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Comparison of Numerical Solution with Analytical Solution of one Dimensional Dirichlet-Helmholtz Boundary Value Problem

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ABSTRACT: In this paper, we have studied the analytical solution of one dimensional Dirichlet-Helmholtz boundary value problem. We found the numerical solution of one dimensional Dirichlet-Helmholtz boundary value problem by Finite Element Method (FEM). And then we compare the solutions.

Keywords: Dirichlet-Helmholtz BVP, Finite Element Method, stiffness matrix, Numerical Simulation, L_1 norm, Relative errors.

I. INTRODUCTION

The Helmholtz equation, named for Hermann von Helmholtz, is a partial differential equation ($-\lambda\Delta u + u = f$, where λ is arbitrary non zero constant and it is called Helmholtz parameter), which often arises in the study of physical problems involving partial differential equations (PDEs) in both space and time. The Helmholtz equation is a time-independent form of the original equation.

The FEM is a novel numerical method used to solve ordinary and partial differential equations. The method is based on the integration of the terms in the equation to be solved, in lieu of point discretization schemes like the finite difference method. The FEM utilizes the method of weighted residuals and integration by parts (Green-Gauss Theorem) to reduce second order derivatives to first order terms. The solution domain is discretized into individual elements, these elements are operated upon individually and then solved globally using matrix solution techniques.

In this work, we present the analytical solution of one dimensional Dirichlet-Helmholtz boundary value problem. We describe the finite element method. And then we find the numerical solution of one dimensional Dirichlet-Helmholtz boundary value problem by finite element method. Also we next show that the numerical solution by finite element method converges with the analytical solution when we take a large number of grid points.

II. DIRICHLET-HELMHOLTZ BVP AND ITS ANALYTICAL SOLUTION

A. Dirichlet-Helmholtz BVP

For $x \in \Omega \subseteq R^n, u : \Omega \to R$ with $u \in C^2(\Omega)$,
the Dirichlet-Helmholtz BVP reads as

$$-\lambda\Delta u + u = f \text{ in } \Omega, \quad f \in C(\Omega) \qquad \text{(Helmholtz equation)}$$

$$u = g \text{ on } \partial\Omega, \quad g \in C(\partial\Omega) \qquad \text{(Dirichlet boundary condition)}$$

Here f is the force function and λ is any arbitrary non zero constant. The constant λ determines the role of convection term in the Helmholtz equation.

The one dimensional form of Dirichlet-Helmholtz BVP is

$$-\lambda u''(x) + u(x) = f(x), \quad x \in (0,1) \tag{1}$$

$$u(0) = u(1) = 0$$

Here the domain $\Omega \equiv (0,1)$. For simplicity, we will find the analytical solution of Dirichlet-Helmholtz BVP for $f(x) = \sin x$.

B. Analytical solution of Dirichlet-Helmholtz BVP when $f(x) = \sin(x)$

If $f(x) = \sin x$, then equation (1) reduces as,

$$-\lambda u''(x) + u(x) = \sin x,$$

$$u(0) = u(1) = 0$$

$$\Rightarrow u''(x) - \frac{1}{\lambda} u(x) = -\frac{\sin x}{\lambda}$$

$$u(0) = u(1) = 0 \qquad (2)$$

The above equation is a linear differential equation of second order.

The Auxiliary Equation of (2) is,

$$D^2 - \frac{1}{\lambda} = 0 \qquad \left[\text{ Here } D \equiv \frac{d}{dx}\right]$$

$$\Rightarrow \left(D + \frac{1}{\sqrt{\lambda}}\right)\left(D - \frac{1}{\sqrt{\lambda}}\right) = 0$$

The Complementary Function is,

$$C.F. = C_1 e^{\frac{x}{\sqrt{\lambda}}} + C_2 e^{-\frac{x}{\sqrt{\lambda}}}$$

And the Particular Integral is,

$$P.I. = \frac{1}{D^2 - \frac{1}{\lambda}}\left(-\frac{\sin x}{\lambda}\right) = \frac{1}{1 - \lambda D^2}(\sin x)$$

$$= \frac{\sin x}{1 - \lambda(-1)} \qquad \left[\because \frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax\right]$$

$$= \frac{\sin x}{1 + \lambda}$$

Hence, the Complete Solution is,

$$u(x) = C.F. + P.I.$$

$$\Leftrightarrow u(x) = c_1 e^{\frac{x}{\sqrt{\lambda}}} + c_2 e^{-\frac{x}{\sqrt{\lambda}}} + \frac{\sin x}{1 + \lambda} \qquad (3)$$

