3 + \frac{(1+z+z^2+z^3)(1-z)}{(1-z)}x^4 + \ldots$
$= x + \frac{(1-z^2)}{(1-z)}x^2 + \frac{(1-z^3)}{(1-z)}x^3 + \frac{(1-z^4)}{(1-z)}x^4 + \ldots$
$= x + \frac{(z^2-1)}{(z-1)}x^2 + \frac{(z^3-1)}{(z-1)}x^3 + \frac{(z^4-1)}{(z-1)}x^4 + \ldots$
$= \sum_{k=1}^{\infty} \left(\frac{z^k-1}{(z-1)}\right) x^k$ = R,H.S. . Hence The Corollary.

Theorem 1: $\sum_{n=1}^{\infty} FFW(z,n)x^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{\frac{n(n+1)}{2}}}{(1-zx^n)(x)_n} = \frac{1}{(1-z)}\{1-\frac{(x)_\infty}{(zx)_\infty}\}$.

Proof: we get; $\frac{x^5}{1-x} = x^5(1+x+x^2+x^3+.....)$

Or, $x^1x^4\frac{1}{(x)_1} = x^5 + x^6 + x^7 + ...$

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is the generating function for partitions into 2 distinct parts with smallest part 2 like the required partitions 3+2, 4+2, …….respectively.

Again $$\frac{x^6}{(1-x)(1-x^2)} = x^6(1+x+x^2+x^3+....)(1+x^2+x^4+....)$$

$$= x^6(1+x+x^2+x^2+.....)$$

Or, $x^3.x^3\frac{1}{(x)_2} = x^6+x^7+2x^8+.....is$ the generating function for partitions into 3 distinct parts with smallest part 1 like the required partitions are 3+2+1, 4+2+1, ……respectively. Now we see that

$x^{\frac{n(n-1)}{2}}x^{nk}\frac{1}{(x)_{n-1}}$ is the generation function for partitions into n distinct parts with smallest part k.

Thus $$\sum_{n=1}^{\infty} FFW(z,n)x^n = \sum_{n=1}^{\infty}(x^n+(1+z)x^{2n}+.....(1+z+...+z^{k-1})x^{kn}+...)\frac{(-1)^{n-1}x^{\frac{n(n-1)}{2}}}{(x)_{n-1}}$$

$$=\sum_{n=1}^{\infty}\{x^n+\frac{(1+z)(1-z)}{(1-z)}x^{2n}+\frac{(1+z+z^2)(1-z)x^{3n}}{(1-z)}+...)\frac{(-1)^{n-1}x^{\frac{n(n-1)}{2}}}{(x)_{n-1}}$$

$$=\sum_{n=1}^{\infty}\{x^n+\frac{z^2-1}{z-1}x^{2n}+\frac{z^3-1}{z-1}x^{3n}+....\}\frac{(-1)^{n-1}x^{\frac{n(n-1)}{2}}}{(x)_{n-1}}$$

$$=\sum_{n=1}^{\infty}(\sum_{k=1}^{\infty}\frac{z^k-1}{z-1}x^{nk})\frac{(-1)^{n-1}x^{\frac{n(n-1)}{2}}}{(x)_{n-1}} \qquad = \sum_{n=1}^{\infty}\frac{x^n}{(1-zx^n)(1-x^n)}\frac{(-1)^{n-1}x^{\frac{n(n-1)}{2}}}{(x)_{n-1}}$$ [by Corollary 1]

$$=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{\frac{n(n+1)}{2}}}{(1-zx^n)(x)_n}$$

[Since $\sum_{n=1}^{\infty}(1-x^n)(x)_{n-1} = (1-x)+(1-x^2)(1-x)+(1-x^3)(1-x^2)(1-x)+..... = \sum_{n=1}^{\infty}(x)_n$]

Or, 1st part = $\sum_{n=1}^{\infty}FFW(z,n)x^n = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{\frac{n(n+1)}{2}}}{(1-zx^n)(x)_n}$ = 2nd part

Now $$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{\frac{n(n+1)}{2}}}{(1-zx^n)(x)_n} = \frac{x}{(1-zx)(1-x)} - \frac{x^3}{(1-zx^2)(1-x)(1-x^2)}+....$$

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$$= \frac{1}{(1-z)}\{\frac{x(1-z)}{(1-zx)(1-x)} - \frac{x^3(1-z)}{(1-zx^2)(1-x)(1-zx^2)} + ...\} = \frac{1}{(1-z)}\{1-1+\frac{x(1-z)}{(1-zx)(1-x)} - ....\}$$

$$= \frac{1}{(1-z)}[1-\{1-\frac{x(1-z)}{(1-zx)(1-x)} + ...\}] = \frac{1}{(1-z)}\{1-\sum_{n=0}^{\infty}\frac{(-1)^n x^{\frac{n(n+1)}{2}}(z)_n}{(x)_n(zx)_n}.....\}$$

$$= \frac{1}{(1-z)}\{1-(1-(1-z)\ x-(1-z^2)x^2+....)\}$$

$$= \frac{1}{(1-z)}\{1-(1-x-x^2+x^5+.....)(1+zx+(z+z^2)x^2+....)\}$$

$$= \frac{1}{(1-z)}\{1-\frac{(1-x)(1-x^2)(1-x^3)....}{(1-zx)(1-zx^2)(1-zx^3).....}\} = \frac{1}{(1-z)}\{1-\frac{(x)_\infty}{(zx)_\infty}\} = 3^{rd} \text{ Part.}$$

