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An *O(n)* Time Algorithm For Maximum Induced Matching In Bipartite $Star_{123}$-free Graphs

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Abstract— A matching in a graph is a set of edges no two of which share a common vertex. A matching M is an induced matching if no edge connects two edges of M. The problem of finding a maximum induced matching is known to be NP-hard in general and specifically for bipartite graphs. Lozin has been proposed an $O(n^3)$ time algorithm for this problem on the class of bipartite $Star_{123}, Sun_4$-free graphs. In this paper we improve and generalize this result in presenting a simple $O(n)$ time algorithm for maximum induced matching problem in bipartite $Star_{123}$-free graphs.

Keywords-Bipartite graph; Decomposition of graphs; Design and analysis of algorithms; Matching; Induced Matching.

I. INTRODUCTION

A *matching* M of a graph $G = (V, E)$ is a subset of edges with the property that no two edges of M share a common vertex. A matching is called *induced* if no two edges in the matching have a third edge connecting them. Equivalently, the subgraph of G induced by M consists of exactly M itself. We study the problem of finding in G an induced matching of maximum cardinality, denoted $i\mu(G)$. This problem has been introduced by Cameron [3], where he has proved its NP-hardness in the class of bipartite graphs. The maximum induced matching problem was shown to be polynomial for several classes of graphs: for chordal graphs and for interval graphs by Cameron [3], for circular-arc graphs by Golumbic and Laskar [5], and for trapezoid graphs, k-interval-dimension graphs and cocomparability graphs by Golombic and Lewenstein [4]. Fricke and Laskar give a linear algorithm for trees [1]. Lozin in [8] describes an $O(n^3)$ time algorithm for the problem on bipartite $Star_{123}, Sun_4$-free graphs where n is the number of vertices. In addition, he studied in [7] the class of bipartite $Star_{123}$-free graphs and conjectured that his result in [8] can be extended to this class of bipartite graphs. In this paper we improve and generalize Lozin's algorithm in presenting a simple $O(n)$ time algorithm for this problem on bipartite $Star_{123}$-free graphs. Our algorithm is based on the recognition algorithm of the class $Star_{123}$-free bipartite graphs introduced by Quaddoura in [6].

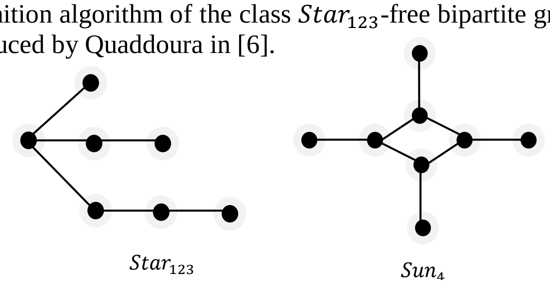

Figure. 1. $\boldsymbol{Star_{123}}$ and $\boldsymbol{Sun_4}$ configurations

II. DEFINITION AND PROPERTIES

A bipartite graph $G = (B \cup W, E)$ is defined by two disjoint vertex subsets B – the black vertices and W - the whites ones, and a set of edges $E \subseteq B \times W$. The *bi-complement* of a bipartite graph $G = (B \cup W, E)$ is the bipartite graph defined by $\bar{G}^{bip} = (B \cup W, B \times W - E)$. If the color classes B and W are both non empty the graph will be called *bichromatic, monochromatic* otherwise. A vertex x will be called *isolated* (resp. *universal*) if x has no neighbors in G (resp. in $\bar{G}^{bip}$). A *complete* bipartite graph is a graph having only universal white vertices and universal black vertices. A *stable* set is a set of isolated vertices. A chordless path on k vertices is denoted by P_k and a chordless cycle on k vertices is denoted by C_k. Given a subset X of the vertex set $V(G)$, the subgraph induced by X will be denoted by $G[X]$ or simply by X if there is no confusion. A K_2 is a complete bipartite graph with two vertices. A $2K_2$ is a two copies of a K_2.

Definition 1 [2] *Given a bipartite graph $G = (B \cup W, E)$ of order at least 2, G is $K + S$ graph if and only if G contains an isolated vertex or its vertex set can be decomposed into two sets K and S such that K induces a complete bipartite graph while S is a stable set.*

Property 1 [2] *Let $G = (B \cup W, E)$ be a bipartite graph of order at least 2. G is $K + S$ graph if and only if there exists a partition of its vertex set into two non empty classes V_1 and V_2 such that all possible edges exists between the black vertices of V_1 and the white vertices of V_2 while there is no edge connecting a white vertex of V_1 with a black vertex of V_2.*

Such partition is referred as associated partition of G and is denoted by the ordered pair (V_1, V_2).

Property 2 [2] *A bipartite graph G is a $K+S$ graph if and only if G admit a unique (up to isomorphism) partition of its vertex set $(V_1 \cup V_2 \dots \cup V_k)$ satisfying the following conditions:*

a) $\forall i = 1, \dots, k-1, (V_1 \cup \dots \cup V_i, V_{i+1} \cup \dots \cup V_k)$ *is an associated partition to the graph G*
b) $\forall i = 1, \dots, k, G[V_i]$ *is not a $K+S$ graph.*

The partition $(V_1, \dots, V_k)$ of the above property is called $K+S$ decomposition while a set V_i said to be $K+S$ -component of the graph.