Now, applying $u(0) = 0$ in (3), we have,

$$c_1 + c_2 = 0$$

$$c_1 = -c_2 \qquad (4)$$

Again applying $u(1) = 0$ in (3), we have,

$$c_1 e^{\frac{1}{\sqrt{\lambda}}} + c_2 e^{-\frac{1}{\sqrt{\lambda}}} + \frac{\sin 1}{1 + \lambda} = 0$$

$$c_1 e^{\frac{1}{\sqrt{\lambda}}} - c_1 e^{-\frac{1}{\sqrt{\lambda}}} + \frac{\sin 1}{1+\lambda} = 0$$

$$c_1 (e^{\frac{1}{\sqrt{\lambda}}} - e^{-\frac{1}{\sqrt{\lambda}}}) = -\frac{\sin 1}{1+\lambda}$$

$$c_1 = -\frac{\sin 1}{(1+\lambda)(e^{\frac{1}{\sqrt{\lambda}}} - e^{-\frac{1}{\sqrt{\lambda}}})}$$

Putting the value of c_1 in (4) we have , $c_2 = \dfrac{\sin 1}{(1+\lambda)(e^{\frac{1}{\sqrt{\lambda}}} - e^{-\frac{1}{\sqrt{\lambda}}})}$

Hence the required analytical solution of (2.2) is:

$$\Leftrightarrow u(x) = -\frac{\sin 1}{(1+\lambda)(e^{\frac{1}{\sqrt{\lambda}}} - e^{-\frac{1}{\sqrt{\lambda}}})} e^{\frac{x}{\sqrt{\lambda}}} + \frac{\sin 1}{(1+\lambda)(e^{\frac{1}{\sqrt{\lambda}}} - e^{-\frac{1}{\sqrt{\lambda}}})} e^{-\frac{x}{\sqrt{\lambda}}} + \frac{\sin x}{1+\lambda}$$

$$\Leftrightarrow u(x) = \frac{(e^{-\frac{x}{\sqrt{\lambda}}} - e^{\frac{x}{\sqrt{\lambda}}})\sin 1}{(1+\lambda)(e^{\frac{1}{\sqrt{\lambda}}} - e^{-\frac{1}{\sqrt{\lambda}}})} + \frac{\sin x}{1+\lambda}$$ here *f(x) = sin(x),* x Ω ≡ (0,1)

C. Complexity of Analytical Solution

It is not easy to obtain the analytical solution of equation (5) for every value of *f(x).* The biggest disadvantage of the analytical method is that formulations can become very complicated. The more complicated a system is more difficult it will be to analytically formulate an expression for the system's reliability. The disadvantage of the analytical solution lies in the oversimplifications needed in the derivations. When *f(x)* is a complicated function of x, we need more difficult calculations to obtain analytical solution of the one dimensional Dirichlet-Helmholt BVP. For example: It is very difficult to obtain the analytical solution of equation (1) when $f(x) = x^3 e^x \sin x$. Moreover, we need different calculations to obtain analytical solution of Dirichlet-Helmholt BVP for different values of *f(x).*

Way out: To avoid the above complexities of obtaining the Analytical solution of Dirichlet-Helmholtz BVP, we need to find the Numerical solution of (1).

III. NUMERICAL SOLUTION OF DIRICHLET-HELMHOLTZ BVP

The weak form is often an integral form and requires a weaker continuity on the field variables. Due to the weaker requirement on the field variables, and the integral operation, a formulation based on a weak form usually produces a set of system equations that give much more accurate result.
So to find the numerical solution of one dimensional Dirichlet-Helmholtz BVP, we first find the weak formulation of equation (2.1).

