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A Study of Tuan and Saigo's Multidimensional Modified Fractional Calculus Operators

Manoj Sharma* and Renu Jain**
**Department of Mathematics, RJIT, BSF Academy, Tekanpur, Gwalior, (MP)*
*** School of Mathematics and Allied Sciences, Jiwaji University Gwalior, (MP)*

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I. INTRODUCTION

This paper deals with the study of Tuan and Saigo's multi dimensional operators [5] of fractional calculus. These modified integral operators have been used in the context of elementary and generalized hyper geometric functions of several variables. We have given all the proofs as it is which earlier given by Taun and Saigo. We also apply these operators on H-functions [1] and I-functions [4].

II. MULTIDIMENSIONAL MODIFIED FRACTIONAL CALCULUS OPERATORS

Tuan and Saigo [5] simplify operators reducing them to sums of single integrals. Since the region R_+^n can be divided for a fixed $x \in R_+^n$ in the following form

$$R_+^n = \bigcup_{k=1}^{n} \left\{ t \in R_+^n \left| \frac{x_k}{t_k} \le \frac{x_j}{t_j} \right. (\mathrm{j=1,...,n;\ j \ne k}) \right\},$$

we have

$$X_+^{\alpha} f(x) = \frac{1}{\Gamma(\alpha+1)} \frac{\partial^n}{\partial x_1 \ldots \partial x_n} \sum_{k=1}^{n} \int_0^{x_k} \left(\frac{x_k}{t_k} - 1 \right)^{\alpha} dt_k \times \int_0^{x_n t_k / x_k} \overset{k}{\underset{---}{v}} \int_0^{x_1 t_k / x_k} f(t_1,...,t_n) dt_1 \overset{k}{\underset{...}{v}} dt_n$$

$$= \frac{1}{\Gamma(\alpha+1)} \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \int_0^{x_k} (x_k - t_k)^{\alpha} t_k^{-\alpha} dt_k \times \left[\prod_{\substack{j=1 \\ j \ne k}}^{n} \frac{\partial}{\partial x_j} \right] \int_0^{x_n t_k / x_k} \overset{k}{\underset{---}{v}} \int_0^{x_1 t_k / x_k} f(t_1,...,t_n) dt_1 \overset{k}{\underset{---}{v}} dt_n$$

$$= \frac{1}{\Gamma(\alpha+1)} \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \int_0^{x_k} (x_k - t_k)^{\alpha} t_k^{n-\alpha-1} x_k^{1-n} f\left(\frac{x_1 t_k}{x_k}, ..., \frac{x_n t_k}{x_k} \right) dt_k,$$

where the notation $\overset{k}{\underset{---}{v}}$ means that the integration with respect to the variables $t_1 ..., t_n$ without t_k.

Hence $$X_+^{\alpha} f(x) = \frac{1}{\Gamma(\alpha+1)} \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \left[x_k^{1-n} \int_0^{x_k} \left(x_k - t_k \right)^{\alpha} t_k^{n-\alpha-1} f\left(\frac{x_1 t_k}{x_k}, ..., \frac{x_n t_k}{x_k} \right) dt_k \right. \qquad \ldots(1.1)$$

$$= \frac{1}{\Gamma(\alpha+1)} \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \left[x_k \int_0^1 t^{n-\alpha-1} \left(1 - t \right)^{\alpha} f(x_1 t, ..., x_n t) dt \right].$$

Analogously by virtue of the division $R_+^n = \bigcup_{k=1}^{n} \left\{ t \in R_+^n \left| \frac{x_k}{t_k} \geq \frac{x_j}{t_j} (j = 1,...,n; \quad j \neq k) \right. \right\}.$

