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Common Fixed Point Theorem on Six Mappings

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ABSTRACT: The purpose of this paper is to present a common fixed point theorem in a metric space which extends the result of Singh and Chauhan for six self maps using the weaker conditions such as Weakly compatible mappings and Associated sequence in place of compatibility and completeness of the metric space. More over the condition of continuity of any one of the mapping is being dropped.

Keywords: Fixed point, self maps, compatible mappings, weakly compatible mappings, associated sequence.

AMS (2000) Mathematics Classification: 54H25, 47H10.

I. INTRODUCTION

G. Jungck [1] introduced the concept of compatible maps which is weaker than weakly commuting maps. After wards Jungck and Rhoades [3] defined weaker class of maps known as weakly compatible maps.

A. Definitions and Preliminaries

Compatible mappings. Two self maps S and T of a metric space (X,d) are said to be compatible mappings if $\lim_{n\to\infty} d(STx_n, TSx_n)=0$, whenever $<x_n>$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Weakly compatible. Two self maps S and T of a metric space (X, d) are said to be weakly compatible if they commute at their coincidence point. *i.e* if Su = Tu for some $u \in X$ then Stu = TSu. It is clear that every compatible pair is weakly compatible but its converse need not be true. Singh and Chauhan proved the following theorem [4].

Theorem (A): Let A, B, S and T be self mappings from a complete metric space (X, d) into itself satisfying the following conditions

$$A(X) \subseteq T(X) \text{ and } B(X) \subseteq S(X) \qquad \dots (1)$$

One of A, B, S or T is continuous……(2)

$$[d(Ax,By)]^2 \le k_1 \begin{bmatrix} d(Ax,Sx)\ d(By,Ty)+ \\ d(By,Sx)\ d(Ax,Ty) \end{bmatrix} + k_2 \begin{bmatrix} d(Ax,Sx)\ d(Ax,Ty)+ \\ d(By,Ty)\ d(By,Sx) \end{bmatrix} \quad \dots (3)$$

$$\forall\ x,y \in X \quad where\ \ 0 \le k_1 + 2k_2 < 1, k_1.k_2 \ge 0.$$

The pairs (A, S) and (B, T) are compatible on X ……(4)

Further if X is a complete metric space …..(5)
Then A, B, S and T have a unique common fixed point in X.
Now we generalize the theorem(A) using weakly compatible mappings and Associated Sequence.

Associated Sequence: Suppose A,B,S,P, Q and T are six self maps of a metric space (X, d) satisfying the conditions $P(X) \subseteq AB(X)$ and $Q(X) \subseteq ST(X)$. Then for an arbitrary $x_0 \in X$ such that $Px_0 = ABx_1$ and for this point x_1, there exist a point x_2 in X such that $Qx_1 = STx_2$ and so on.

Proceeding in the similar manner, we can define a sequence <y_n> in X such that $y_{2n} = Px_{2n} = ABx_{2n+1}$ and $y_{2n+1} = Qx_{2n+1} = STx_{2n+2}$ for n ≥ 0. We shall call this sequence as an "*Associated sequence of x_0* "relative to the six self maps A, B, S ,T, P and Q.

Lemma: Suppose A,B,S,P,Q and T are six self maps of a complete metric space (X, d) satisfying the conditions

P(X) ⊆ AB(X) and Q(X) ⊆ ST(X)……(1)

and

$$\left[d(Px,Qy)\right]^2 \le k_1\begin{bmatrix} d(Px,STx)\ d(Qy,ABy)+ \\ d(Qy,STx)\ d(Px,ABy \end{bmatrix}$$
$$+k_2\begin{bmatrix} d(Px,STx)\ d(Px,ABy)+ \\ dQy,ABy)\ d(Qy,STx) \end{bmatrix}$$
$$Forall\ \ x,y\in X \quad where\ \ 0\le k_1+2k_2<1, k_1,k_2\ge 0. \quad ……(2)$$

Then the sequence $Px_0,Qx_1…..Px_{2n},Qx_{2n+1}$….converges to z εX.

