



Numerical Simulation of a Dimensionless Form of Two Lane Traffic Flow Model

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ABSTRACT: In this work, a numerical solution of a dimensionless form of the macroscopic two lane traffic flow model based on a linear density-velocity relationship is studied. We make the two lane traffic flow model dimensionless by introducing some dimensionless parameters. In order to compute the numerical solution, we present the discretization of the considered model which leads to the explicit upwind difference scheme. The numerical simulation of 2.5km highway of two lanes is performed for 1.5 minutes using the explicit upwind difference scheme based on artificially generated initial and boundary data. An experimental result for the stability condition of the numerical scheme is also presented. Since the scaled model is simpler than the non-scaled model, therefore we get more computational efficiency for the scaled model.

Keywords: Two Lane Traffic Flow Model; Dimensionless form; Numerical Simulation.

I. INTRODUCTION

Traffic flow, in Mathematics and Civil Engineering is the study of interactions between vehicles, drivers, and infrastructure (including highways, signals and traffic control devices), with the aim of understanding and developing an optimal road network with efficient movement of traffic and minimal traffic congestion problems. Attempts to produce a mathematical theory of traffic flow date back to the 1920s, when Frank Knight first produced an analysis of traffic equilibrium, which was refined into Wardrop's first and second principles of equilibrium in 1952.

Traffic phenomena are complex and nonlinear, depending on the interactions of a large number of vehicles. Due to the individual reactions of human drivers, vehicles do not interact simply following the laws of mechanics, but rather show phenomena of cluster formation and shock wave propagation, both forward and backward, depending on vehicle density in a given area. Some mathematical models in traffic flow make use of a vertical queue assumption, where the vehicles along a congested link do not spill back along the length of the link.

Axel [1] presented a hierarchy of multilane traffic flow models and described the derivation of macroscopic multilane traffic flow model. The dimensionless form of the macroscopic traffic flow model for single lane highway was studied [2]. The numerical experiments were performed in order to verify some qualitative traffic flow behaviors with respect to the traffic flow parameters, [3], [4], [5]. The fluid dynamic traffic flow model as an initial boundary value problem (IBVP) with two sided boundary conditions was studied and a new version of the Lax-Friedrichs scheme for the fluid dynamic traffic flow model was also presented [6-7]. A computational study of the non-scaled multilane traffic flow model was presented [9].

In this research article, we choose a numerical scheme named explicit upwind difference scheme to compute the numerical solution of the two lane traffic flow model as an IBVP. We discretize the scaled two lane traffic flow model by using finite difference formula which leads to the explicit upwind difference scheme. We build up a code in MatLab programming language of scientific computing for the Explicit Upwind difference scheme. We present the density profile as well as computed velocity and flux profiles of our considered model which is scaled (dimensionless) model without transformation back and the scaled model with transformation back by using the Explicit Upwind difference scheme. Some experimental results are also presented for the stability restriction of the numerical scheme for the dimensionless form of two lane traffic flow model.

II. MACROSCOPIC MULTILANE TRAFFIC FLOW MODEL

We assume a highway with N lanes. They are numbered by $\alpha = 1, 2, \dots, N$. The maximal velocity is denoted by w ; i.e., the velocities range between 0 and w . The kinetic multilane traffic flow model as in [1] is given by

$$\partial_t f_\alpha + v \partial_x f_\alpha = C_\alpha(f_1, \dots, f_N) \quad (1)$$

In our research, we focus on the multilane traffic model (1). The multilane model (1) can be written in generalized form as follows from [1].

$$\begin{aligned} \frac{\partial \rho_1}{\partial t} + \frac{\partial q_1}{\partial x} &= \frac{\rho_2}{T_2^1} - \frac{\rho_1}{T_1^2} \\ \frac{\partial \rho_j}{\partial t} + \frac{\partial q_j}{\partial x} &= \frac{\rho_{j-1}}{T_{j-1}^j} - \frac{\rho_j}{T_j^{j-1}} + \frac{\rho_{j+1}}{T_{j+1}^j} - \frac{\rho_j}{T_j^{j+1}} \\ \frac{\partial \rho_N}{\partial t} + \frac{\partial q_N}{\partial x} &= \frac{\rho_{N-1}}{T_{N-1}^N} - \frac{\rho_N}{T_N^{N-1}} \end{aligned} \quad (2)$$

