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Complex Order – Distribution and Caputo Fractional Derivatives of the I-Function

Dr. Arif M. Khan
Department of Mathematics,
Jodhpur Institute of Engineering and Technology, Jodhpur, (RJ), India

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ABSTRACT: The concept of complex order-distribution & conjugated differ integrals was developed by Adams, Hartley & Lorenzo [1]. The present paper deals with the complex order-distribution using conjugated differintegrals of the I-Function. These conjugate-order differ integrals involving the I-Function allows the generalization of fractional system identification to enables the search for complex order-derivatives that may better describe real-time behaviors involving special functions. Further Caputo fractional derivative of I-function also obtained. Due to generous nature of I-function, this paper may have vast applications in signal processing & electrical systems.

Keywords: Fractional-order systems, complex order distributions, I-function, conjugated-order differintegrals, Caputo fractional derivatives.
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I. INTRODUCTION AND DEFINITIONS

Order distributions were introduced by Hartley & Lorenzo as the continuum extension of collections of fractional order operators for real or non real orders. The idea of conjugate-order differintegrals is utilized to ensure that only real time responses are considered, while using complex order distributions.

Fractional operators for non integers real or complex have been studied by few [5, 7].

The aim of this paper is the development of complex order differ integrals which yield purely real time-response and construct Caputo fractional derivatives of the I-function.

The I-function which was introduced by Saxena [9] is an extension of Fox's H-function. On specializing the parameters, I-function can be reduced to almost all the known as well as unknown special functions.

Definition 1.1: The definition of I-function given by Saxena [9] is as follows:

$$\mathrm{I}(z) = I^{m,n}_{p_i,q_i:r}\,[z]$$

$$= \mathrm{I}^{m,n}_{p_i,q_i:r}\left[Z \left| \begin{matrix} (a_j,\alpha_j)_{1,n}\,(a_{ji},\alpha_{ji})_{n+1,p_i} \\ (b_j,\beta_j)_{1,m}\,(b_{ji},\beta_{ji})_{m+1,q_i} \end{matrix} \right. \right]$$

$$= \frac{1}{2\pi_i}\int_C t(s) z^s\, ds \qquad \text{... (1.1)}$$

where

$$t(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_j+\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}s)\right\}}$$

p_i (i = 1, 2, ... r), q_i (i = 1, 2, ...r), m, n are integers satisfying $0 \le n \le p_i$, $0 \le m \le q_i$ (i = 1, 2, ... r) r is finite α_i, β_i, α_{ji}, β_{ji} are real and positive and a_j, b_j, a_{ji}, b_{ji} are complex numbers such that $\alpha_j (b_h + v) \ne \beta_j (a_h - 1 - k)$ with all necessary conditions for existence as given by Saxena [9].

Definition 1.2: Euler transform of the I-function can easily be established using result given by Srivastava [10]

$$\int_0^x y^{-\alpha}(x-y)^{\alpha-\beta-1}\, I^{m,n}_{p_i,q_i;r}[y]dy = \frac{\Gamma(\alpha-\beta)}{x^{\beta}} I^{m,n+1}_{p_i+1,q_i+1;r}\left[x\left|\begin{matrix}(\alpha,1)(a_j,\alpha_j)(a_{ji},\alpha_{ji})\\(b_j,\beta_j)(b_{ji},\beta_{ji})(\beta,1)\end{matrix}\right.\right] \quad \text{... (1.2)}$$

with the convergence conditions [10].

Definition 1.3: Laplace transform of an I-function given by Vaishya, Jain & Verma [11]

$$\int_0^{\infty} e^{-\beta x} x^{-\rho}\, I^{m,n}_{p_i,q_i;r}[\alpha x]dx = \beta^{\rho-1} I^{m,n+1}_{p_i+1,q_i;r}\left[\frac{\alpha}{\beta}\left|\begin{matrix}(\alpha,1)(a_j,\alpha_j)(a_{ji},\alpha_{ji})\\(b_j,\beta_j)(b_{ji},\beta_{ji})\end{matrix}\right.\right] \quad \text{... (1.3)}$$

where Ra (β) > 0

II. COMPLEX DIFFER INTEGRALS OF I-FUNCTION

If $f(\tau) = (\tau)^{-\alpha}\, I^{m,n}_{p_i,q_i;r}[\tau]$ ... (2.1)

then its q[th] order fractional integral is defined as

g(t) = ${}_0d_t^{-q} f(\tau)$

(Let q = $\alpha-\beta$)

g(t) = ${}_0d_t^{-(\alpha-\beta)} f(\tau)$

g(t) = $\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-\tau)^{\alpha-\beta-1} f(\tau)d\tau$

g(t) = $\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-\tau)^{\alpha-\beta-1}\tau^{-\alpha} I^{m,n}_{p_i,q_i;r}[\tau]d\tau$