A. Weak Formulation of Dirichlet-Helmholtz BVP

Multiplying equation (2.1) with the test function $v \in C_0^{\infty}(\Omega)$ and then integrating by parts in Ω we obtain

$$-\lambda \int_0^1 u''(x) v(x) dx + \int_0^1 u(x) v(x) dx = \int_0^1 f(x) v(x) dx$$

$$\left[-\lambda u'v\right]_0^1+\lambda\int_0^1 u'(x)v'(x)dx+\int_0^1 u(x)v(x)dx=\int_0^1 f(x)v(x)dx$$

$$\Rightarrow\ \lambda\int_0^1 u'(x)v'(x)dx+\int_0^1 u(x)v(x)dx=\int_0^1 f(x)v(x)dx$$

Therefore the weak formulation of simplified Dirichlet-Helmholt BVP is

Find $u\in H_0^{\ 1}(\Omega)$ so that

$$\lambda\int_0^1 u'(x)v'(x)dx+\int_0^1 u(x)v(x)dx=\int_0^1 f(x)v(x)dx;\quad \forall u\in H_0^{\ 1}(\Omega) \tag{5}$$

Here $H_0^{\ 1}(\Omega)$ is the Hilbert space.

B. Numerical Solution by Finite Element Method

The discrete abstract formulation of equation (4.1) is:

Find $u_N\in V_N$ so that $\quad a(u_{N,}v_N)=l(v_N);\quad \forall v_N\in V_N$

Where, $a(u_{N,}v_N)=\lambda\int_0^1 u'(x)v'(x)dx+\int_0^1 u(x)v(x)dx$

$$l(v_N)=\int_0^1 f(x)v(x)dx$$

V_N is a finite dimensional subspace of V so that $\quad u_N=\sum\alpha_i b_i(x)$;

$b_i(x)$ denote the basis function of the finite dimensional space.

Then

$$a(b_i,b_j)\alpha_i=l(b_j);\qquad \forall i,j=1,2,\ldots\ldots\ldots\ldots,N-1$$

$$\Rightarrow (a_{ij})_{i,j=1}^{N-1}\alpha_i=l_j;\qquad \forall i,j=1,2,\ldots\ldots\ldots\ldots,N-1$$

$$\Rightarrow L\alpha=l \tag{6}$$

Where $L=(a_{ij})_{i,j=1}^{N-1}$; $\quad \alpha=\alpha_i$; $\quad l=l_j$; $\forall i,j=1,2,\ldots\ldots\ldots\ldots,N-1$

For N equidistant grid, the basis function is

$$b_i(x):=\begin{cases}\frac{1}{h}(x-x_i)+1 & ; \quad x_{i-1}\le x\le x_i\\ \frac{1}{h}(x_i-x)+1 & ; \quad x_i\le x\le x_{i+1}\\ 0 & ; \quad else\end{cases}\qquad \text{for } i=1,2,\ldots\ldots\ldots\ldots,N-1$$

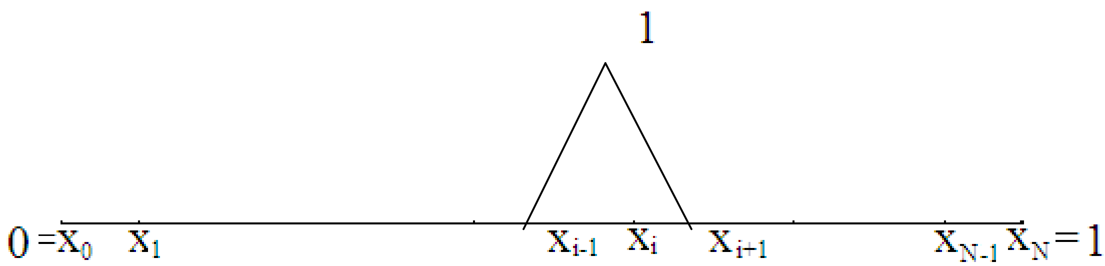

Fig. 3.1: Discretization of basis function.

