IJRST
Volume: 2 | Issue: 3 | March 2015 | ISSN: 2349-0845

A Study on the Local Property of indexed Summability of a factored Fourier Series

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Article Info

Article history:

Received on 7th March 2015
Accepted on 11th March 2015
Published on 18th March 2015

Keyword:

$\left|\overline{N}, p_n\right|_k$ - summability ,

$\left|\overline{N}, p_n, \alpha_n\right|_k$ - summability ,

$\left|\overline{N}, p_n, \alpha_n; \delta\right|_k$ - summability ,

Fourier series.

***AMS Classification No:* 40D25**

ABSTRACT

In this paper we have established a theorem on the local property of $\left|\overline{N}, p_n, \alpha_n; \delta\right|_k$ summability of a factored Fourier series.

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I. INTRODUCTION

Let $\sum a_n$ be a given infinite series with sequence of partial sums $\{s_n\}$.Let $\{p_n\}$be a sequence of positive real constants such that

(1.1) $$P_n = \sum_{\nu=0}^{n} p_\nu \to \infty \ as\ n \to \infty \ (P_{-i} = p_{-i} = 0, i \geq 1)$$

The sequence –to-sequence transformation

(1.2) $$t_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu s_\nu$$

defines the sequence $\{t_n\}$ of the $\left|\overline{N}, p_n\right|$-means of the sequence $\{s_n\}$ generated by the sequence of coefficients $\{p_n\}$.The series $\sum a_n$ is said to be summable $\left|\overline{N}, p_n\right|_k$, $k \geq 1$, if

(1.3) $$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}\left|t_n - t_{n-1}\right|^k < \infty .$$

For k=1, $\left|\overline{N}, p_n\right|_k$ - summability is same as $\left|\overline{N}, p_n\right|$ -summability.

When $p_n = 1$ for all n and $k = 1$, $\left|\overline{N}, p_n\right|_k$ - summability is same as $|C,1|$ -summability.

Also if we take $k = 1$ and $p_n = \dfrac{1}{(n+1)}$, $\left|\overline{N}, p_n\right|_k$ -summability is equivalent to the summability $|R, \log n, 1|$.

Let $\{\alpha_n\}$ be any sequence of positive numbers. The series $\sum a_n$ is said to be summable $\left|\overline{N}, p_n, \alpha_n\right|_k$, $k \geq 1$, if

(1.4) $$\sum_{n=1}^{\infty}\alpha_n^{k-1}\left|t_n - t_{n-1}\right| < \infty ,$$

where $\{t_n\}$is as defined in (1.2).The series $\sum a_n$ is said to be $\left|\overline{N}, p_n, \alpha_n; \delta, \gamma\right|_k$, $k \geq 1, \delta \geq 0$, summable if

(1.5) $$\sum_{n=1}^{\infty}\alpha_n^{\delta k+k-1}\left|t_n - t_{n-1}\right|^k < \infty .$$

For $\delta = 0$, the summability metod $\left|\overline{N}, p_n, \alpha_n; \delta\right|_k$, $k \geq 1, \delta \geq 0$, reduces to the summabilty method $\left|\overline{N}, p_n, \alpha_n\right|_k$, $k \geq 1$.

For any real number γ, the series $\sum a_n$ is said to be summable by the summabilty method $\left|\overline{N}, p_n, \alpha_n; \delta, \gamma\right|_k$, $k \geq 1, \delta \geq 0$, if

(1.6) $$\sum_{n=1}^{\infty}\alpha_n^{\gamma(\delta k+k-1)}\left|t_n - t_{n-1}\right|^k < \infty .$$

For $\gamma = 1$, the summability method $\left|\overline{N}, p_n, \alpha_n; \delta, \gamma\right|_k$, $k \geq 1, \delta \geq 0$, any real γ, reduces to the method $\left|\overline{N}, p_n, \alpha_n; \delta, \gamma\right|_k$, $k \geq 1, \delta \geq 0$.

For any sequence $\{c_n\}$we use the following notation:

$$\Delta c_n = c_n - c_{n-1} , \ \Delta^2 c_n = \Delta(\Delta c_n) .$$

A sequence $\{\lambda_n\}$is said to be convex if $\Delta^2\lambda_n \geq 0$ for every positive integer 'n'.