Here, the subscripts $1, j = 2, \dots, N-1$ and N refer to the lane numbers. The quantities ρ_j and $q_j = \rho_j v_j$ are the vehicle density and the vehicle flux in the j -th lane respectively whereas v_j is the vehicle velocity at the j -th lane for $j = 1, 2, \dots, N$; at last $T_j^k = T_j^k(\rho_j, \rho_k)$ is the vehicle transition rate from lane j to lane k , with $|j - k| = 1$. In particular, we choose macroscopic multilane traffic flow model (2) for two lanes that is for $j = 1, 2, (N = 2)$:

$$\begin{aligned} \frac{\partial \rho_1}{\partial t} + \frac{\partial q_1}{\partial x} &= \frac{\rho_2}{T_2^1} - \frac{\rho_1}{T_1^2} \\ \frac{\partial \rho_2}{\partial t} + \frac{\partial q_2}{\partial x} &= \frac{\rho_1}{T_1^2} - \frac{\rho_2}{T_2^1} \end{aligned} \quad (3)$$

A. Density-Velocity Relationship

The macroscopic two lane traffic flow model (3) is approximated by the Greenschild's linear density-velocity relationship as follows:

$$V(\rho) = V_{\max} \left(1 - \frac{\rho}{\rho_{\max}} \right) \quad (4)$$

Where, V_{\max} is the maximum velocity and ρ_{\max} is the maximum density which is based on bumper to bumper traffic. By using the Density-Velocity relationship (4), the equations (3) take the following form.

$$\begin{aligned} \frac{\partial \rho_1}{\partial t} + \left[V_1^{\max} \rho_1 \left(1 - \frac{\rho_1}{\rho_1^{\max}} \right) \right]_x &= \frac{\rho_2}{T_2^1} - \frac{\rho_1}{T_1^2} \\ \frac{\partial \rho_2}{\partial t} + \left[V_2^{\max} \rho_2 \left(1 - \frac{\rho_2}{\rho_2^{\max}} \right) \right]_x &= \frac{\rho_1}{T_1^2} - \frac{\rho_2}{T_2^1} \end{aligned} \quad (5)$$

III. DIMENSIONLESS FORM OF THE TWO LANE TRAFFIC FLOW MODEL

The dimensionless form of the two lane traffic flow model presented here. Now equations (5) can be simplified by bringing it in a dimensionless form. Let L and τ be a typical length and time respectively, such that $\frac{L}{\tau} = V_{\max}$.

Introducing the following dimensionless parameters

$$x_s = \frac{x}{L}, \quad t_s = \frac{t}{T}, \quad u_1 = 1 - \frac{2\rho_1}{\rho_1^{\max}}, \quad u_2 = 1 - \frac{2\rho_2}{\rho_2^{\max}}$$

By using these dimensionless parameters, from (5) we get

$$\frac{\partial \rho_1}{\partial t} = \frac{1}{\tau} \frac{\partial}{\partial t_s} \left[\frac{\rho_1^{\max}}{2} (1 - u_1) \right] = -\frac{\rho_1^{\max}}{2\tau} \frac{\partial u_1}{\partial t_s}$$

$$\left[V_1^{\max} \rho_1 \left(1 - \frac{\rho_1}{\rho_1^{\max}} \right) \right] = \frac{1}{L} \frac{\partial}{\partial x_s} \left[V_1^{\max} \frac{\rho_1^{\max}}{2} (1 - u_1) \frac{1}{2} (1 + u_1) \right] = -\frac{\rho_1^{\max}}{2} \frac{\partial}{\partial x_s} \left(\frac{u_1^2}{2} \right)$$