We get $X_-^\alpha f(x) = -\frac{1}{\Gamma(\alpha+1)} \sum_{k=1}^{n} \frac{\partial}{\partial x} \left[x_k^{1-n} \int_{x_k}^{\infty} (t_k - x_k)^\alpha t_k^{n-\alpha-1} f\left(\frac{x_1 t_k}{x_k}, ..., \frac{x_n t_k}{x_k} \right) dt_k \right]$

$$= -\frac{1}{\Gamma(\alpha+1)} \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \left[x_k \int_{x_1}^{\infty} t^{n-\alpha-1} (t-1)^\alpha f(x_1 t, ..., x_n t) dt \right]. \qquad ...(1.2)$$

III. MODIFIED FRACTIONAL INTEGRALS OF SOME SPECIAL FUNCTIONS

For evaluating modified fractional integrals of various functions, Tuan and Saigo's considered integrals of functions of max $\{x_1,...,x_n\}$ and min $\{x_1...,x_n\}$. Let $s = (s_1,....s_n) \in C^n$ with $\mathrm{Re}(s_j) > 0$ $(j = 1,...,n)$, and suppose that

$$g(y) y^{|s|-1} \in L_1(R_+). \text{ We have } = \sum_{k=1}^{n} \int_0^\infty x_k^{s_{k-1}} g(x_k) dx_k \int_0^{x_k} \overset{k}{\underset{...}{v}} \int_0^{x_k} \frac{x^{s-1}}{x_k^{s_k - 1}} dx_1 \overset{k}{\underset{...}{v}} dx_n \qquad ...(1.3)$$

$$= \sum_{k=1}^{n} \int_0^\infty x_k^{s_k-1} g(x_k) \left[\prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{x_k^{s_j}}{s_j} \right] dx_k = \sum_{k=1}^{n} \frac{s_k}{s_1 ... s_n} \int_0^\infty x_k^{|s|-1} g(x_k) dx_k.$$

Therefore, $\int_{R_+^n} x^{s-1} g(\max\{x_1,...,x_n\}) dx = \frac{|s|}{s_1 ... s_n} g^*(|s|)$ ...(1.4)

For $\mathrm{Re}(s_j) > 0 (j = 1,...,n),$ *where* $g^*(\sigma)$ function g(y) : is the one-dimensional Mellin transformation of a function g(y) : $g^*(\sigma) = M[g](\sigma) = \int_0^\infty y^{\sigma-1} g(y) dy.$

Analogously, let $\mathrm{Re}(s_j) < 0 \ (j = 1,...,n)$ *and* $g(y) y^{|s|-1} \in L_1(R_+).$

Then we have $\int_{R_+^n} x^{s-1} g(\min\{x_1,...,x_n\}) dx = (-1)^{n-1} \frac{|s|}{s_1 ... s_n} g^*(|s|).$ ...(1.5)

In particular, there hold following relation :

$$\int_{R_+^n} x^{s-1} [1 - \max\{x_1,...,x_n\}]_+^\alpha dx = \frac{\Gamma(\alpha+1)\Gamma(1+|s|)}{\Gamma(1+\alpha+|s|) s_1 ... s_n} \ ... (1.6)$$

For $\mathrm{Re}(s_j) > 0$ (j=1,...,n) and

$$\int_{R_+^n} x^{s-1} [\min\{x_1,...,x_n\} - 1]_+^\alpha dx = (-1)^n \frac{\Gamma(\alpha+1)\Gamma(-\alpha-|s|)}{\Gamma(-|s|) s_1 .... s_n} \qquad ...(1.7)$$

For Re(s_i) <0 (j=1,...n) and $\mathrm{Re}\left(\alpha + |s| \right) < 0,$

$$M[(a-x)_+^b](\sigma) = a^{b+\sigma} \frac{\Gamma(b+1)\Gamma(\sigma)}{\Gamma(b+\sigma+1)} \text{ for } \mathrm{Re}(\sigma) > 0 \text{ and } \mathrm{Re(b)} > \text{-}1,$$

$$M[(x-a)_+^b](\sigma) = a^{b+\sigma} \frac{\Gamma(b+1)\Gamma(-b-\sigma)}{\Gamma(-\sigma+1)}$$