Proof: From the conditions (1), (2) and from the definition of associated sequence we have

$$put\ x=x_{2n}\ \ ,y=x_{2n-1}$$
$$[d(y_{2n},y_{2n-1})]^2=[d(Px_{2n},Qx_{2n-1})]^2$$
$$\le k_1\begin{bmatrix} d(Px_{2n},STx_{2n})d(Qx_{2n-1},ABx_{2n-1})+ \\ d(Qx_{2n-1}\,STx_{2n})d(Px_{2n},ABx_{2n-1}) \end{bmatrix}$$
$$+k_2\begin{bmatrix} d(Px_{2n},STx_{2n})\ d(Px_{2n},ABx_{2n-1})+ \\ d(Qx_{2n-1},ABx_{2n-1})\ d(Qx_{2n-1}\,STx_{2n}) \end{bmatrix}$$
$$[d(y_{2n},y_{2n-1})]^2\le k_1[d(y_{2n},y_{2n-1})d(y_{2n-1},y_{2n-2})+0]$$
$$+k_2[d(y_{2n},y_{2n-1})d(y_{2n},y_{2n-2})+0]$$
$$[d(y_{2n},y_{2n-1})]^2\le\begin{Bmatrix} k_1[\,d(y_{2n-1},y_{2n-2})]\ + \\ k_2[d(y_{2n},y_{2n-2})] \end{Bmatrix}d(y_{2n},y_{2n-1})$$
$$[d(y_{2n},y_{2n-1})]\le\{k_1[\,d(y_{2n-1},y_{2n-2})]\ +k_2[d(y_{2n},y_{2n-2})]\}$$
$$[d(y_{2n},y_{2n-1})]\le\begin{Bmatrix} k_1[\,d(y_{2n-1},y_{2n-2})]\ + \\ k_2[d(y_{2n},y_{2n-1})+(y_{2n-1},y_{2n-2})] \end{Bmatrix}$$
$$[d(y_{2n},y_{2n-1})][1-k_2]\le\{k_1+k_2\}[\,d(y_{2n-1},y_{2n-2})]$$

This implies

$$[d(y_{2n},y_{2n-1})]\le\frac{k_1+k_2}{[1-k_2]}[\,d(y_{2n-1},y_{2n-2})]$$
$$\le h[\,d(y_{2n-1},y_{2n-2})]$$
$$where\ \ h=\frac{k_1+k_2}{1-k_2}<1$$

$$
\begin{aligned}
&For\ every\ \ integer\ p>0, we\ get\\
&d(y_n, y_{n+p}) \le\ d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \\
&\qquad\qquad ....+\ \ d(y_{n+p-1}, y_{n+p})\\
&\qquad \le h^n d(y_0, y_1) + h^{n+1} d(y_0, y_1) + \\
&\qquad\qquad ....+ h^{n+p-1} d(y_0, y_1)\\
&\qquad \le \left(h^n + h^{n+1} + ...+ h^{n+p-1}\right) d(y_0, y_1)\\
&\qquad \le h^n \left(1 + h + h^2 + ...+ h^{p-1}\right) d(y_0, y_1)
\end{aligned}
$$

Since $h < 1$, $h^n \to 0$ as $n \to \infty$, so that $d(y_n, y_{n+}) \to 0$. This shows that the sequence $\{y_n\}$ is a cauchy sequence in X and since X is a complete metric space, it converges to a limit, say $z \in X$.
The converse of the Lemma is not true, that is A,B,S, T,P and Q are self maps of a metric space (X,d) satisfying (1) and (2), even if for $x_0 \in X$ and for associated sequence of x_0 converges, the metric space (X,d) need not be complete. The following example establishes this.

Example: Let X = (-1,1) with $d(x,y) = \left| x - y \right|$

$$
Px = Qx = \begin{cases} \dfrac{1}{5} & if\ -1 < x < \dfrac{1}{6} \\ \dfrac{1}{6} & if\ \dfrac{1}{6} \le x < 1 \end{cases}
$$

$$
Bx = \begin{cases} \dfrac{1}{5} & if\ -1 < x < \dfrac{1}{6} \\ \dfrac{1}{3} - x & if\ \dfrac{1}{6} \le x < 1 \end{cases}
$$

$$
Ax = Sx = \begin{cases} x & if\ -1 \le x < \dfrac{1}{6} \\ x & if\ \dfrac{1}{6} \le x < 1 \end{cases}
$$

$$
Tx = \begin{cases} \dfrac{1}{5} & if\ -1 < x < \dfrac{1}{6} \\ \dfrac{6x+5}{36} & if\ \dfrac{1}{6} \le x < 1 \end{cases}
$$

$$
ABx = \begin{cases} \dfrac{1}{5} & if\ -1 < x < \dfrac{1}{6} \\ \dfrac{1}{3} - x & if\ \dfrac{1}{6} \le x < 1 \end{cases}
$$

$$STx = \begin{cases} \frac{1}{5} & if \ -1 < x < \frac{1}{6} \\ \frac{6x+5}{36} & if \ \frac{1}{6} \le x < 1 \end{cases}$$