Let $f(t)$ be a periodic function with period 2π and integrable in the sense of Lebesgue over $(-\pi, \pi)$ .Without loss of generality we may assume that the constant term in the Fourier series of $f(t)$ is zero, so that

(1.7)

$$f(t) \sim \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty}A_n(t).$$

II. KNOWN THEOREMS

Dealing with $\left|\overline{N}, p_n\right|_k$ -summability of an infinite series Bor**[1]** proved the following theorem:

2.1. THEOREM- 1:

Let $k \geq 1$and let the sequences $\{p_n\}$ and $\{\lambda_n\}$ be such that

(2.1.1) $$\Delta X_n = O\left(\frac{1}{n}\right),$$

(2.1.2) $$\sum_{n=1}^{\infty}X_n^{k-1}\frac{|\lambda_n|^k + |\lambda_{n+1}|^k}{n} < \infty ,$$

(2.1.3) $$\sum_{n=1}^{\infty}(X_n^k + 1)|\Delta\lambda_n| < \infty ,$$

Where $X_n = \frac{P_n}{np_n}$. Then the summability $\left|\bar{N}, p_n\right|_k$ of the series $\sum_{n=1}^{\infty} A_n(t)\lambda_n X_n$ at a point can be ensured by the local property.

Subsequently, extending the result of Bor, Misra et al [2] established the following theorem on summabilty $\left|\bar{N}, p_n, \alpha_n\right|_k$:

2.2. THEOREM- 2:

Let $k \ge 1$. Suppose $\{\lambda_n\}$ be a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent. Let $\{p_n\}$ and $\{\alpha_n\}$ be any sequence of positive numbers such that

(2.2.1) $$\Delta X_n = O\left(\frac{1}{n}\right),$$

(2.2.2) $$\sum_{n=\nu+1}^{m+1} \alpha_n^{k-1}\left(\frac{p_n}{P_n}\right)^k \left(\frac{1}{P_{n-1}}\right) = O\left(\frac{1}{P_\nu}\right),$$

(2.2.3)

$$\sum_{n=1}^{\infty} X_n^{k-1} \frac{\left|\lambda_n\right|^k + \left|\lambda_{n+1}\right|^k}{n} < \infty,$$

(2.2.4) $$\sum_{n=1}^{\infty} (X_n^k + 1)\left|\Delta\lambda_n\right| < \infty,$$

and

(2.2.5) $$\sum_{n=2}^{m+1} \alpha_n^{k-1} \frac{\left|\lambda_n\right|^k}{n^k} < \infty,$$

where $X_n = \frac{P_n}{np_n}$. Then the summability $\left|\bar{N}, p_n, \alpha_n\right|_k$ of the series $\sum_{n=1}^{\infty} A_n(t)\lambda_n X_n$ at a point can be ensured by the local property.

Further, extending to $\left|\bar{N}, p_n, \alpha_n; \delta\right|_k$ -summability, Padhy et al [3] established the following theorem:

2.3. THEOREM- 3:

Let $k \ge 1$. Suppose $\{\lambda_n\}$ be a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent. Let $\{\alpha_n\}$ and $\{p_n\}$ be a sequence of positive numbers such that

(2.3.1) $$\Delta X_n = O\left(\frac{1}{n}\right),$$

(2.3.2)
$$\sum_{n=\nu+1}^{m+1} \alpha_n^{(\delta k+k-1)}\left(\frac{p_n}{P_n}\right)^k \left(\frac{1}{P_{n-1}}\right) = O\left(\frac{1}{P_\nu}\right),$$

(23.3)

$$\sum_{n=1}^{\infty} X_n^{k-1} \frac{\left|\lambda_n\right|^k + \left|\lambda_{n+1}\right|^k}{n} < \infty,$$

(2.3.4) $$\sum_{n=1}^{\infty} (X_n^k + 1)\left|\Delta\lambda_n\right| < \infty,$$

and

(2.3.5) $$\sum_{n=2}^{\infty} \alpha_n^{\delta k+k-1} \frac{\left|\lambda_n\right|^k}{n^k} < \infty,$$

where $X_n = \frac{P_n}{np_n}$. Then the summability $\left|\bar{N}, p_n, \alpha_n; \delta\right|_k, k \ge 1, \delta \ge 0$ of the series $\sum_{n=1}^{\infty} A_n(t)\lambda_n X_n$ at a point can be ensured by the local property.

In what follows in the present paper, extending the above theorems to $\left|\bar{N}, p_n, \alpha_n; \delta, \gamma\right|_k$ summabilty, we establish the following theorem:

III.MAIN THEOREM

Let $k \geq 1$. Suppose $\{\lambda_n\}$ be a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent. Let $\{\alpha_n\}$and $\{p_n\}$ be a sequence of positive numbers such that

(3.1) $$\Delta X_n = O\left(\frac{1}{n}\right),$$

(3.2)
$$\sum_{n=\nu+1}^{m+1} \alpha_n^{\gamma(\delta k+k-1)} \left(\frac{p_n}{P_n}\right)^k \left(\frac{1}{P_{n-1}}\right) = O\left(\frac{1}{P_\nu}\right),$$