For $\mathrm{Re}(\sigma) < -\mathrm{Re}(b) \text{ and } \mathrm{Re\ (b)} > \text{-}1,$

Tuan and Saigo's calculated modified fractional integrals of some elementary and special functions:

They considered the function $f(x) = x^\beta$ for $\beta \in C^n$ with Re $(\beta_j) > -1$ for j = 1,…,n and n+Re $|\beta| > \mathrm{Re}(\alpha)$, then they obtained

$$X_+^\alpha x^\beta = \frac{1}{\Gamma(\alpha+1)} \frac{\partial^n}{\partial x_1,...\partial x_n} x^{\beta+1} \int_{R_+^n} [\min\{y_1,...,y_n\} - 1]_+^\alpha y^{-\beta-2} dy.$$

Then, by virtue of the formula (1.4), they obtained $X_+^\alpha x^\beta = \dfrac{\Gamma(n-\alpha+|\beta|)}{\Gamma(n+|\beta|)} x^\beta,$

…(1.8)

Provided that $\mathrm{Re}(\beta_j) > -1,...,n) \text{ and } \mathrm{n} + \mathrm{Re}(|\beta|) > \mathrm{Re}(\alpha).$

Analogously, the formula

$$X_-^\alpha x^\beta = \frac{\Gamma(1-n-|\beta|)}{\Gamma(1+\alpha-n-|\beta|)} x^\beta$$

…(1.9)

is valid , when $\mathrm{Re}(\beta_j) < -1\,(\mathrm{j} = 1,...,\mathrm{n}).$

We consider the function

$$\frac{1}{x_1,...,x_n} H_{p,q}^{m,r}\left(\min\{x_1,...,x_n\} \middle| \begin{matrix} (a_1,A_1),...,(a_p,A_p) \\ (b_1,B_1),...,(b_q,B_q) \end{matrix} \right),$$

where $H_{p,q}^{m,r}$ is the Fox's H-function defined by

$$H_{p,q}^{m,r}\left(t \middle| \begin{matrix} (a_p,A_p) \\ (b_q,B_q) \end{matrix} \right) = \frac{1}{2\pi i} \int_{Q-i\infty}^{Q+i\infty} \frac{\prod_{j=1}^{m} \Gamma(b_j + B_j\sigma) \prod_{j=1}^{r} \Gamma(1-a_j - A_j\sigma)}{\prod_{j=r+1}^{p} \Gamma(a_j + A_j\sigma) \prod_{j=m+1}^{q} \Gamma(1-b_j - B_j\sigma)} t^{-\sigma} d\sigma$$

…(1.10)

Where, 2(m+r)>p+q, or 2(m+r) = p +q ,

$$\text{Re}\left(\sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j\right) + (p-q)\rho > 1;$$

$$\text{Re}(b_j) > -\rho \ \ (\text{j}=1,\dots,\text{m}) \text{ and } \text{Re}(\text{a}_\text{j}) < 1-\rho\,(\text{j}=1,\dots,\text{r}).$$

Let $\lambda \in R^n, \lambda_\text{j} < 0\,(\text{j}=1,\dots,\text{n})$ and set $|\lambda| = \sigma.$

Applying the formula (1.2), we have

$$x^{-1}H_{p,q}^{m,r}\left(\min\{x_1,\dots,x_n\} \left| \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right. \right)$$

$$= \frac{(-1)^{n-1}}{(2\pi i)^n} \int_{(\lambda)-i\infty}^{(\lambda)+i\infty} \frac{\prod_{j=1}^{m}\Gamma(b_j + B_j|s|)\prod_{j=1}^{r}\Gamma(1-a_j - A_j|s|)}{\prod_{j=r+1}^{p}\Gamma(a_j + A_j|s|)\ \prod_{\text{j=m+1}}^{\text{q}}\Gamma(1-b_j - B_j|s|)}\ \frac{|\text{s}|}{s_1 \dots s_n}\ \text{x}^{\text{-s-1}}\ \text{ds}. \quad \dots(1.11)$$