Similarly,

$$\frac{\partial \rho_2}{\partial t} = \frac{1}{\tau} \frac{\partial}{\partial t_s} \left[\frac{\rho_2^{\max}}{2} (1 - u_2) \right] = -\frac{\rho_2^{\max}}{2\tau} \frac{\partial u_2}{\partial t_s}$$

$$\left[V_2^{\max} \rho_2 \left(1 - \frac{\rho_2}{\rho_2^{\max}} \right) \right] = \frac{1}{L} \frac{\partial}{\partial x_s} \left[V_2^{\max} \frac{\rho_2^{\max}}{2} (1 - u_2) \frac{1}{2} (1 + u_2) \right] = -\frac{\rho_2^{\max}}{2} \frac{\partial}{\partial x_s} \left(\frac{u_2^2}{2} \right)$$

Finally, the equations in (5) may also be rewritten as follows.

$$\frac{\partial u_1}{\partial t_s} + \frac{\partial}{\partial x_s} \left(\frac{u_1^2}{2} \right) = \frac{\tau}{T_2^1} (u_2 - 1) - \frac{\tau}{T_1^2} (u_1 - 1)$$

$$\frac{\partial u_2}{\partial t_s} + \frac{\partial}{\partial x_s} \left(\frac{u_2^2}{2} \right) = \frac{\tau}{T_1^2} (u_1 - 1) - \frac{\tau}{T_2^1} (u_2 - 1)$$

Here $u^0 = 1 - \frac{2\rho_0}{\rho_{\max}}$, If the highway is empty ($\rho = 0$), we have $u = 1$ in a tailback ($\rho = \rho_{\max}$), $u = -1$

holds. The left hand sides of the above equations are analogous as the Inviscid Burgers equation.

IV. NUMERICAL SCHEME FOR TWO LANE TRAFFIC FLOW MODEL

Explicit upwind difference scheme is one of the finite difference methods to find the numerical approximation of hyperbolic partial differential equation (PDE). In order to apply the explicit upwind difference scheme, we have to make the considered model as an IBVP in the following form.

$$\begin{aligned}
\frac{\partial u_1}{\partial t_s} + \frac{\partial}{\partial x_s} \left(\frac{u_1^2}{2} \right) &= \frac{\tau}{T_1^1} (u_2 - 1) - \frac{\tau}{T_1^2} (u_1 - 1) \\
\frac{\partial u_2}{\partial t_s} + \frac{\partial}{\partial x_s} \left(\frac{u_2^2}{2} \right) &= \frac{\tau}{T_1^2} (u_1 - 1) - \frac{\tau}{T_2^1} (u_2 - 1) \\
u_1(x, 0) &= u_1^0(x), \quad u_2(x, 0) = u_2^0(x) \\
u_1(a, t) &= u_1^a(t), \quad u_2(a, t) = u_2^a(t)
\end{aligned} \tag{6}$$

The equations in the above IBVP are similar to the Forced Burgers' Equation.

We discretize the time derivatives $\frac{\partial u_1}{\partial t_s}$ and $\frac{\partial u_2}{\partial t_s}$ by the first order forward difference in time while the space

derivatives $\frac{\partial}{\partial x_s} \left(\frac{u_1^2}{2} \right)$ and $\frac{\partial}{\partial x_s} \left(\frac{u_2^2}{2} \right)$ by the backward difference in space.

Forward difference in time:

$$\frac{\partial u_1}{\partial t} \approx \frac{(u_1)_i^{n+1} - (u_1)_i^n}{\Delta t} \quad \text{and} \quad \frac{\partial u_2}{\partial t} \approx \frac{(u_2)_i^{n+1} - (u_2)_i^n}{\Delta t}$$

Backward difference in space:

$$\frac{\partial}{\partial x} \left(\frac{u_1^2}{2} \right) \approx \frac{((u_1)_i^n)^2 - ((u_1)_{i-1}^n)^2}{2\Delta x} \quad \text{and} \quad \frac{\partial}{\partial x} \left(\frac{u_2^2}{2} \right) \approx \frac{((u_2)_i^n)^2 - ((u_2)_{i-1}^n)^2}{2\Delta x}$$

We assume the uniform grid spacing with step size $k \equiv \Delta t$ and $h \equiv \Delta x$ for time and space respectively $t^{n+1} = t^n + k$ and $x_{i+1} = x_i + h$.

