International Journal of Theoretical & Applied Sciences, **5**(1): 47-52 (2013)
ISSN No. (Print): 0975-1718
ISSN No. (Online): 2249-3247

Common Fixed Point Theorem in Hilbert Space Using Rational Inequlity

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(Received 05 February, 2013, Accepted 02 March, 2013)

ABSTRACT: Our aim of this paper to generalize the result due to Hema Yadav [9]. In this paper we will prove a common fixed point theorem using rational inequality in Hillbert space.
AMS 2000 Subject classification : Primary 47H10; secondary 54H25.

Keywords : Hillbert Space , Contraction mapping, common fixed point.

I. INTRODUCTION

In linear spaces there are two general iterations which have been successfully applied to fixed point problem of operators and also for obtaining solution of operator equations. These are Ishikawa scheme [2] and Mann iteration scheme [4]. The Ishikawa iteration scheme [2] was first used to establish the strong convergence for a pseudo contractive self mapping of a convex compact subset of a Hillbert Space. The Mann iteration scheme was introduced by Mann [4] and has been applied extensively to fixed point problems. In [5], [1], [6], [7], and [8] it has shown that for a mapping T satisfying certain conditions, if the sequence of Mann iterates converges, then it converges to a fixed point of T.

Let C be a non empty convex subset of a normed space E and $T: C \to C$ be a mapping. The mann iteration process is defined by the sequence $\{x_n\}$[4]:

$$x_1 = x \in C,$$
$$x_{n+1} = (1 - b_n)x_n + b_n T x_n, \quad n \in N \qquad \ldots(1)$$

where N denote the set of all positive integer. Where $\{b_n\}$ is a sequence in [0,1].

Liu [3] introduced the concept of Mann iteration process with errors by the sequence $\{x_n\}$ defined as follows:

$$\left.\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= (1 - b_n)x_n + b_n T x_n + u_n, \quad n \in N. \end{aligned}\right\} \qquad \ldots(2)$$

where $\{b_n\}$ is a sequence in [0,1] and $\{u_n\}$ satisfy $\sum_{n=1}^{\infty} \|u_n\| < \infty$. This surely contains (1).

Let X be a Banach space and and C be a non- empty subset of X. Let $T_1 T_2 : C \to C$ be two mappings. The iteration scheme called I- scheme is defined as follows:

$$x_0 \in C \qquad \ldots(3)$$
$$\left.\begin{aligned} y_{2n} &= \beta_{2n} T_1 x_{2n} + (1 - \beta_{2n}) x_{2n}, \quad n \geq 0 \\ x_{2n+1} &= (1 - \alpha_{2n}) x_{2n} + \alpha_{2n} T_2 y_{2n}, \quad n \geq 0 \end{aligned}\right\} \qquad \ldots(4)$$
$$\left.\begin{aligned} y_{2n+1} &= \beta_{2n+1} T_1 x_{2n+1} + (1 - \beta_{2n+1}) x_{2n+1}, \quad n \geq 0 \\ x_{2n+2} &= (1 - \alpha_{2n+1}) x_{2n+1} + \alpha_{2n+1} T_2 y_{2n+1}, \quad n \geq 0 \end{aligned}\right\} \qquad \ldots(5)$$

In this paper we shall make the assumption that

(i) $0 \leq \alpha_{2n} \leq \beta_{2n} \leq 1$, for all n ,

(ii) $\lim_{n\to\infty} \alpha_{2n} = \alpha_{2n} > 0$, and

(iii) $\lim_{n\to\infty} \beta_{2n} = \beta_{2n} < 1$.

We know that Banach space is Hilbert if and only if its norm satisfies the parallelogram law i.e. for every $x, y \in X$(Hilbert space).

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \qquad \ldots(6)$$

which implies

$$\|x+y\|^2 \leq 2\|x\|^2 + 2\|y\|^2. \qquad \ldots(7)$$

We often use this inequality throughout the result.