By using (1.2) we get

$${}_0d_t^{-(\alpha-\beta)} f(\tau) = \frac{1}{t} I^{m,n+1}_{p_i+1,q_i+1;r}\left[t\left|\begin{matrix}(\alpha,1)(a_j,\alpha_j)(a_{ji},\alpha_{ji})\\(b_j,\beta_j)(b_{ji},\beta_{ji})(\beta,1)\end{matrix}\right.\right] \quad \text{... (2.2)}$$

In general the complex differintegral for f(τ) given by (2.1) is defined as

$$g_1(t) = {}_0d_t^{-q} f(\tau) = {}_0d_t^{-(u+iv)}(\tau)$$

Laplace transform of g_1 (t) is given by Kober [6]

$$L[g_1(t)] = G_1(s) = s^{u+iv}F(s) = s^u \cdot s^{iv}F(s)$$

$$= s^u e^{iv\log_e s}F(s) \quad \text{... (2.3)}$$

where F(s) = L[f(τ)]

$$F(s) = \int_0^{\infty} e^{-s\tau}\tau^{-\alpha} I^{m,n}_{p_i,q_i;r}[\tau]d\tau$$

By Using (1.3)

$$F(s) = s^{\alpha-1} I^{m,n+1}_{p_i+1,q_i;r}\left[\frac{1}{s}\left|\begin{matrix}(1,1)(a_j,\alpha_j)(a_{ji},\alpha_{ji})\\(b_j,\beta_j)(b_{ji},\beta_{ji})\end{matrix}\right.\right] \quad \text{... (2.4)}$$

where, Re(s) > 0

From (2.3)

$$G_1(s) = s^u\left[\cos(v\log s) + i\sin(v\log s)\right]F(s) \quad \text{... (2.5)}$$

III. CONJUGATED ORDER DIFFERINTEGRALS

The conjugated order fractional integral may be expressed for negative real order as

$$g(t) = {}_0d_t^{-q(u,v)} f(t) = {}_0d_t^{-(u+iv)} f(t) + {}_0d_t^{-(u-iv)} f(t)$$

with the Laplace transform given by

$$L[g(t)] = L\left[{}_0d_t^{-q(u,v)} f(t)\right] = \left(s^{-(u+iv)} + s^{-(u-iv)}\right)F(s) \quad \text{... (3.1)}$$

$$= s^{-u}\left[s^{-iv}+s^{iv}\right]\cdot F(s)$$
$$= s^{-u}\left[e^{-iv\log s}+e^{iv\log s}\right]F(s)$$
$$= s^{-u}\left[2\cos(v\log s)\right]\left[F(s)\right]$$

Now taking inverse Laplace to both side

$$g(t)=L^{-1}\left[\left\{s^{-(u+iv)}+s^{-(u-iv)}\right\}F(s)\right]$$
$$g(t)=2t^{u-\alpha}\cos(v\log t)\, I^{m,n}_{p_i,q_i;r}\left[t\left|\begin{matrix}(a_j,\alpha_j)(a_{ji},\alpha_{ji})\\(b_j,\beta_j)(b_{ji},\beta_{ji})\end{matrix}\right.\right] \quad \text{... (3.2)}$$

since

$$L^{-1}\left[s^{\alpha-1}I^{m,n+1}_{p_i+1,q_i;r}\left[s^{-1}\left|\begin{matrix}(1,1)(a_j,\alpha_j)(a_{ji},\alpha_{ji})\\(b_j,\beta_j)(b_{ji},\beta_{ji})\end{matrix}\right.\right]\right]$$
$$= t^{-\alpha}\, I^{m,n}_{p_i,q_i;r}(t)$$