The Explicit upwind difference scheme for the dimensionless form of the model is as follows

$$(u_1)_i^n = (u_1)_i^n - \frac{\Delta t}{2\Delta x} \left[((u_1)_i^n)^2 - ((u_1)_{i-1}^n)^2 \right] + \frac{\Delta t}{T_1^1} (1 - (u_2)_i^n) - \frac{\Delta t}{T_1^2} (1 - (u_1)_i^n) \tag{7}$$

$$(u_2)_i^n = (u_2)_i^n - \frac{\Delta t}{2\Delta x} \left[((u_2)_i^n)^2 - ((u_2)_{i-1}^n)^2 \right] + \frac{\Delta t}{T_1^2} (1 - (u_1)_i^n) - \frac{\Delta t}{T_2^1} (1 - (u_2)_i^n) \tag{8}$$

Therefore, the equations (7) and (8) are the discrete versions of the PDEs in the IBVP (6).

V. NUMERICAL SIMULATION

In this section, we present the numerical results for some specific cases of traffic flow focusing on the dimensionless parameters by using the numerical scheme named Explicit Upwind difference scheme. We present the density profile as well as computed velocity and flux profiles of our considered model which is scaled (dimensionless) model without transformation back and the scaled model with transformation back by the Explicit Upwind difference scheme.

In particular, we choose maximum velocity, $V_1^{\max} = V_2^{\max} = 100$ km/hour. For satisfying the CFL criterion we pick the unit of velocity as km/sec. We consider $\rho_1^{\max} = \rho_2^{\max} = 175$ /km, Transition rates, $T_2^1 = 10\%$, $T_1^2 = 20\%$ for 90 seconds in $\Delta t = 1500$ time steps for a two lanes highway of 2.5 km. Here we present the scaled initial profile (dimensionless) without transformation back in Fig. 1 while Fig. 2 shows the scaled density profile. The constant boundary data is $u_1^a = 30$ /km and $u_2^a = 27$ /km for first and second lane.

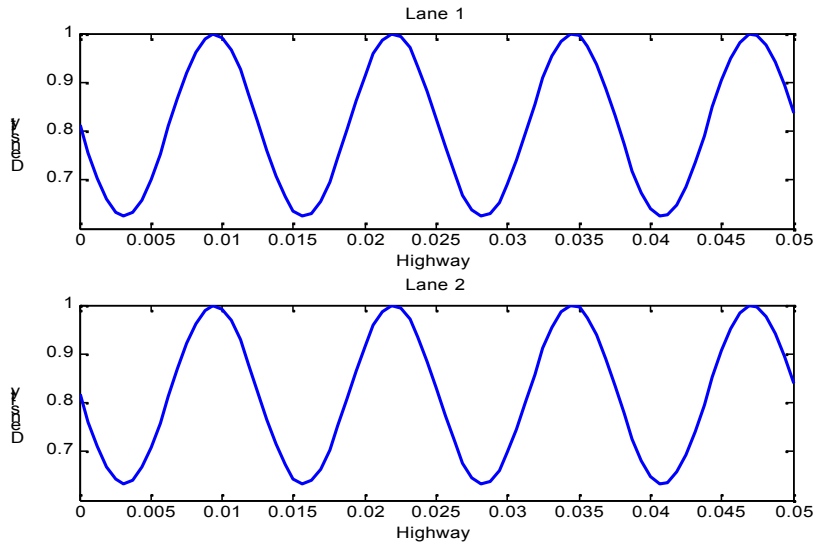


Fig. 1. Scaled Initial Density Profile.