Now $\quad a_{ij} = \lambda \int_0^1 b'_i(x) b'_j(x)\,dx + \int_0^1 b_i(x) b_j(x)\,dx$

$l_j = \int_0^1 f(x) b_j(x)\,dx$

$$\therefore\ b'_i(x) := \begin{cases} \dfrac{1}{h} & ; \quad x_{i-1} \le x \le x_i \\ -\dfrac{1}{h} & ; \quad x_i \le x \le x_{i+1} \\ 0 & ; \quad else \end{cases}$$

Now, $\quad a_{ij} = \lambda \int_0^1 b'_i(x) b'_i(x)\,dx + \int_0^1 b_i(x) b_i(x)\,dx$

Let $a_{ij} = I_1 + I_2$ (7)

where, $I_1 = \lambda \int_0^1 b'_i(x) b'_i(x)\,dx$

$$= \lambda \int_{x_{i-1}}^{x_i} b'_i(x) b'_i(x)\,dx + \lambda \int_{x_i}^{x_{i+1}} b'_i(x) b'_i(x)\,dx$$

$$= \lambda \int_{x_{i-1}}^{x_i} \frac{1}{h}\frac{1}{h}\,dx + \lambda \int_{x_{i-1}}^{x_i} \left(-\frac{1}{h}\right)\left(-\frac{1}{h}\right) dx$$

$$= \frac{\lambda}{h^2}\left[x_i - x_{i-1}\right] + \frac{\lambda}{h^2}\left[x_{i+1} - x_i\right] \quad = \frac{\lambda}{h} + \frac{\lambda}{h} = \frac{2\lambda}{h}$$

And $I_2 = \lambda \int_0^1 b_i(x) b_i(x)\,dx$

$$= \int_{x_{i-1}}^{x_i} \left\{\frac{1}{h}(x - x_i) + 1\right\}^2 dx + \int_{x_i}^{x_{i+1}} \left\{\frac{1}{h}(x_i - x) + 1\right\}^2 dx$$

Let $I_2 = I'_2 + I''_2$ (8)

Where, $I'_2 = \int_{x_{i-1}}^{x_i} \left\{\frac{1}{h}(x - x_i) + 1\right\}^2 dx$

Let $\frac{1}{h}\left(x-x_i\right)+1=t$

$$\therefore I'_2 = h\int_0^1 t^2\,dt \;=\frac{h}{3}$$

And $I''_2 = \int_{xi}^{x_{i-1}} \left\{\frac{1}{h}\left(x-x_i\right)+1\right\}^2 dx$

$$\therefore I''_2 = h\int_0^1 t^2\,dt = \frac{h}{3}$$

From equation (8) we have

$$I_2 = I'_2 + I''_2 = \frac{h}{3}+\frac{h}{3} = \frac{2h}{3}$$

Putting the values of I_1 and I_2 in (7), we have,

$$a_{ii} = \frac{2\lambda}{h}+\frac{2h}{3} \quad \forall i = 1,2,\ldots\ldots\ldots\ldots, N-1$$

If $\left|i-j\right|=1$

$$a_{ii} = \lambda\int_0^1 b'_i(x)b'_{i+1}(x)\,dx + \int_0^1 b_i(x)b_{i+1}(x)\,dx$$

Let

$$a_{ii} = I_3 + I_4 \tag{9}$$

where, $I_3 = \lambda\int_0^1 b'_i(x)b'_{i+1}(x)\,dx$

$$= \lambda\int_{x_i}^{x_{i+1}} \left(-\frac{1}{h}\right)\left(\frac{1}{h}\right)dx$$

$$= \frac{-\lambda}{h^2}\left(x_{i+1}-x_i\right) \quad = \frac{-\lambda}{h}$$

And $I_4 = \int_0^1 b_i(x)b_{i+1}(x)\,dx \;= \int_{x_i}^{x_{i+1}} \left\{\frac{1}{h}\left(x_i - x\right)+1\right\}\left\{\frac{1}{h}\left(x-x_{i+1}\right)+1\right\}dx$

Let $\frac{1}{h}\left(x-x_{i+1}\right)+1=t$

$$\Rightarrow \frac{1}{h}\left(x_i-x\right)+1 = 1-t$$

$$\therefore I_4 = h\int_0^1 t(1-t)\,dt \quad = \frac{h}{6}$$

Putting the values of I_3 and I_4 in (9), we have,

$$a_{ij} = -\frac{\lambda}{h}+\frac{h}{6} \quad ; \quad \text{when } \left|i-j\right|=1$$