Hence by (3.2) the conjugated differintegral has purely real time response.
Likewise, the complementary conjugated differintegral is define has

$$\overline{g}(t)={}_0d_t^{q\overline{(u,v)}}\, f(t)={}_0d_t^{q}f(t)-{}_0d_t^{\overline{q}}f(t)$$
$$={}_0d_t^{u+iv}f(t)-{}_0d_t^{u-iv}f(t)$$

Representing in Laplace domain

$$L\left[\overline{g}(t)\right]=L\left[{}_0d_t^{q\overline{(u,v)}}\, f(t)\right]==\left(s^{u+iv}-s^{u-iv}\right)F(s)$$
$$=2i\, s^{u}\sin(v\log s)\, F(s)$$

which is purely imaginary operator.

IV. COMPLEX ORDER DISTRIBUTION DEFINITION

Adams, Hartley & Lorenzo [1] defined the complex order distribution as

$$h(t)\equiv\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}k(u,v)\left\{{}_0d_t^{u+iv}f(t)\right\}dudv \quad \text{... (4.1)}$$

The equation can be Laplace transformed as

$$H(s)\equiv\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}k(u,v)s^{u+iv}F(s)dudv$$
$$H(s)\equiv\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}k(u,v)\, s^{u+iv}\, s^{\alpha-1}I^{m,n+1}_{p_i+1,q_i;r}\left[s^{-1}\left|\begin{matrix}(1,1)(a_j,\alpha_j)(a_{ji},\alpha_{ji})\\(b_j,\beta_j)(b_{ji},\beta_{ji})\end{matrix}\right.\right]dudv \quad \text{... (4.2)}$$

A. Blockwise constant complex order-distribution

Let consider complex order-distribution that are constant intensity, k symmetric about the real axis from $\overline{u}-\delta u$ to $\overline{u}+\delta u$ and from $-i\delta v$ to $+i\delta v$.

So from (4.2)

$$H(s)=\int_{-\delta v}^{\delta v}\int_{\overline{u}-\delta u}^{\overline{u}+\delta u}k\, s^{u+iv}\, s^{\alpha-1}I^{m,n+1}_{p_i+1,q_i;r}\left[s^{-1}\right]dudv$$
$$H(s)=\int_{-\delta v}^{\delta v}\int_{-\delta u}^{\delta u}k\, s^{\overline{u}+w+iv}\, s^{\alpha-1}I^{m,n+1}_{p_i+1,q_i;r}\left[s^{-1}\right]dwdv$$

where $u=w-\overline{u}$

$du = dw$

$$H(s)=k\, s^{\overline{u}+\alpha-1}\, I^{m,n+1}_{p_i+1,q_i;r}\left[s^{-1}\right]\int_{-\delta v}^{\delta v}\int_{-\delta u}^{\delta u}s^{w}\cdot s^{iv}\, dwdv$$
$$H(s)=k\, s^{\overline{u}+\alpha-1}\, I^{m,n+1}_{p_i+1,q_i;r}\left[s^{-1}\right]\int_{-\delta v}^{\delta v}e^{iv\log s}\, dv\int_{-\delta u}^{\delta u}e^{w\log s}\, dw$$

$$H(s) = 4k\, s^{\bar{u}+\alpha-1}\, I^{m,n+1}_{p_i+1,q_i;r}\left[s^{-1}\right] \frac{\sin h(su \log s)}{\log s} \cdot \frac{\sin(sv \log s)}{\log s}.$$

V. CAPUTO FRACTIONAL DERIVATIVES OF THE I-FUNCTION

Lorenzo and Hartley (LH) [5] have discussed the following initialization of Riemann-Liouville Fractional Differintegral

$$_a d_t^{-q} f(t) = \frac{1}{\Gamma(q)} \int_a^t (t-\tau)^{q-1} f(\tau) d\tau, \quad t > a \qquad \text{... (5.1)}$$

and

the generalized fractional derivative

$$_c D_t^{-v} f(t) = {}_c d_t^{-v} f(t) + \psi(f, -v, a, c, t), \quad v \geq 0 \qquad \text{... (5.2)}$$

$$t > c \geq a \qquad \& \qquad f(t) = 0 \qquad \text{for all } t \leq a$$

Where, $\psi(f, -v, a, c, t) = \dfrac{1}{\Gamma v} \displaystyle\int_a^c (t-\tau)^{v-1} f(\tau) d\tau$. ...(5.3)