Below we prove the result concerning the existence of common fixed point of pairs of mappings satisfying the contraction condition of the type.

$$\|T_x - T_y\|^2 \leq K \max \left\{ \begin{array}{c} (\|y-T_y\|^2), \frac{1}{4}\left(\|x-T_y\|^2 + \|y-T_x\|^2\right), \\ \frac{1}{2}\left(\|x-T_x\|^2 + \|y-T_y\|^2\right), \\ \frac{\|y-T_y\|^2[1+\|x-T_x\|^2]}{1+\|x-y\|^2}, \\ \frac{\|x-T_x\|^2[1+\|x-y\|^2]}{1+\|y-T_y\|^2}, \\ \frac{\|x-y\|^2[1+\|x-T_x\|^2]}{1+\|y-T_y\|^2}, \\ \frac{\|x-T_x\|^2\left[1+\|y-T_y\|^2\right]}{1+\|x-y\|^2}, \\ \frac{(1+\|y-T_y\|^2)[1+\|x-T_x\|^2]}{1+\|x-y\|^2} \end{array} \right\} \qquad \ldots(8)$$

Theorem: Let X be a Hilbert space and C be a closed convex subset of X. Let T_1 and T_2 be two sets of mapping satisfying

$$\|T_1x - T_2y\|^2 \leq K \max \left\{ \begin{array}{c} \|y-T_2y\|^2, \frac{1}{4}(\|x-T_2y\|^2 + \|y-T_1x\|^2), \\ \frac{1}{2}(\|x-T_1x\|^2 + \|y-T_2y\|^2), \\ \frac{\|y-T_2y\|^2[1+\|x-T_1x\|^2]}{1+\|x-y\|^2}, \\ \frac{\|x-T_1x\|^2[1+\|x-y\|^2]}{1+\|y-T_2y\|^2}, \\ \frac{\|x-y\|^2[1+\|x-T_1x\|^2]}{1+\|y-T_2y\|^2} \\ \frac{\|x-T_1x\|^2[1+\|y-T_2y\|^2]}{1+\|x-y\|^2}, \end{array} \right.$$

$$\left.\frac{(1+\|y-T_2y\|^2)[1+\|x-T_1x\|^2]}{1+\|x-y\|^2}\right\} \quad \ldots (9)$$

Where, $0 \le k < \frac{1}{4}$. If there exist a point x_0 such that the I - scheme for point of T_1 and T_2 defined by (4) & (5), converges to a point p,then p is a common fixed point of T_1 and T_2.

Proof: It follows that from (4) $x_{2n+1} - x_{2n} = \alpha_{2n}(T_2y_{2n} - x_{2n})$. since $x_{2n} \to p$, $\|x_{2n+1} - x_{2n}\| \to 0$ since $\{\alpha_{2n}\}$ is bounded away from zero, $\|T_2y_{2n} - x_{2n}\| \to 0$ .It also follows that $\|p - T_2y_{2n}\| \to 0$. Since T_1 and T_2 satisfies (9), we have

$$\|T_1x_{2n} - T_2y_{2n}\|^2 \le K \max\left\{\begin{array}{l} \|y_{2n} - T_2y_{2n}\|^2, \frac{1}{4}(\|x_{2n} - T_2y_{2n}\|^2 + \|y_{2n} - T_1x_{2n}\|^2), \\ \frac{1}{2}(\|x_{2n} - T_1x_{2n}\|^2 + \|y_{2n} - T_2y_{2n}\|^2), \\ \frac{\|y_{2n} - T_2y_{2n}\|^2[1+\|x_{2n} - T_1x_{2n}\|^2]}{1+\|x_{2n} - y_{2n}\|^2}, \\ \frac{\|x_{2n} - T_1x_{2n}\|^2[1+\|x_{2n} - y_{2n}\|^2]}{1+\|y_{2n} - T_2y_{2n}\|^2}, \\ \frac{\|x_{2n} - y_{2n}\|^2[1+\|x_{2n} - T_1x_{2n}\|^2]}{1+\|y_{2n} - T_2y_{2n}\|^2}, \\ \frac{\|x_{2n} - T_1x_{2n}\|^2[1+\|y_{2n} - T_2y_{2n}\|^2]}{1+\|x_{2n} - y_{2n}\|^2}, \\ \frac{(1+\|y_{2n} - T_2y_{2n}\|^2)[1+\|x_{2n} - T_1x_{2n}\|^2]}{1+\|x_{2n} - y_{2n}\|^2} \end{array}\right\} \quad \ldots (10)$$