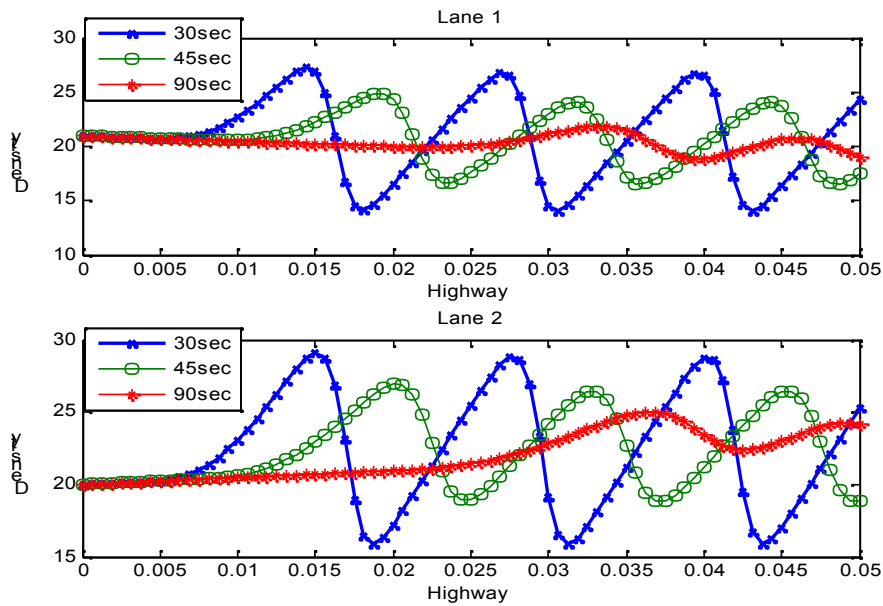


Fig. 2. Scaled Density Profile.

The solution surface of the scaled two lane traffic flow model is presented in the Fig. 3. In this figure, the density is depicted w

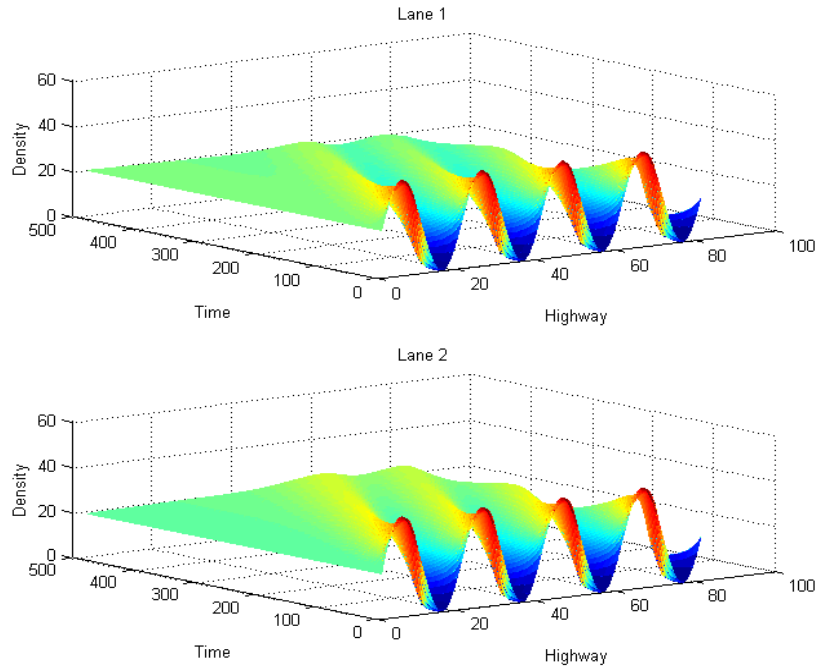


Fig. 3.