Since a_{ij} is symmetric

$$\therefore\ a_{ij} = a_{ji} = -\frac{\lambda}{h}+\frac{h}{6} \quad ; \quad \text{for all } i, j \text{ satisfying } |i-j| = 1$$

Hence
$$a_{ij} = \begin{cases} \dfrac{2\lambda}{h}+\dfrac{2h}{3} & ; \quad i = j \\ -\dfrac{\lambda}{h}+\dfrac{h}{6} & ; \quad |i-j| = 1 \\ 0 & ; \quad else \end{cases}$$

Therefore, the stiffness matrix is

$$L=\begin{bmatrix} \frac{2\lambda}{h}+\frac{2h}{3} & -\frac{\lambda}{h}+\frac{h}{6} & 0 & & 0 & 0 & 0 \\ -\frac{\lambda}{h}+\frac{h}{6} & \frac{2\lambda}{h}+\frac{2h}{3} & -\frac{\lambda}{h}+\frac{h}{6} & \cdots & 0 & 0 & 0 \\ 0 & -\frac{\lambda}{h}+\frac{h}{6} & \frac{2\lambda}{h}+\frac{2h}{3} & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & \frac{2\lambda}{h}+\frac{2h}{3} & -\frac{\lambda}{h}+\frac{h}{6} & 0 \\ 0 & 0 & 0 & \cdots & -\frac{\lambda}{h}+\frac{h}{6} & \frac{2\lambda}{h}+\frac{2h}{3} & -\frac{\lambda}{h}+\frac{h}{6} \\ 0 & 0 & 0 & & 0 & -\frac{\lambda}{h}+\frac{h}{6} & \frac{2\lambda}{h}+\frac{2h}{3} \end{bmatrix}$$

Now, $l_j = \int_0^1 f(x) b_i(x)\,dx$

$$= f\int_0^1 b_i(x)\,dx = f\int_{x_{i-1}}^{x_i}\left\{\frac{1}{h}(x-x_i)+1\right\}dx + f\int_{x_i}^{x_{i+1}}\left\{\frac{1}{h}(x_i - x)+1\right\}dx$$

Let $l_j = I_5 + I_6$ (10)

Where, $I_5 = f\int_{x_{i-1}}^{x_i}\left\{\frac{1}{h}(x-x_i)+1\right\}dx$

Let $\frac{1}{h}(x-x_i)+1 = t$

$$\therefore I_5 = f\int_0^1 ht\,dt = fh\left[\frac{t^2}{2}\right]_0^1 = \frac{1}{2}fh$$

Similarly, we have $I_6 = \frac{1}{2} fh$

$$\therefore l_j = I_5 + I_6 = \frac{1}{2} fh + \frac{1}{2} fh = fh$$

i.e, $$l = \begin{bmatrix} fh \\ fh \\ fh \\ \vdots \\ fh \end{bmatrix}_{N-1}$$

From equation (4.2) we have, $L\alpha = l$

$$\Rightarrow \alpha = L^{-1} l$$

$$\Rightarrow \alpha = \begin{bmatrix} \frac{2\lambda}{h}+\frac{2h}{3} & -\frac{\lambda}{h}+\frac{h}{6} & 0 & & 0 & 0 & 0 \\ -\frac{\lambda}{h}+\frac{h}{6} & \frac{2\lambda}{h}+\frac{2h}{3} & -\frac{\lambda}{h}+\frac{h}{6} & \cdots & 0 & 0 & 0 \\ 0 & -\frac{\lambda}{h}+\frac{h}{6} & \frac{2\lambda}{h}+\frac{2h}{3} & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & \frac{2\lambda}{h}+\frac{2h}{3} & -\frac{\lambda}{h}+\frac{h}{6} & 0 \\ 0 & 0 & 0 & \cdots & -\frac{\lambda}{h}+\frac{h}{6} & \frac{2\lambda}{h}+\frac{2h}{3} & -\frac{\lambda}{h}+\frac{h}{6} \\ 0 & 0 & 0 & & 0 & -\frac{\lambda}{h}+\frac{h}{6} & \frac{2\lambda}{h}+\frac{2h}{3} \end{bmatrix}^{-1} \begin{bmatrix} fh \\ fh \\ fh \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ fh \end{bmatrix}$$

which is our required numerical solution of one dimensional Dirchlet-Helmholtz boundary value problem by finite element method.