Now,

$$\begin{aligned} \|y_{2n} - x_{2n}\|^2 &= \|\beta_{2n}T_1x_{2n} + (1-\beta_{2n})x_{2n} - x_{2n}\|^2 \\ &= \|\beta_{2n}T_1x_{2n} + x_{2n} - \beta_{2n}x_{2n} - x_{2n}\|^2 \\ &= \|\beta_{2n}(T_1x_{2n} - x_{2n})\|^2 \\ &= \beta^2_{2n}\|(T_1x_{2n} - T_2y_{2n}) + (T_2y_{2n} - x_{2n})\|^2 \\ &\le 2\beta^2_{2n}\|T_1x_{2n} - T_2y_{2n}\|^2 + 2\beta^2_{2n}\|T_2y_{2n} - x_{2n}\|^2 \\ &\le 2\|T_1x_{2n} - T_2y_{2n}\|^2 + 2\|T_2y_{2n} - x_{2n}\|^2 \end{aligned} \quad \ldots (11)$$

$$\begin{aligned} \|y_{2n} - T_2y_{2n}\|^2 &= \|\beta_{2n}T_1x_{2n} + (1-\beta_{2n})x_{2n} - T_2y_{2n}\|^2 \\ &= \|\beta_{2n}T_1x_{2n} + (1-\beta_{2n})x_{2n} - T_2y_{2n} + \beta_{2n}T_2y_{2n} - \beta_{2n}T_2y_{2n}\|^2 \\ &= \|\beta_{2n}(T_1x_{2n} - T_2y_{2n}) + (1-\beta_{2n})(x_{2n} - T_2y_{2n})\|^2 \\ &\le 2\beta^2_{2n}\|(T_1x_{2n} - T_2y_{2n})\|^2 + 2(1-\beta_{2n})\|(x_{2n} - T_2y_{2n})\|^2 \\ &\le 2\|(T_1x_{2n} - T_2y_{2n})\|^2 + 2\|(x_{2n} - T_2y_{2n})\|^2 \end{aligned} \quad \ldots (12)$$

$$\begin{aligned} \|y_{2n} - T_1x_{2n}\|^2 &= \|\beta_{2n}T_1x_{2n} + (1-\beta_{2n})x_{2n} - T_1y_{2n}\|^2 \\ &= \|(1-\beta_{2n})(x_{2n} - T_1x_{2n})\|^2 \end{aligned}$$

$$= (1-\beta_{2n})^2\|(x_{2n}-T_1x_{2n})\|^2$$
$$== (1-\beta_{2n})^2\|(x_{2n}-T_2y_{2n})+(T_2y_{2n}-T_1x_{2n})\|^2$$
$$\leq 2(1-\beta_{2n})^2\|x_{2n}-T_2y_{2n}\|^2+2(1-\beta_{2n})^2\|T_2y_{2n}-T_1x_{2n}\|^2$$
$$\leq 2\|(x_{2n}-T_2y_{2n})\|^2+2\|(T_2y_{2n}-T_1x_{2n})\|^2. \quad \ldots(13)$$