Then P(X) = Q(X) = $\left\{\frac{1}{5}, \frac{1}{6}\right\}$ while AB(X) = $\left\{\frac{1}{5} \cup \left[\frac{1}{6}, \frac{-2}{3}\right)\right\}$ ST(X) = $\left\{\frac{1}{5} \cup \left[\frac{1}{6}, \frac{11}{36}\right)\right\}$

so that the conditions P(X) $\subseteq$ AB(X and Q(X) $\subseteq$ ST(X) .Also the metric in equality can be easily verified with appropriate values of k_1 and k_2 lies between 0 and 1 .Also The sequence Px_0, Qx_1, Px_2, Qx …. Converges to $\frac{1}{6}$ but (X,d) is not a complete metric space.

Now we generalize the above Theorem (A) in the following form.

Theorem (B): Let A, B, P, Q, S and T are self maps of a metric space (X, d) satisfying the conditions P(X) $\subseteq$ AB(X) and Q(X) $\subseteq$ ST(X) …..(1)

$$[d(Px,Qy)]^2 \le k_1 \begin{bmatrix} d(Px,STx)\ d(Qy,ABy) + \\ d(Qy,STx)\ d(Px,ABy \end{bmatrix}$$
$$+ k_2 \begin{bmatrix} d(Px,STx)\ d(Px,ABy) + \\ dQy,ABy)\ d(Qy,STx) \end{bmatrix}$$
$$Forall\ \ x,y \in X \quad where\ \ 0 \le k_1 + 2k_2 < 1, k_1, k_2 \ge 0. \quad \text{…..(2)}$$

The pairs (P, ST) is Reciprocally continuous and compatible and (Q, AB) are weakly compatible …..(3) and

AB = BA, ST = TS,TP = PT,QA = AQ …….(4)

The squence $Px_0, Qx_1 \ldots.. Px_{2n}, Qx_{2n+1} \ldots.$converges to z εX….(5)

Then A, B, P, Q, S and T have a unique common fixed point z in X.

Proof: Since $Px_{2n} \to z$, $ABx_{2n+1} \to z$, $Qx_{2n+1} \to z$, $STx_{2n+2} \to z$ as $n \to \infty$.

From the condition P(X) $\subseteq$ AB(X) implies there exists u$\in$ X such that z = ABu.

Also from the condition Q(X $\subseteq$ ST(X) implies there exists v$\in$ X such that z = STv.

Since (P, ST) is reciprocally continuous $P(ST)x_{2n} \to Pz$ and $(ST)Px_{2n} \to STz$ as $n \to \infty$.

Also since (P, ST) is compatible $\lim_{n \to \infty} d(P(ST)x_{2n}, (ST)Px_{2n}) =$

$\lim_{n \to \alpha}$= STz. This implies Pz = STz

To prove Pz = z, Put x = z, y = x_{2n+1}

$$[d(Pz,Qx_{2n+1})]^2 \le k_1\begin{bmatrix} d(Pz,STz)\ d(Qx_{2n+1},ABx_{2n+1}) \\ +d(Qx_{2n+1},STz)\ d(Pz,ABx_{2n+1} \end{bmatrix} + k_2\begin{bmatrix} d(Pz,STz)\ d(Pz,ABx_{2n+1})+ \\ d(Qx_{2n+1},ABx_{2n+1})\ d(Qx_{2n+1},STz) \end{bmatrix}$$

$$[d(Pz,z)]^2 \le k_1\begin{bmatrix} d(Pz,Pz)\ d(z,z)+ \\ d(z,Pz)\ d(Pz,z) \end{bmatrix} + k_2\begin{bmatrix} d(Pz,Pz)\ d(Pz,z)+ \\ d(z,z)\ d(z,Pz) \end{bmatrix}$$

$$[d(Pz,z)]^2 \le k_1[d(Pz,z)]^2$$

$$[d(Pz,z)]^2(1-k_1) \le 0,\ \text{since}\ k_1+2k_2<1\ \textit{this gives}$$

$$[d(Pz,z)] \le 0$$

$$Pz = z$$

$$\therefore Pz = STz = z$$

To prove Qu = z, put x = z and y = u.