C. Error Estimation of the Numerical solution

We compute the Relative error in L_1 norm defined by

$$\| e \|_1 := \frac{\| u_e - u_n \|_1}{\| u_e \|_1} \tag{11}$$

for all time, where u_e is the exact solution and u_n is the numerical solution computed by the finite element method.

If $f(x) = \sin x$, then the relative errors of the numerical solution with respect to analytical solution of (1) for different values of λ and for different number of grid point are given below:

Table 1: Relative errors in L_1 norm of Numerical solution with respect to Analytical solution (*N is the number of grid points***).**

	N=25	N=50	N=100	N=200	N=400	N=800	N=1600	N=3200
λ=1	.0685	.0351	.0177	.0089	.0043	.0021	.0010	.0006
λ=.1	.0386	.0197	.0098	.0048	.0023	.0011	.0005	.0003
λ=.01	.0109	.0047	.0023	.0010	.0005	.0003	. 0002	. 0001

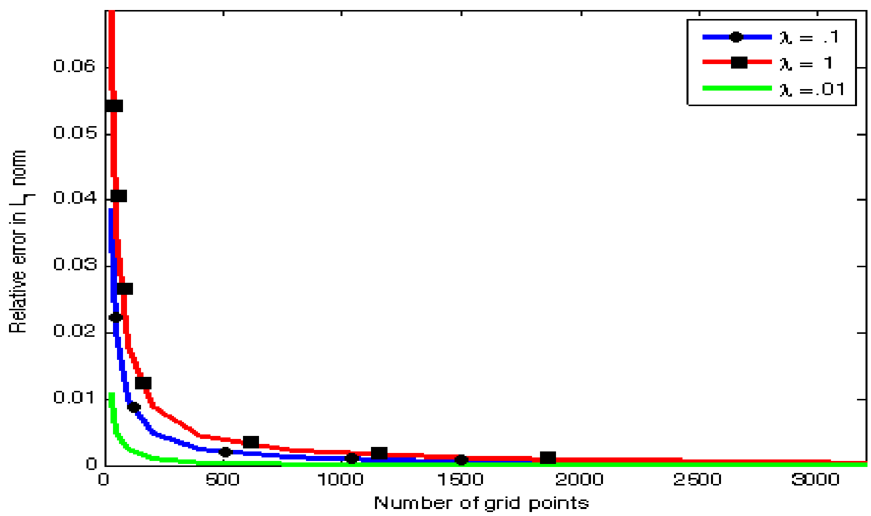

Fig. 1: Relative error decreases as the number of grid points increases.

Table.1 and the figure.1 shows the relative error of numerical solution of Dirichlet-Helmholtz BVP. Figure 1 presents that the error is decreasing with respect to increasing of number of grid points. This shows the convergence of the numerical solution. As number of grid point increases our calculation steps are also increasing. Therefore we have to calculate every small part of our domain. So as we increase the number of grid points the error will be decreased proportionally.

Also we see from Table 3.1 and the figure 1 that, the relative error decreases as we decrease ë. When we decrease ë the role of second order derivative in convection term “$-\lambda u''(x)$” of Helmholtz equation is also decreases. And for this reason the relative error is also decreases.

D. Comparison of Numerical Solution with Analytical Solution

We will see in the following figures that as the number of grid point increases the numerical solution converge to analytical solution.