The diagram of the scaled initial density profile with transformation back and the scaled density profile for 1.5 minutes are featured in Fig. 4 and Fig. 5 respectively. Here $\rho_1^{\max} = \rho_2^{\max} = 175/\text{km}$, perform numerical experiment for 90 seconds in $\Delta t = 1500$ time steps for a two lanes highway of 2.5 km.

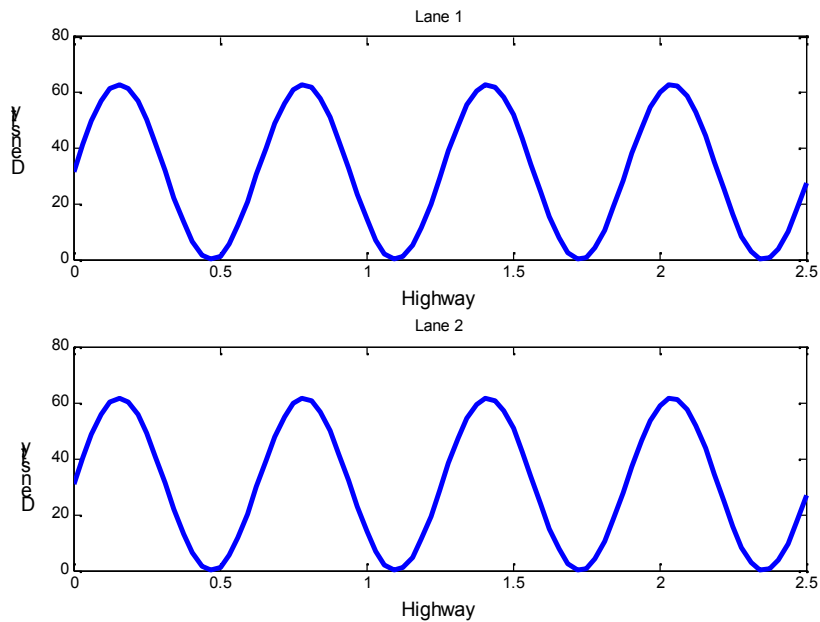


Fig. 4. Scaled Initial density profile with Transformation Back.

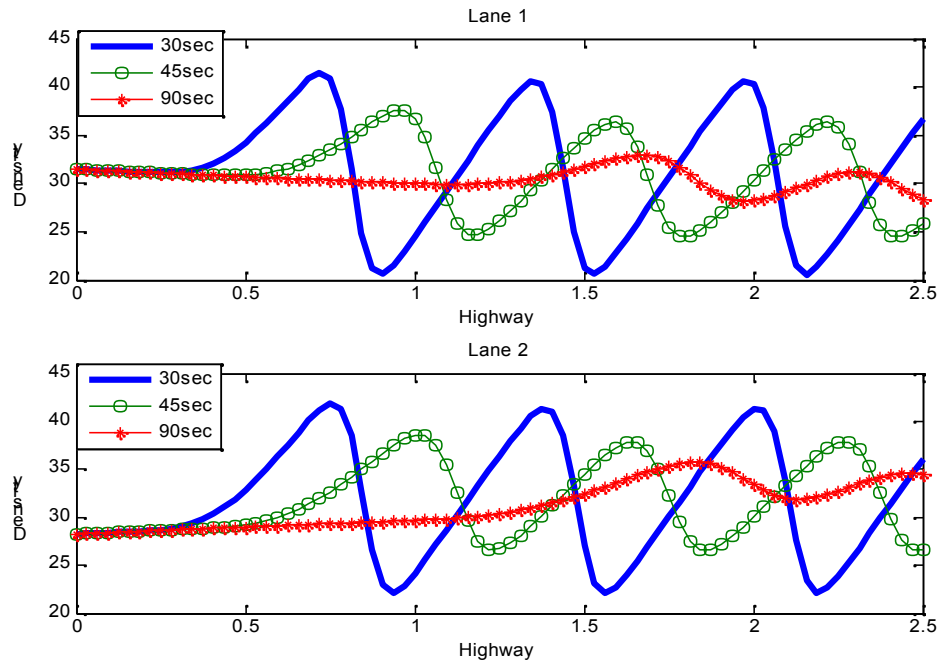


Fig. 5. Density Profile.