$$[d(Pz,Qu)]^2 \le k_1\begin{bmatrix} d(Pz,STz)\ d(Qu,ABu)+ \\ d(Qu,STz)\ d(Pz,ABu \end{bmatrix} + k_2\begin{bmatrix} d(Pz,STz)\ d(Pz,ABu)+ \\ d(Qu,ABu)\ d(Qu,STz) \end{bmatrix}$$

$$[d(z,Qu)]^2 \le k_1\begin{bmatrix} d(z,z)\ d(Qu,z)+ \\ d(Qu,z)\ d(z,z) \end{bmatrix} + k_2\begin{bmatrix} d(z,z)\ d(z,z)+ \\ d(Qu,z)\ d(Qu,z) \end{bmatrix}$$

$$[d(z,Qu)]^2 \le k_2[d(Qu,z)]^2$$

$$[d(Qu,z)]^2(1-k_2) \le 0$$

$$[d(Qu,z)] \le 0$$

$$Qu = z$$

Since (Q, AB) is weakly compatible implies Q[(AB)]u = [AB(Q)]u $\Rightarrow$ Qz = ABz

To prove z = Qz, put x = x_{2n}, y = z

$$[d(Px_{2n},Qz)]^2 \le k_1\begin{bmatrix} d(Px_{2n},STx_{2n})\, d(Qz,ABz)+ \\ d(Qz,STx_{2n})\, d(Px_{2n},ABz) \end{bmatrix}$$
$$+k_2\begin{bmatrix} d(Px_{2n},STx_{2n})\, d(Px_{2n},ABz)+ \\ d(Qz,ABz)\, d(Qz,STx_{2n}) \end{bmatrix}$$

$$[d(z,Qz)]^2 \le k_1[d(z,z)\, d(Qz,Qz)+d(Qz,z)\, d(z,Qz)]$$
$$+k_2[d(z,z)\, d(z,Qz)+d(Qz,Qz)\, d(Qz,z)]$$

$$[d(z,Qz)]^2 \le k_1[d(Qz,z)]^2$$

$$[d(Qz,z)]^2(1-k_1) \le 0, \text{since } k_1+2k_2<1$$

$$[d(Qz,z)] \le 0 \text{ this gives}$$

$$Qz = z$$

$$\therefore Qz = ABz = z$$

To prove Az = z, put x = z, and y = Az.

$$[d(Pz,Q(Az))]^2 \le$$
$$k_1\begin{bmatrix} d(Pz,STz)\, d(Q(Az),AB(Az))+ \\ d(Q(Az),STz)\, d(Pz,AB(Az) \end{bmatrix}$$
$$+k_2\begin{bmatrix} d(Pz,STz)\, d(Pz,AB(Az))+ \\ d(QAz,ABAz)\, d(QAz,STz) \end{bmatrix}$$

$$[d(Pz,Az)]^2 \le$$
$$k_1\begin{bmatrix} d(Pz,STz)\, d(Az,(AB)Az))+ \\ d(Az,STz)\, d(Pz,(AB)Az) \end{bmatrix}$$
$$+k_2\begin{bmatrix} d(Pz,STz)\, d(Pz,(AB)Az)+ \\ d(Az,(AB)Az)\, d(Az,STz) \end{bmatrix}$$

$$[d(z,Az)]^2 \le k_1\begin{bmatrix} d(z,z)\, d(Az,Az))+ \\ d(Az,z)\, d(z,Az) \end{bmatrix}$$
$$+k_2\begin{bmatrix} d(z,z)\, d(z,Az)+ \\ d(Az,Az)\, d(Az,z) \end{bmatrix}$$