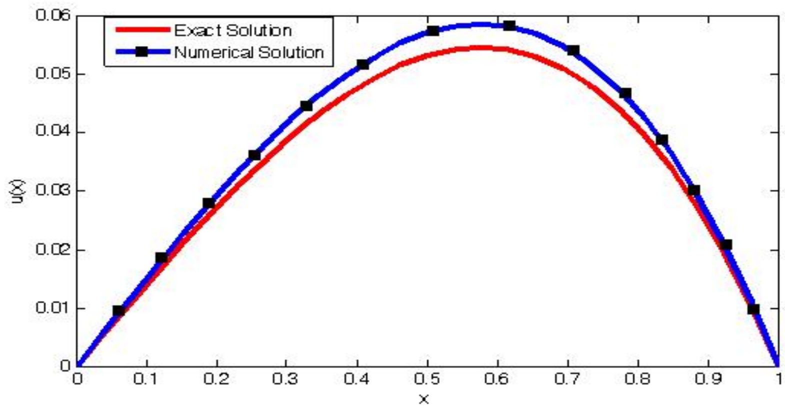

Fig. 2: Comparison of Numerical solution with Analytical solution.[f(x) = sin(x), N=25 and λ=1].

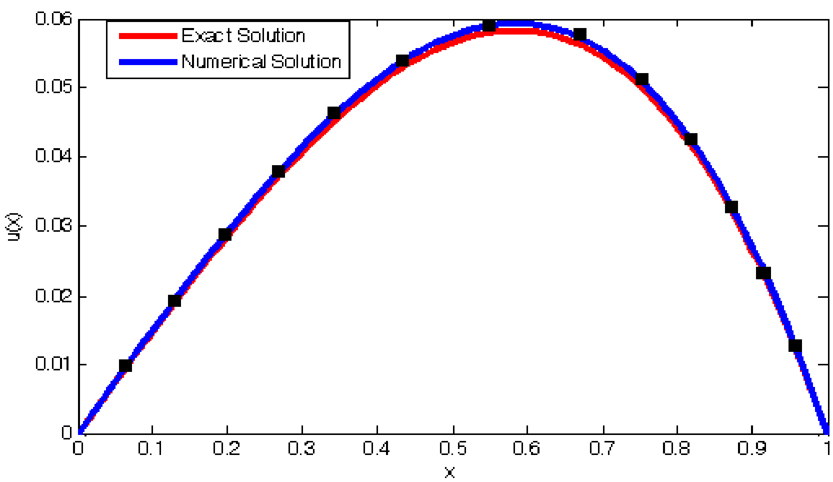

Fig. 3: Comparison of Numerical solution with Analytical solution.
[f(x)=sin(x), N=100 and λ=1].

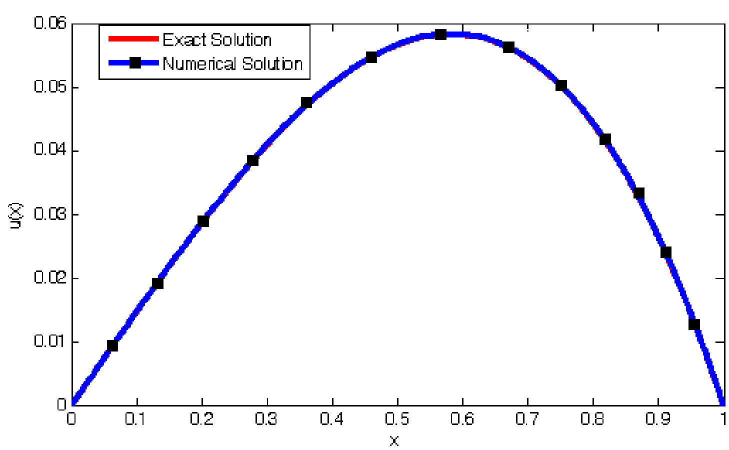

Fig. 4: Comparison of Numerical solution with Analytical solution.
[f(x) = sin(x), N=1600 and λ=1].

In figure 4 the number of grid points is 1600. We see that the both solution are coincide each other. This shows numerical solution converges with the analytical solution when we use large number of grid points.

IV. CONCLUSION

In this paper, we have presented analytical solution of one dimensional Dirichlet-Helmholtz boundary value problem and we find numerical solution of this simplified boundary value problem by finite element method. On implementing the numerical solution in MATLAB we have found the relative errors, which show a good rate of convergence of the solutions.

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