The Caputo fractional derivative defined by [3] as

$$_a^c d_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-\tau)^{m-\alpha-1} f^{m}(\tau) d\tau. \qquad (m-1 < \alpha < m) \qquad \text{... (5.4)}$$

Further Achar, Lorenzo and Hartley [8] given following relation

$$_0 D_t^{\alpha} f(t) = {}_0^c d_t^{\alpha} f(t) + \frac{t^{-\alpha} f(0)}{\Gamma(1-\alpha)} + \frac{d}{dt} \psi(f, -(1-\alpha), a, 0, t) \qquad t > 0 \qquad \text{... (5.5)}$$

In this section we try to construct an above relation for I-function

Let f (τ) is given by (2.1)

$$_0 D_t^{\alpha} f(t) = \frac{d}{dt}\left\{ {}_0 d_t^{-(1-\alpha)} f(t) + \psi(f, -(1-\alpha), a, 0, t) \right\}$$

$$_0 D_t^{\alpha} f(t) = \frac{d}{dt}\left\{ \frac{1}{\Gamma 1-\alpha} \int_0^t (t-\tau)^{-\alpha} \tau^{-\alpha}\, I^{m,n}_{p_i,q_i;r}[\tau] d\tau \right\}$$

$$+ \frac{d}{dt}\left\{ \frac{1}{\Gamma 1-\alpha} \int_a^0 (t-\tau)^{-\alpha} \tau^{-\alpha}\, I^{m,n}_{p_i,q_i;r}[\tau] d\tau \right\} \qquad \text{... (5.6)}$$

$$= \frac{d}{dt}\left\{ t^{-2\alpha+1} I^{m,n+1}_{p_i+1,q_i+1;r}\left[t \left| \begin{matrix} (\alpha,1)(a_j,\alpha_j)(a_{ji},\alpha_{ji}) \\ (b_j,\beta_j)(b_{ji},\beta_{ji})(2\alpha-1,1) \end{matrix} \right. \right] \right\}$$

$$+ \frac{d}{dt} \psi\left(I^{m,n}_{p_i,q_i;r}[\tau], -(1-\alpha), a, 0, t\right), \qquad t > 0$$

$$_0 D_t^{\alpha} f(t) = t^{-2\alpha}\, I^{m,n+2}_{p_i+2,q_i+2;r}\left[t \left| \begin{matrix} (\alpha,1)(2\alpha-1,1)(a_j,\alpha_j)(a_{ji},\alpha_{ji}) \\ (b_j,\beta_j)(b_{ji},\beta_{ji})(2\alpha-1,1)(2\alpha,1) \end{matrix} \right. \right]$$

$$+ \psi'\left(I^{m,n}_{p_i,q_i;r}(\tau), -(1-\alpha), a, 0, t\right), \qquad t > 0 \qquad \text{... (5.7)}$$

Comparing above result with equation (5.5) we get an interesting result

$$_0^c d_t^{\alpha} f(t) = t^{-2\alpha}\, I^{m,n+2}_{p_i+2,q_i+2;r}\left[t \left| \begin{matrix} (\alpha,1)(2\alpha-1,1)(a_j,\alpha_j)(a_{ji},\alpha_{ji}) \\ (b_j,\beta_j)(b_{ji},\beta_{ji})(2\alpha-1,1)(2\alpha,1) \end{matrix} \right. \right] - \frac{t^{-\alpha} f(0)}{\Gamma(1-\alpha)} \qquad \text{... (5.8)}$$

where $0 < \alpha < 1$

which is Caputo fractional derivative of function involving I-function

VI. CONCLUSIONS

Conjugated order differintegrals with I-function have been defined in the time-domain and their Laplace transforms obtained. The conjugated order differintegrals discussed in the paper allows the use of complex order operators while retaining real time responses.

Further complex order distribution involving I-function introduced and block wise constant complex order distribution presented in the Laplace domain.

From this study of complex order distributions involving I-function we can better describe the behavior of some real dynamic systems. In final section paper addresses the Caputo fractional derivatives of the I-function that has vast applications to solution of fractional differential equation.

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