We compute the velocity by using Greenschild's velocity density relation, $V(\rho) = V^{max} \left(1 - \frac{\rho}{\rho^{max}}\right)$ which is depicted in Fig. 6.

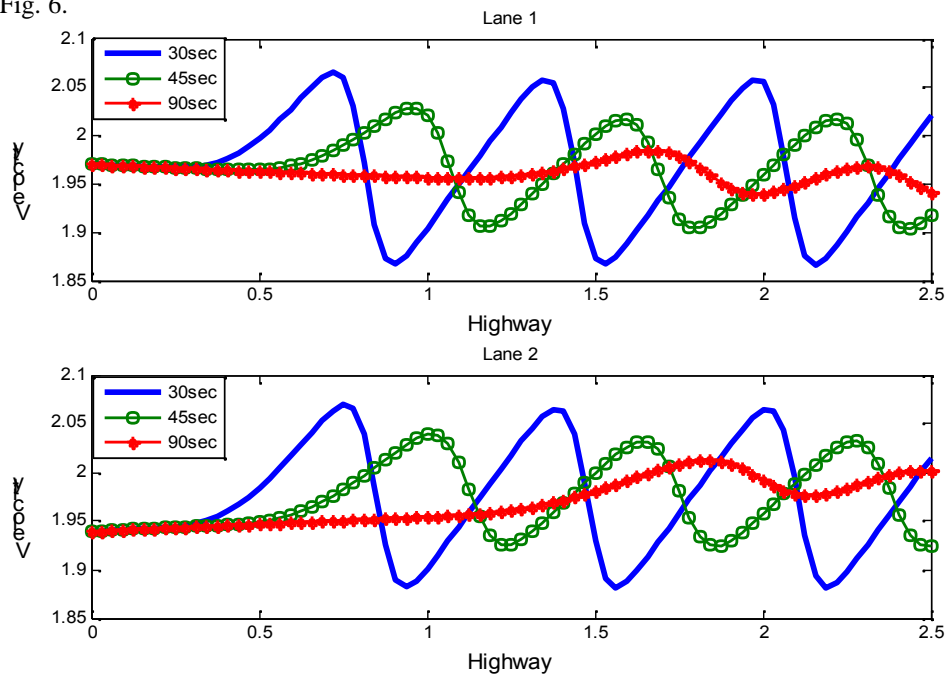


Fig. 6. Computed Velocity Profile.

We are known about the density and velocity of two lane traffic flow for a certain point of highway. The flux of multilane traffic can be computed with the support of the relation $q = \rho v$ and Fig. 7 represents the computed flux with respect to the distance.

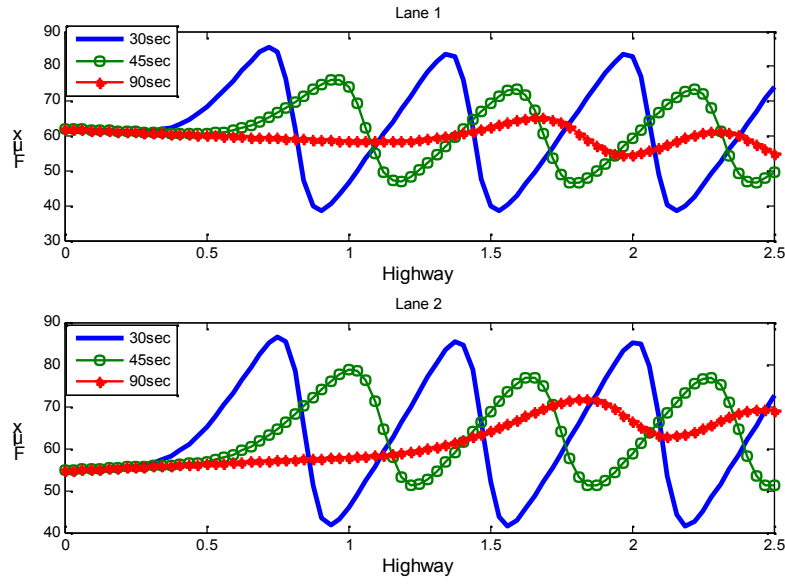


Fig. 7. Computed Flux Profile.