$$[d(z,Az)]^2 \le k_1[d(Az,z)\, d(z,Az)]$$

$$[d(z,Az)]^2(1-k_1) \le 0, \text{since } k_1+2k_2<1$$

$$[d(z,Az)] \le 0 \text{ this gives}$$

$$Az = z$$

$$(AB)z = z \Rightarrow (BA)z = z \Rightarrow Bz = z$$

To prove Tz = z, put x = Tz, y = z

$$[d(PTz,Qz)]^2 \le k_1\begin{bmatrix} d(PTz,STTz)\ d(Qz,ABz)+ \\ d(Qz,STTz)\ d(PTz,ABz) \end{bmatrix} + k_2\begin{bmatrix} d(PTz,STTz)\ d(PTz,ABz)+ \\ d(Qz,ABz)\ d(Qz,STTz) \end{bmatrix}$$

$$[d(P(Tz),Qz)]^2 \le k_1\begin{bmatrix} d(P(Tz),ST(Tz))\ d(Qz,ABz)+ \\ d(Qz,ST(Tz))\ d(P(Tz),ABz) \end{bmatrix} + k_2\begin{bmatrix} d(P(Tz),ST(Tz))\ d(P(Tz),ABz) \\ +d(Qz,ABz)\ d(Qz,ST(Tz)) \end{bmatrix}$$

$$[d(Tz,z)]^2 \le k_1\begin{bmatrix} d(Tz,Tz)\ d(z,z)+ \\ d(z,Tz)\,d(Tz,z) \end{bmatrix} + k_2\begin{bmatrix} d(Tz,Tz)\ d(Tz,z)+ \\ d(z,z)\ d(z,Tz) \end{bmatrix}$$

$$[d(Tz,z)]^2 \le k_1[\,d(z,Tz)\,d(Tz,z)]$$

$$[d(Tz,z)]^2 \le k_1[d(Tz,z)]^2$$

$$[d(Tz,z)]^2(1-k_1) \le 0\ ,\text{since } k_1+2k_2<1 \text{ this gives}$$

$$d(Tz,z) \le 0$$

$$Tz = z$$

$$STz = z \Rightarrow Sz = z$$

$$\therefore Az = Bz = Sz = Tz = Pz = Qz = z$$

We get z in a common fixed point of A, B, P, Q, S and T. The uniqueness of the fixed point can be easily proved.

Remark: From the example given above, clearly the pairs (P, ST) is reciprocally continuous and compatible and (Q, AB) are weakly compatible as they commute at coincident point $\frac{1}{6}$. But the pair (Q, AB) are not compatible

For this, take a sequence $x_n=\left(\frac{1}{6}+\frac{1}{n}\right)$ for n≥1, then $\lim_{n\to\infty} Px_n = \lim_{n\to\infty} STx_n = \frac{1}{6}$ and $\lim_{n\to\infty} P(ST)x_n = \frac{1}{6}$ also $\lim_{n\to\infty} (ST)Px_n = \frac{1}{6}$. So that $\lim_{n\to\infty} d(P(ST)x_n,(ST)Px_n) = 0$. Also note that none of the mappings are continuous and the rational inequality holds for the values of $0 \le k_1+2k_2<1,\ \textit{where}\ \ k_1,k_2 \ge 0$. Clearly $\frac{1}{6}$ is the unique common fixed point of A, B, P, Q, S and T.

Remark: Theorem (B) is a generalization of Theorem (A) by virtue of the weaker conditions such as weakly compatibility of the pairs (P, ST) and (Q, AB) in place of compatibility of (A, S) and (B, T); The continuity of any one of the mappings is being dropped .

REFERENCES

[1]. Jungck.G, (1986) Compatible mappings and common fixed points, *Internat. J. Math. & Math. Sci.* **9**, 771- 778.

[2]. R.P.Pant,(1999) A Common fixed point theorem under a new condition, *Indian J. of Pure and App. Math.*, **30**(2), 147-152.

[3]. Jungck.G. and Rhoades.B.E.(1998) Fixed point for set valued functions without continuity, *Indian J. Pure. Appl. Math.*, **29** (3), 227-238.

[4]. Bijendra Singh and S.Chauhan, (1998) On common fixed poins of four mappings, *Bull. Cal. Math. Soc*., **88**, 301-308.

[5] Srinivas V, B.V.B Reddy & R.Umamaheshwar Rao, (2013) Anaysis on a common fixed point theorem , *IOSR journal of Mathematics*, vol. **5**, 1-4.

[6] Srinivas V, B.V.B Reddy & R. Umamaheshwar Rao , (2012), A focus on a common fixed point theorem using weakly compatible mappings, *Mathematical Theory and Modeling*, vol. **2**, No.3, 60-65.