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Linearization of Nonlinear Differential Equation by Taylor's Series Expansion and Use of Jacobian Linearization Process

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ABSTRACT : In this paper, we show how to perform linearization of systems described by nonlinear differential equations. The procedure introduced is based on the Taylor's series expansion and on knowledge of Jacobian linearization process. We develop linear differential equation by a specific point, called an equilibrium point.

Keywords : Nonlinear differential equation, Equilibrium Points, Jacobian Linearization, Taylor's Series Expansion.

I. INTRODUCTION

In order to linearize general nonlinear systems, we will use the Taylor Series expansion of functions. Consider a function $f(x)$ of a single variable x, and suppose that $\bar{x}$ is a point such that $f(\bar{x}) = 0$. In this case, the point $\bar{x}$ is called an equilibrium point of the system $\dot{x} = f(x)$, since we have $\dot{x} = 0$ when $x = \bar{x}$ (*i.e.*, the system reaches an equilibrium at $\bar{x}$). Recall that the Taylor Series expansion of $f(x)$ around the point $\bar{x}$ is given by,

$$f(x) = f(\bar{x}) - \left(\frac{\partial f}{\partial x}\right)_{x=\bar{x}} (x-\bar{x}) - \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}\right)_{x=\bar{x}}$$
$$(x-\bar{x})^2 - \ldots .$$

This can be written as

$$f(x) = f(\bar{x}) - \left(\frac{\partial f}{\partial x}\right)_{x=\bar{x}} (x-\bar{x}) - \text{ higher order terms.}$$

For x sufficiently close to $\bar{x}$, these higher order terms will be very close to zero, and so we can drop them to obtain the approximation

$$f(x) \approx f(\bar{x}) - a(x-\bar{x}), \text{ where } a = \left(\frac{\partial f}{\partial x}\right)_{x=\bar{x}}$$

Since $f(\bar{x}) = 0$, the nonlinear differential equation $\dot{x} = f(x)$ can be approximated near the equilibrium point by

$$\dot{x} = a(x-\bar{x})$$

To complete the linearization, we define the perturbation state (also known as delta state) $\delta x = x - \bar{x}$, and using the fact that $\delta\dot{x} = \dot{x}$, we obtain the linearized model

$$\delta\dot{x} = a\delta x$$

This linear model is valid only near the equilibrium point.

II. EQUILIBRIUM POINTS

Consider a nonlinear differential equation

$$\dot{x}(t) = f[x(t), u(t)] \qquad \ldots (1)$$

where $f : R^n \times R^m \to R^n$. A point $\bar{x} \in R^n$ is called an equilibrium point if there is a specific $\bar{u} \in R^m$ (called the equilibrium input) such that $f(\bar{x}, \bar{u}) = 0_n$.

Suppose $\bar{x}$ is an equilibrium point (with equilibrium input $\bar{u}$). Consider starting system (1) from initial condition $x(t_0) = \bar{x}$, and applying the input $u(t) \equiv \bar{u}$ for all $t \geq t_0$. The resulting solution $x(t)$ satisfies $x(t) - \bar{x}$; for all $t \geq t_0$. That is why it is called an equilibrium point.

III. DEVIATION VARIABLES

Suppose $(\bar{x}, \bar{u})$ is an equilibrium point and input. We know that if we start the system at $x(t_0) = \bar{x}$, and apply the constant input $u(t) \equiv \bar{u}$, then the state of the system will remain fixed at $x(t) = \bar{x}$ for all t. Define deviation variables to measure the difference.

$$\delta_x(t) = x(t) - \bar{x} \text{ and } \delta_x(t) = u(t) - \bar{u}$$

In this way, we are simply relabling where we call 0. Now, the variables $x(t)$ and $u(t)$ are related by the differential equation

$$\dot{x}(t) = f[x(t), u(t)]$$

Substituting in, using the constant and deviation variables, we get

$$\delta_x(t) = f[\bar{x} - \dot{\delta}_x(t), \bar{u} - \delta_x(t)]$$

This is exact. Now however, let's do a Taylor expansion of the right hand side, and neglect all higher (higher than 1st) order terms

$$\dot{\delta}_x(t) \approx f(\bar{x},\bar{u}) - \left.\frac{\partial f}{\partial x}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}} \delta_x(t) - \left.\frac{\partial f}{\partial u}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}} \delta_u(t)$$

But $f(\bar{x},\bar{u}) = 0,$

$$\dot{\delta}_x(t) \approx \left.\frac{\partial f}{\partial x}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}} \delta_x(t) - \left.\frac{\partial f}{\partial u}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}} \delta_u(t)$$

This differential equation approximately governs (we are neglecting 2nd order and Higher order terms) the deviation variables $\delta_x(t)$ and $\delta_u(t)$, as long as they remain small. It is a linear, time-invariant, differential equation, since the derivatives of δ_x are linear combinations of the δ_x variables and the deviation inputs, $\delta_{u.}$ The matrices,

$$A = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}} \in R^n \times R^n,$$

$$B = \left.\frac{\partial f}{\partial u}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}} \in R^n \times R^m \qquad \text{... (2)}$$

are constant matrices. With the matrices A and B as defined in (2), the linear system

$$\dot{\delta}_x(t) \approx A\delta_x(t) - B\delta_u(t)$$

is called the Jacobian Linearization of the original nonlinear system (1), about the equilibrium point $(\bar{x},\bar{u})$. For "small" values of δ_x and δ_u, the linear equation approximately governs the exact relationship between the deviation variables δ_x and δ_u.