The numerical solution of the scaled two lane traffic flow model with transformation back is presented in Fig. 8 as a surface using the explicit upwind difference scheme.

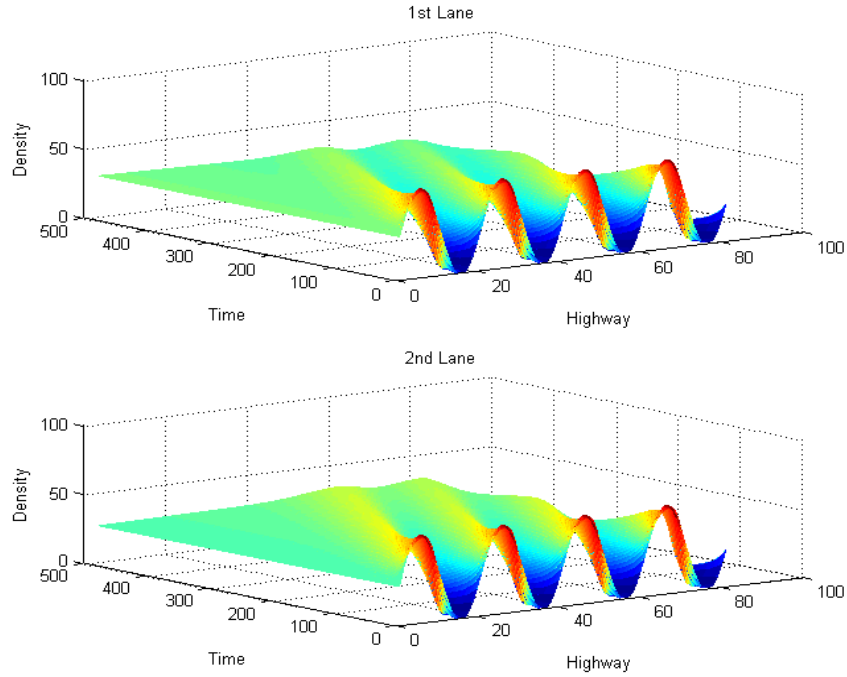


Fig. 8.

In this frame work, we investigate the stability condition for the numerical scheme. The stability condition of Explicit Upwind Difference Scheme for the dimensionless form of the single lane traffic flow model is $\left| \sigma \frac{\Delta t}{\Delta x} \right| \leq 1$

where $\sigma := \max_x u^0(x)$.

In case of the dimensionless form of two lane traffic flow model we examine experimentally that the stability condition of Explicit Upwind Difference Scheme for our considered model is $\left| \sigma \frac{\Delta t}{\Delta x} \right| \leq 0.6$.

VI. CONCLUSION

We have described the explicit upwind difference scheme for the scaled two lane traffic flow model. The numerical simulation of 2.5 km highway of two lanes has been performed for 1.5 minutes using the numerical scheme based on artificially generated initial and boundary data. We have presented the numerical experiments of two lane traffic flow with respect to the dimensionless parameters. An experimental result for the stability restriction of the numerical scheme has also been verified. The stability condition of explicit upwind scheme for our considered model is $\left| \sigma \frac{\Delta t}{\Delta x} \right| \leq 0.6$ whereas the stability condition for scaled single lane traffic model is $\left| \sigma \frac{\Delta t}{\Delta x} \right| \leq 1$; where $\sigma := \max_x u^0(x)$. Since the scaled model is simpler than the non-scaled model, therefore, more computational efficiency for the scaled model has been observed.

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