But $\sum_{n=0}^{\infty}\frac{(-1)^n x^{\frac{n(n+1)}{2}}(z)_n}{(x)_n(zx)_n} = \frac{(x)_\infty}{(zx)_\infty}$ [Andrews (1976)]

$$\therefore \sum_{n=1}^{\infty} FFW(z,n)x^n = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{\frac{n(n+1)}{2}}}{(1-zx^n)(x)_n} = \frac{1}{(1-z)}\{1-\frac{(x)_\infty}{(zx)_\infty}\}.\text{Hence The Theorem.}$$

Theorem 2: FFW (z, n) = $\sum_{\substack{\pi\in D \\ |\pi|=n \\ k+1\le s(\pi).}} (-1)^{\#(\pi)-1}(1+z+......+z^{s(\pi)-1})$.

Proof: We get; $\sum_{k=0}^{\infty}\frac{z^k}{(x)_k}[(x)_k-(x)_\infty]$

$$=\{1-(x)_\infty\}+\frac{z}{(1-x)}\{(1-x)-(x)_\infty\}+\frac{z^2}{(1-x)(1-x^2)}\{(1-x)(1-x^2)-(x)_\infty\}+\frac{z^3}{(1-x)(1-x^2)(1-x^3)}$$
$$\{(1-x)(1-x^2)(1-x^3)-(x)_\infty\}+.....$$

$$=\{1-(x)_\infty\}+z\{1-(1-x^2)(1-x^3)....\}+z^2\{1-(1-x^3)(1-x^4)....\}$$

$$=\{1-(x)_\infty\}+z\{1-(x^2;x)_\infty\}+z^2\{1-(x^3;x)_\infty\}+.... = \sum_{k=0}^{\infty} z^k\{1-(x^{k+1};x)_\infty\}$$

$$\therefore \sum_{n=1}^{\infty} FFW(z,n)x^n = \sum_{k=0}^{\infty} z^k\{1-(x^{k+1};x)_\infty\}.$$

We see that the co-efficient of $z^k x^n$ in right hand side

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is $\sum_{\substack{\pi\in D\\ |\pi|=n\\ k+1\le s(\pi).}} (-1)^{\#(\pi)-1}(1+z+......+z^{s(\pi)-1})$

which is also co-efficient of $z^k x^n$ in left hand side.

$\therefore$ FFW (z, n) = $\sum_{\substack{\pi\in D\\ |\pi|=n\\ k+1\le s(\pi).}} (-1)^{\#(\pi)-1}(1+z+......+z^{s(\pi)-1})$ .Hence the Theorem.

Corollary 2: FFW (1, n) = FFW (n)

Proof: We get; $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{\frac{n(n+1)}{2}}}{(x)_n(1-zx^n)}$

$$= x+(1+z)x^2+(z+z^2)x^3+(z+z^2+z^3)x^4+(-1+z^2+z^3+z^4)x^5+(-1+1+z^2+z^3+z^4+z^5)x^6+....$$

Or, $\sum_{n=1}^{\infty} FFW(z,n)x^n = x+(1+z)x^2+(z+z^2)x^3+(z+z^2+z^3)x^4+(-1+z^2+z^3+z^4)x^5+.....$

If z=1, we get;

$$\sum_{n=1}^{\infty} FFW(1,n)x^n = x+2x^2+2x^3+3x^4+2x^5+4x^6+.....$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{\frac{n(n+1)}{2}}}{(x)_n(1-x^n)} \quad \text{(by above)}$$

Or, $\sum_{n=1}^{\infty} FFW(1,n)x^n = \sum_{n=1}^{\infty} FFW(n)x^n$.

Equating the co-efficient of x^n form both sides we get;

$\therefore$ FFW (1, n) = FFW (n).Hence the Corollary.

5. CONCLUSION

In this study we have found the number of partitions of n into distinct parts with required conditions. We have already shown the numbers of partitions for n = 5 and 6 respectively and have found the number of partitions from the relation FFW (n) = d (n) for any positive integral of n. We have proved the Theorem 1 containing two generating functions

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{\frac{n(n+1)}{2}}}{(x)_n(1-zx^n)} \text{ and } \frac{1}{(1-z)}\left\{1-\frac{(x)_\infty}{(zx)_\infty}\right\}$$

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with the help of generating functions for partitions into n distinct parts with smallest part k and have proved the Theorem 2 by taking the co-efficient from various two generating functions. Finally we have established the Corollary FFW (1, n) = FFW (n) by taking z=1.

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