For "small" δ_x [*i.e.*, while $u(t)$ remains close to $\bar{u}$], and while δ_x remains "small" [*i.e.*, while $x(t)$ remains close to $\bar{x}$], the variables δ_x and δ_u are related by the differential equation,

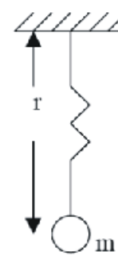
$$\dot{\delta}_x(t) \approx A\delta_x(t) - B\delta_u(t)$$

In some of the rigid body problems we considered earlier, we treated problems by making a small-angle approximation, taking θ and its derivatives $\dot{\theta}$ and $\ddot{\theta}$ very small, so that certain terms were ignored $(\dot{\theta}^2, \ddot{\theta}\sin\theta)$ and other terms simplified $(\sin\theta \approx \theta, \cos\theta \approx 1)$. In the context of this discussion, the linear models we obtained were, in fact, the Jacobian linearization around the equilibrium point $\theta = 0, \dot{\theta} = 0.$

If we design a controller that effectively controls the deviations δ_x, then we have designed a controller that works well when the system is operating near the equilibrium point (x, u). This is somewhat effective way to deal with nonlinear systems in a linear manner.

IV. EXAMPLE

Consider the system shown below.



The governing differential equations of motion for the above system is given by

$$m\ddot{r} - kr - kl_0 = mr\dot{\theta}^2 - mg\cos\theta = 0 \qquad \text{... (1)}$$

$$mr\ddot{\theta} - 2m\dot{r}\dot{\theta} - mg\sin\theta = 0 \qquad \text{... (2)}$$

where, l_0 is the initial length of the spring and 'k' is the stiffness constant of the spring.

Note that the above differential equations are non-linear in nature. First, to find the equilibrium point, equate all the derivative terms to zero. Therefore equation (2) reduces to

$mg\sin\theta = 0,$

$= \sin\theta = 0,$

$= \theta = n\pi.$

There $\theta_0 = 0$ is one equilibrium point for the above system.

Following the same procedure for equation (1), we get

$kr - kl_0 - mg\cos\theta = 0,$

$= kr - kl_0 - mg = 0,$

$$= r = \frac{kl_0 - mg}{k} = r_0 \qquad \text{... (3)}$$

Therefore $r = r_0$ is the equilibrium value for the variable 'r'.

Expanding each term in equation (1) by Taylor's series about the equilibrium point and neglecting the higher order terms, we have

$$m\ddot{r} - kr - kl_0 - mr\dot{\theta}^2 - mg\cos\theta = 0,$$

$$= m\ddot{r} - kr - kl_0 - (mr\dot{\theta}^2)$$
$$\Big|_{\substack{r=r_0\\ \theta=\theta_0}} - \frac{\partial}{\partial r}(mr\dot{\theta}^2)$$
$$\Big|_{\substack{r=r_0\\ \theta=0}} (r - r_0) - \frac{\partial}{\partial \theta}(mr\dot{\theta}^2)$$
$$\Big|_{\substack{r=r_0\\ \theta=0}} (\dot{\theta} - 0) - (mg\cos\theta)$$
$$\Big|_{\theta=\theta_0} - \frac{\partial}{\partial \theta}(mg\cos\theta)$$
$$\Big|_{\theta=0} (\theta - 0) = 0,$$

$$= m\ddot{r} - kr - kl_0 - mg = 0 \qquad \text{... (4)}$$

Following the same procedure for equation (2), we get

$$mr\dot{\theta} - 2m\dot{r}\dot{\theta} - mg\sin\theta = 0,$$

$$= (mr\ddot{\theta})\Big|_{\substack{r=r_0\\ \theta=\theta_0}} - \frac{\partial}{\partial r}(2m\dot{r}\dot{\theta})$$
$$\Big|_{\substack{r=r_0\\ \theta=0}} (r - r_0) - \frac{\partial}{\partial \theta}(mr\ddot{\theta})$$
$$\Big|_{\substack{r=r_0\\ \theta=0}} (\ddot{\theta} - 0) - (2m\dot{r}\dot{\theta})$$
$$\Big|_{\substack{r=r_0\\ \theta=\theta_0}} - \frac{\partial}{\partial r}(2m\dot{r}\dot{\theta})$$
$$\Big|_{\substack{r=r_0\\ \theta=0}} (\dot{r} - r_0) - \frac{\partial}{\partial \theta}(2m\dot{r}\dot{\theta})$$
$$\Big|_{\substack{r=r_0\\ \theta=0}} -(\dot{\theta} - 0) - (mg\sin\theta)$$
$$\Big|_{\theta-\theta_0} - \frac{\partial}{\partial \theta}(mg\sin\theta)$$
$$\Big|_{\theta=0_0} (\theta - 0) = 0$$

$$mr_0\ddot{\theta} - mg\theta = 0 \qquad \text{... (5)}$$

Equations (4) and (5) represent the linearized differential equation of motion for the above system.

V. CONCLUSION

Our method is to find linear differential equation by Taylor's series expansion and use of Jacobian linearization process. But here find linear system only at equilibrium points. This method is useful for check the stability of system of differential system and stability is depends upon the nature of the eigenvalue. This method is used for nonlinear model.

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