As $\text{Re}(-s_j - 1) > -1\ \ (\text{j}=1,\dots,\text{n}),$ then by using (1.5) we obtain as follows:

$$X_+^{\alpha} x^{-1}H_{p,q}^{m,r}\left(\min\{x_1,\dots.,x_n\} \left| \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right. \right)$$

$$= \frac{(-1)^{n-1}}{(2\pi i)^n} \int_{(\lambda)-i\infty}^{(\lambda)+i\infty} \frac{\prod_{j=1}^{m}\Gamma(b_j + B_j|s|)\prod_{\text{j=1}}^{\text{r}}\Gamma(1-a_j - A_j|s|)}{\prod_{j=r+1}^{p}\Gamma(a_j + A_j|s|)\ \prod_{j=m+1}^{q}\Gamma(1-b_j - B_j|s|)}\ \frac{|\text{s}|}{s_1 \dots . s_n}\ \text{X}_+^{\alpha} x^{-s-1} ds$$

$$= \frac{(-1)^{n-1}}{(2\pi i)^n} \int_{(\lambda)-i\infty}^{(\lambda)+i\infty} \frac{|s|}{s_1 \dots s_n}\ \frac{\prod_{j=1}^{m}\Gamma(b_j + B_j|s|)\prod_{j=1}^{r}\Gamma(1-a_j - A_j|s|)}{\prod_{j=r+1}^{p}\Gamma(a_j + A_j|s|)\ \prod_{j=m+1}^{q}\Gamma(1-b_j - B_j|s|)}\ \text{x}^{\text{-s-1}} ds$$

for $\text{Re}(\alpha) + |\lambda| < 0,$ where we set $a_0 = 1 + \alpha, \text{b}_{\text{q+1}} = 1$.

Using the definition (1.7), this reduces to

$$X_+^{\alpha} x^{-1}H_{p,q}^{m,r}\left(\min\{x_1,\dots,x_n\} \left| \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right. \right) \quad \dots(1.12)$$

$$= x^{-1}H_{p+1,q+1}^{m.r.+1}\left(\min\{x_1,\dots,x_n\} \left| \begin{matrix} (1+\alpha, 1), (a_p, A_p) \\ (b_q, B_q), (1,1) \end{matrix} \right. \right)$$

with $\mathrm{Re}(b_j) > -\sigma$ (j = 1...,m); $\mathrm{Re}(a_j) < 1-\sigma$ (j = 1,...r), $\mathrm{Re}(\alpha) < -\sigma$;
and either $2(m+r) > p+q$; or $2(m+r) = p+q$ holds ...(1.13)

and $\mathrm{Re}\left(\sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j\right) + \sigma(p-q) > 1.$ ...(1.14)

Similarly we obtain

$$X_{-}^{\alpha} x^{-1} H_{p,q}^{m,r}\left(\max\{x_1,...,x_n\} \middle| \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right)$$
$$= x^{-1} H_{p+1,q+1}^{m+1,r}\left(\max\{x_1,...,x_n\} \middle| \begin{matrix} (a_p, A_p),(1+\alpha,1) \\ (1,1),(b_q, B_q) \end{matrix} \right) \quad \text{…(1.15)}$$

with $\mathrm{Re}(b_j) > -\sigma$ (j = 1,...,m); $\mathrm{Re}(a_j) < 1-\sigma$ (j = 1,...,r), $\sigma > 0$; and either $2(m+r) > p+q$; or $2(m+r) = p+q$, holds with

$$\mathrm{Re}\left(\sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j\right) + \sigma(p-q) > 1.$$