From (11), (12) and (13), (10) can be written as

$$\|T_1x_{2n}-T_2y_{2n}\|^2 \leq k\max\left\{(2\|T_1x_{2n}-T_2y_{2n}\|^2+2\|x_{2n}-T_2y_{2n}\|^2),\right.$$
$$\frac{1}{4}(3\|x_{2n}-T_2y_{2n}\|^2+2\|T_2y_{2n}-T_1x_{2n}\|^2),$$
$$\frac{1}{2}(\|x_{2n}-T_1x_{2n}\|^2+2\|T_1x_{2n}-T_2y_{2n}\|^2+2\|x_{2n}-T_2y_{2n}\|^2),$$
$$\frac{(2\|T_1x_{2n}-T_2y_{2n}\|^2+2\|x_{2n}-T_2y_{2n}\|^2)(1+\|x_{2n}-T_1x_{2n}\|^2)}{(1+2\|T_1x_{2n}-T_2y_{2n}\|^2+2\|T_2y_{2n}-x_{2n}\|^2)},$$
$$\frac{(\|x_{2n}-T_1x_{2n}\|^2)(1+2\|T_1x_{2n}-T_2y_{2n}\|^2+2\|T_2y_{2n}-x_{2n}\|^2)}{(1+2\|T_1x_{2n}-T_2y_{2n}\|^2+2\|x_{2n}-T_2y_{2n}\|^2)},$$
$$\frac{(2\|T_1x_{2n}-T_2y_{2n}\|^2+2\|T_2y_{2n}-x_{2n}\|^2)(1+\|x_{2n}-T_1x_{2n}\|^2)}{(1+2\|T_1x_{2n}-T_2y_{2n}\|^2+2\|x_{2n}-T_2y_{2n}\|^2)},$$

$$\frac{(\|x_{2n}-T_1x_{2n}\|^2)(1+2\|T_1x_{2n}-T_2y_{2n}\|^2+2\|x_{2n}-T_2y_{2n}\|^2)}{(1+2\|T_1x_{2n}-T_2y_{2n}\|^2+2\|T_2y_{2n}-x_{2n}\|^2)},$$
$$\left.\frac{(1+2\|T_1x_{2n}-T_2y_{2n}\|^2+2\|x_{2n}-T_2y_{2n}\|^2)(1+\|x_{2n}-T_1x_{2n}\|^2)}{(1+2\|T_1x_{2n}-T_2y_{2n}\|^2+2\|T_2y_{2n}-x_{2n}\|^2)}\right\}$$

$$\|T_1x_{2n}-T_2y_{2n}\|^2 \leq k\ (2\|T_1x_{2n}-T_2y_{2n}\|^2+2\|x_{2n}-T_2y_{2n}\|^2)$$
$$\|T_1x_{2n}-T_2y_{2n}\|^2 \leq k\ (2\|T_1x_{2n}-T_2y_{2n}\|^2+2\|T_2y_{2n}-x_{2n}\|^2) .$$

Taking the limit as $n \to \infty$, we get

$$\|T_1x_{2n}-T_2y_{2n}\|^2 \to 0.$$

It follows that

$$\|(x_{2n}-T_1x_{2n})\|^2 \leq (2\|x_{2n}-T_2y_{2n}\|^2+2\|T_2y_{2n}-T_1x_{2n}\|^2) \to 0$$

and

$$\|(p-T_1x_{2n})\|^2 \leq (2\|p-x_{2n}\|^2+2\|x_{2n}-T_1y_{2n}\|^2) \to 0 \text{ as } n \to \infty .$$

If

x_{2n}, p satisfies eq. (9), we have

$$\|T_1x_{2n}-T_2p\|^2 \leq k\max\left\{\|p-T_2p\|^2, \frac{1}{4}(\|x_{2n}-T_2p\|^2+\|p-T_1x_{2n}\|^2),\right.$$
$$\frac{1}{2}(\|x_{2n}-T_1x_{2n}\|^2+\|p-T_2p\|^2,$$

$$\frac{\|p-T_2p\|^2(1+\|x_{2n}-T_1x_{2n}\|^2)}{1+\|x_{2n}-p\|^2},$$

$$\frac{\|x_{2n}-T_1x_{2n}\|^2(1+\|x_{2n}-p\|^2)}{1+\|p-T_2p\|^2},$$

$$\frac{\|x_{2n}-p\|^2(1+\|x_{2n}-T_1x_{2n}\|^2)}{1+\|p-T_2p\|^2},$$

$$\frac{\|x_{2n}-T_1x_{2n}\|^2(1+\|p-T_2p\|^2)}{1+\|x_{2n}-p\|^2},$$

$$\left.\frac{(1+\|p-T_2p\|^2)(1+\|x_{2n}-T_1x_{2n}\|^2)}{1+\|x_{2n}-p\|^2}\right\}$$