(3.3) $$\sum_{n=1}^{\infty} X_n^{k-1} \frac{|\lambda_n|^k}{n} < \infty ,$$

(3.4) $$\sum_{n=1}^{\infty} (X_n^k + 1)|\Delta\lambda_n| < \infty,$$

and

(3.5) $$\sum_{n=2}^{\infty} \alpha_n^{\gamma(\delta k+k-1)} \frac{|\lambda_n|^k}{n^k} < \infty ,$$

where $X_n = \dfrac{P_n}{np_n}$. Then the summability $\left|\bar{N}, p_n, \alpha_n; \delta\right|_k, k \geq 1, \delta \geq 0$ of the series $\sum_{n=1}^{\infty} A_n(t)\lambda_n X_n$ at a point can be ensured by the local property.

In order to prove the above theorem we require the following lemma:

IV.LEMMA

Let $k \geq 1$. Suppose $\{\lambda_n\}$ be a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent and $\{p_n\}$ be a sequence such that the conditions **(3.1)-(3.5)** are satisfied. If $\{s_n\}$ is bounded then the series $\sum_{n=1}^{\infty} a_n \lambda_n X_n$ is $\left|\bar{N}, p_n, \alpha_n; \delta\right|_k, k \geq 1, \delta \geq 0$-summable when $\{\alpha_n\}$ is any sequence of positive numbers.

V.PROOF OF THE LEMMA

Let $\{T_n\}$denote the $\left|\bar{N}, p_n\right|$ -mean of the series $\sum_{n=1}^{\infty} a_n \lambda_n X_n$ .Then by definition we have

$$T_n = \frac{1}{P_n}\sum_{\nu=0}^{n} p_\nu \sum_{r=0}^{\nu} a_r \lambda_r X_r$$

$$= \frac{1}{P_n}\sum_{\nu=0}^{n} (P_\nu - P_{\nu-1}) a_\nu \lambda_\nu X_\nu \ , X_0 = 0.$$

For $n \geq 1$,we have

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n} P_{\nu-1} a_\nu \lambda_\nu X_\nu .$$

So,

$$T_n - T_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} p_\nu s_\nu \lambda_\nu X_\nu + \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} P_\nu s_\nu X_\nu \Delta\lambda_\nu$$
$$+ \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} P_\nu s_\nu \lambda_{\nu+1} \Delta X_\nu + \frac{p_n s_n \lambda_n X_n}{P_n}.$$

(by Abel's transformation)

$$= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4} \quad (say)$$

To complete the proof of the Lemma using Minokowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \alpha_n^{\gamma(\delta k+k-1)} |T_{n,i}|^k < \infty \quad for\ i = 1,2,3,4.$$

Now, we have

$$\sum_{n=2}^{m+1} \alpha_n^{\gamma(\delta k+k-1)} |T_{n,1}|^k$$

$$=\sum_{n=2}^{m+1}\alpha_n^{\gamma(\delta k+k-1)}\left|\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}p_v s_v\lambda_v X_v\right|^k$$

$$\leq\sum_{n=2}^{m+1}\alpha_n^{\gamma(\delta k+k-1)}\left(\frac{p_n}{P_n}\right)^k\left(\frac{1}{P_{n-1}}\right)\left(\sum_{v=1}^{n-1}p_v|\lambda_v|^k|s_v|^k X_v^k\right)\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right)^{k-1}$$

$$=O(1)\sum_{v=1}^{m}p_v|\lambda_v|^k|s_v|^k X_v^k\sum_{n=v+1}^{m+1}\alpha_n^{\gamma(\delta k+k-1)}\left(\frac{p_n}{P_n}\right)^k\left(\frac{1}{P_{n-1}}\right)$$

$$=O(1)\sum_{v=1}^{m}|\lambda_v|^k X_v^k\frac{p_v}{P_v}$$, **by (3.2)**

$$=O(1)\sum_{v=1}^{m}|\lambda_v|^k X_v^{k-1}\frac{p_v}{P_v}\frac{P_v}{v\,p_v}$$, as $X_n=\dfrac{P_n}{np_n}$

$$=O(1)\sum_{v=1}^{m}X_v^{k-1}\frac{|\lambda_v|^k}{v}$$

$=O(1)\quad as\ m\to\infty$, **by (3.3).**

Again,

$$\sum_{n=2}^{m+1}\alpha_n^{\gamma(\delta k+k-1)}|T_{n,2}|^k$$

$$=\sum_{n=2}^{m+1}\alpha_n^{\gamma(\delta k+k-1)}\left|\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_v s_v X_v\Delta\lambda_v\right|^k$$

$$\leq\sum_{n=2}^{m+1}\alpha_n^{\gamma(\delta k+k-1)}\left(\frac{p_n}{P_n}\right)^k\left(\frac{1}{P_{n-1}}\right)\left(\sum_{v=1}^{n-1}P_v|\Delta\lambda_v|\,|s_v|^k X_v^k\right)\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}P_v|\Delta\lambda|\right)^{k-1}$$