Similarly, we have obtained the result for I-function which can be written as

$$X_{+}^{\alpha} x^{-1} I_{P_i,Q_i:R}^{M,N}\left[\min\{x_1,...x_n\} \middle| \begin{matrix} \{(a_j,\alpha_j)_{1,n}\}, & \{(a_{ji},\alpha_{ji})_{n+1,p_i}\} \\ \{(b_j,\beta_j)_{1,m}\}, & \{(b_{ji},\beta_{ji})_{m+1,q_i}\} \end{matrix} \right]$$
$$= x^{-1} I_{P_i+1,Q_{i+1}:R}^{M,N+1}\left[\min\{x_1,...x_n\} \middle| \begin{matrix} \{(1+\alpha,1),(a_j,\alpha_j)_{1,n}\}, & \{(a_{ji},\alpha_{ji})_{n+1,p_i}\} \\ \{(b_j,\beta_j)_{1,m}\}, & \{(b_{ji},\beta_{ji})_{m+1,q_i},(1,1)\} \end{matrix} \right]$$
…(1.16)

and also

$$X_{-}^{\alpha} x^{-1} I_{P_i,Q_i:R}^{M,N}\left[\min\{x_1,...x_n\} \middle| \begin{matrix} \{(a_j,\alpha_j)_{1,n}\}, & \{(a_{ji},\alpha_{ji})_{n+1,p_i}\} \\ \{(b_j,\beta_j)_{1,m}\}, & \{(b_{ji},\beta_{ji})_{m+1,q_i}\} \end{matrix} \right]$$
$$= x^{-1} I_{P_i+1,Q_{i+1}:R}^{M+1,N}\left[\min\{x_1,...x_n\} \middle| \begin{matrix} \{(a_j,\alpha_j)_{1,n}\}, & \{(a_{ji},\alpha_{ji})_{n+1,p_i},(1+\alpha,1)\} \\ \{(1,1),(b_j,\beta_j)_{1,m}\}, & \{(b_{ji},\beta_{ji})_{m+1,q_i}\} \end{matrix} \right]$$
...(1.17)

The conditions of convergence for these results are same as before except that conditions (1.10) and (1.11) are replace by following conditions.

$$A_i > 0, \left|\arg z\right| < \frac{1}{2} A_i \pi \qquad \forall i \in \{1,2,...,R\}$$

or

$$A_i \geq 0, \left|\arg z\right| \leq \frac{1}{2} A_i \pi, \; \mathrm{Re}(B+1) < 0 \;\; \forall i \in \{1,2,...,R\} \qquad \text{…(1.18)}$$

Where

$$A_i = \sum_{j=1}^{N} \alpha_j - \sum_{j=N+1}^{P_i} \alpha_{ji} + \sum_{j=1}^{M} \beta_j - \sum_{j=M+1}^{Q_i} \beta_{j_i} \qquad \forall i \in \{1,2,...R\},$$

$$B = \frac{1}{2}(P_i - Q_i) + \sum_{j=1}^{Q_i} b_j - \sum_{j=1}^{P_i} a_j \qquad \forall i \in \{1,2,...R\},$$

This completes the analysis.

REFERENCES

[1]. Fox, C. The G and H-functions as symmetrical Fourier kernels. *Trans. Amer. Math. Soc*., **98** (1961), 395-429.

[2]. Fox, C. Integral transforms based upon fractional integration, *Proc. Combridge Philosophical Society*, **59**(1963), 63-71.

[3]. Mathai, A.M. and Saxena, R.K. The H-function with Applications in Statistics and other Disciplines. John Wiley & Sons. Inc., (1978), New York,.

[4]. Saxena, V.P. A formal solution of certain new pair of dual integral equations involving H-functions, *Proc. Nat. Acad. Sci. India Sect.* A **52**(1982), 366-375.

[5]. Tuan, V., Saigo, M. Multidimensional Modified Fractional Calculus Operators. *Math. Nachr*. **161**(1993), 253-270.