Using inequality (7) we have

$$\|T_1x_{2n}-T_2p\|^2 \le kmax\left\{ (2\|p-x_{2n}\|^2+2\|x_{2n}-T_2p\|^2), \right.$$
$$\frac{1}{4}(2\|x_{2n}-T_1x_{2n}\|^2+2\|T_1x_{2n}-T_2p\|^2+\|p-T_1x_{2n}\|^2),$$
$$\frac{1}{2}(2\|x_{2n}-T_1x_{2n}\|^2+2\|p-x_{2n}\|^2+2\|x_{2n}-T_2p\|^2),$$

$$\frac{(2\|p-x_{2n}\|^2+2\|x_{2n}-T_2p\|^2)(1+\|x_{2n}-T_1x_{2n}\|^2)}{1+\|x_{2n}-p\|^2}$$
$$\frac{\|x_{2n}-T_1x_{2n}\|^2(1+\|x_{2n}-p\|^2)}{1+2\|p-x_{2n}\|^2+2\|x_{2n}-T_2p\|^2},$$
$$\frac{\|x_{2n}-p\|^2(1+\|x_{2n}-T_1x_{2n}\|^2)}{1+2\|p-x_{2n}\|^2+2\|x_{2n}-T_2p\|^2},$$

$$\frac{\|x_{2n}-T_1x_{2n}\|^2(1+2\|p-x_{2n}\|^2+2\|x_{2n}-T_2p\|^2)}{1+\|x_{2n}-p\|^2},$$

$$\left.\frac{(1+2\|p-x_{2n}\|^2+2\|x_{2n}-T_2p\|^2)(1+\|x_{2n}-T_1x_{2n}\|^2)}{1+\|x_{2n}-p\|^2}\right\}$$

Taking the limit as $n\to\infty$, we get $\|T_1x_{2n}-T_2p\|\to 0$.

Finally,

$$\|p-T_2p\|^2 = \|p-T_1x_{2n}+T_1x_{2n}-T_2p\|^2$$
$$\le 2\,\|p-T_1x_{2n}\|^2+2\,\|T_1x_{2n}-T_2p\|^2\to 0, \text{ as } n\to\infty.$$

Showing that $p=T_2p$. Similarly, we can prove that $p=T_1p$. Thus p is the common fixed point of T_1 and T_2. This completes the proof.

Letting $T_1 = T_2 = T$ in above theorem, we obtain the following

Corollary: Let X be a Hilbert space and C be a closed convex subset of X. Let T be a self- mapping satisfying (8) where $0\le k<\frac{1}{4}$. If there exists a point x_0 such that the I - scheme for T defined by

$$y_n=\beta_nTx_n+(1-\beta_n)x_n, \quad n\ge 0$$
$$x_{n+1} = (1-\alpha_{2n})x_n+\alpha_nTy_n, \quad n\ge 0$$ converges to a point p, then p is the fixed point of T.

In the I-scheme, $\{\alpha_n\}$ $\{\beta_n\}$ satisfy $0 \leq \alpha_n \leq \beta_n \leq 1$ for all $n.\lim_{n\to\infty} \beta_n = 0, \sum \alpha_n \beta_n = 0$.
Assuming that

(i) $0 \leq \alpha_n, \beta_n \leq 1,$ for all n,

(ii) $\lim \alpha_n = \alpha > 0,$

(iii) $\lim \beta_n = \beta < 1.$

The proof is similar to above Theorem, hence we omit the details.

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