Since

$$\sum_{r=1}^{n-1}P_v|\Delta\lambda_r|\leq P_{n-1}\sum_{v=1}^{n-1}|\Delta\lambda_v|\Rightarrow\frac{1}{P_{n-1}}\sum_{r=1}^{n-1}P_v|\Delta\lambda_r|\leq\sum_{v=1}^{n-1}|\Delta\lambda_v|=O(1)$$

$$=O(1)\sum_{v=1}^{m}P_v|\Delta\lambda_v|X_v^k\sum_{n=v+1}^{m+1}\alpha_n^{\gamma(\delta k+k-1)}\left(\frac{p_n}{P_n}\right)^k\left(\frac{1}{P_{n-1}}\right)$$

$$=O(1)\sum_{v=1}^{m}|\Delta\lambda_v|X_v^k$$, **by (3.2)**

$=O(1)\quad as\ m\to\infty$, **by (3.4).**

Further,

$$\sum_{n=2}^{m+1}\alpha_n^{\gamma(\delta k+k-1)}|T_{n,3}|^k$$

$$=\sum_{n=2}^{m+1}\alpha_n^{\gamma(\delta k+k-1)}\left|\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_v s_v\lambda_{v+1}\Delta X_v\right|^k$$

$$=O(1)\sum_{n=2}^{m+1}\alpha_n^{\delta k+k-1}\left(\frac{p_n}{P_nP_{n-1}}\right)^k\left\{\sum_{v=1}^{n-1}P_v|\lambda_{v+1}||s_v|\frac{1}{v}\right\}^k ,$$

by (3.1)

$$=O(1)\sum_{n=2}^{m+1}\alpha_n^{\delta k+k-1}\left(\frac{p_n}{P_nP_{n-1}}\right)^k\left\{\sum_{v=1}^{n-1}p_v X_v|\lambda_{v+1}|\right\}^k$$

$$=O(1)\sum_{n=2}^{m+1}\alpha_n^{\delta k+k-1}\left(\frac{p_n}{P_n}\right)^k\left(\frac{1}{P_{n-1}}\right)\left\{\sum_{v=1}^{n-1}p_v X_v^k|\lambda_{v+1}|^k\right\}\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v$$

$$=O(1)\sum_{v=1}^{m}p_v|\lambda_{v+1}|^k X_v^k\sum_{n=v+1}^{m+1}\alpha_n^{\delta k+k-1}\left(\frac{p_n}{P_n}\right)^k\left(\frac{1}{P_{n-1}}\right)$$

$$=O(1)\sum_{v=1}^{m}\frac{p_v}{P_v}|\lambda_{v+1}|^k X_v^k$$, **by (3.2)**

$$=O(1)\sum_{v=1}^{m}|\lambda_{v+1}|^k X_v^k\frac{p_{vr}}{P_v}\frac{P_v}{v\,p_v},$$

as $X_n=\dfrac{P_n}{np_n}$

$$=O(1)\sum_{v=1}^{m}X_v^{k-1}\frac{|\lambda_{v+1}|^k}{v}$$

$=O(1)\quad as\ m\to\infty$, **by (3.3).**

Now,

$$\sum_{n=2}^{m+1} \alpha_n^{\delta k+k-1} \left|T_{n,4}\right|^k \quad = \sum_{n=2}^{m+1} \alpha_n^{\delta k+k-1} \left| \frac{p_n s_n \lambda_n X_n}{P_n} \right|^k$$

$$= O(1) \sum_{n=2}^{m+1} \alpha_n^{\delta k+k-1} X_n^k \left|\lambda_n\right|^k \left(\frac{p_n}{P_n} \right)^k$$

$$= O(1) \sum_{n=2}^{m+1} \alpha_n^{\delta k+k-1} \left(\frac{P_n}{np_n} \right)^k \left|\lambda_n\right|^k \left(\frac{p_n}{P_n} \right)^k$$, as

$$X_n = \frac{P_n}{np_n}$$

$$= O(1) \sum_{n=2}^{m+1} \alpha_n^{\delta k+k-1} \frac{\left|\lambda_n\right|^k}{n^k}$$

$$= O(1) \quad as\ m \to \infty$$, **by (3.5) .**

This completes the proof of the Lemma.

VI. PROOF OF THE THEOREM

Since the behavior of the Fourier series, as far as convergence is concerned, for a particular value of x depends on the behavior of the function in the immediate neighborhood of this point only, the truth of the theorem is necessarily the consequence of the Lemma.

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