From $K+S$ decomposition together with the decomposition of bipartite graph G into its connected components (parallel decomposition) or those of $\bar{G}^{bip}$ (series decomposition) yield a new decomposition scheme for G called *canonical decomposition*. It is show in [2] that whatever the order in which the decomposition operators are applied ($K+S$ decomposition, series decomposition or parallel decomposition), a unique set of indecomposable graphs with respect to canonical decomposition is obtained. Obviously, a unique tree is associated to this decomposition. The internal nodes are labeled according to the type of decomposition applied, while every leaf correspond to a vertex of G. Hence there are four types of internal nodes, parallel node (labeled P), series node (labeled S), $K+S$ node (labeled $K+S$) , and indecomposable node (labeled N). By convention, the set of vertices corresponding to the set of leaves having an internal node α as their least common ancestor as well as the subgraph induced by this set of leaves will be denoted simply by α.

Observation 1 let G be a bipartite graph and T be its canonical decomposition tree. According to the order in which the decomposition operations are applied, every child of a P-node or a S-node cannot be a vertex. Such node would have either an isolated or a universal vertex and thus would induce a $K+S$ graph.

Following the recognition algorithm given in [6], bipartite $Star_{123}$ -free graphs are bipartite graphs whose indecomposable graphs within canonical decomposition are reduced to signal vertices or to an extended path EP_k or the bi-complement of an extended path EP_k or an extended cycle EC_k or the bi-complement of an extended cycle EC_k. In all cases $k \geq 7$. More precisely

Definition 2 [6] *A graph G is said to be an extended path EP_k if there is a partition of the vertex set of G into a monochromatic sets $\{V_1, \dots, V_k\}$ such that* $E = \bigcup_{i=1}^{k-1} V_i \times V_{i+1}$ and $k \geq 7$.

Definition 3 [6] *A graph G is said to be an extended cycle EC_k if there is a partition of the vertex set of G into a monochromatic sets $\{V_1, \dots, V_k\}$ such that* $E = \bigcup_{i=1}^{k-1} V_i \times V_{i+1} \cup V_1 \times V_k$ and $k \geq 7$.

The construction of the canonical decomposition tree of a bipartite $Star_{123}$-free graph tree can be obtained in linear time from the algorithm given by Quaddoura in [6]. According to this algorithm, every child of a N-node is a node marked by P' corresponding to a set $V_i, i = 1 \dots k$, if $|V_i| > 1$, or to a vertex of G otherwise. Figure 2 illustrate a bipartite $Star_{123}$-free graph and its canonical decomposition tree.

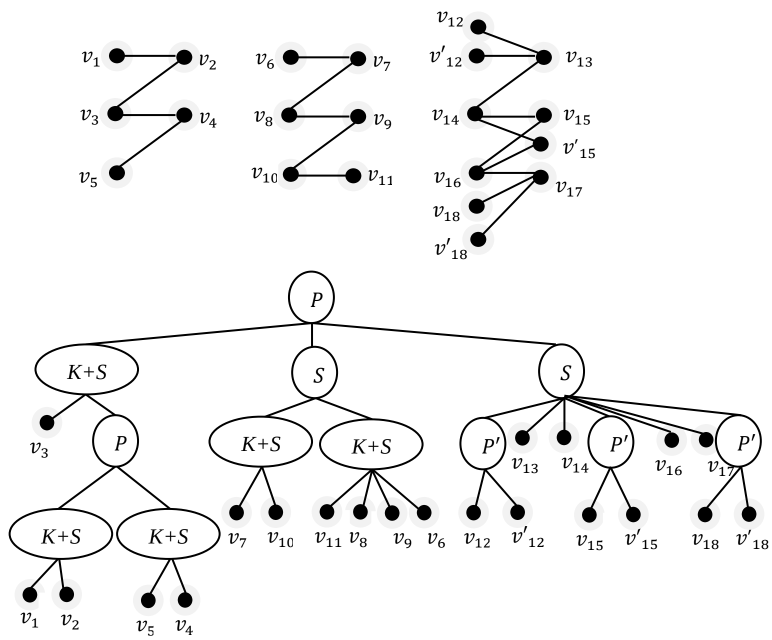

Figure. 2. A bipartite $\boldsymbol{Star_{123}}$-free graph and its canonical decomposition tree

III. MAXIMUM INDUCED MATCHING IN BIPARTITE $Star_{123}$-FREE GRAPHS

Let G be a bipartite $Star_{123}$-free graph and $T(G)$ be its canonical decomposition tree. Our algorithm uses post order traversal to visit all the nodes of $T(G)$. Whenever an internal node α is visited, we compute a maximum induced matching of the subgraph induced by α from the maximum induced matching's of its children say $\alpha_1, \alpha_2, \dots, \alpha_k$. For this purpose we distinguish several cases according to the type of α.

Obviously, If α is a P-node then $i\mu(\alpha) = \bigcup_{i=1}^{k} i\mu(\alpha_i)$. Also, if α is a P'-node then $i\mu(\alpha) = \emptyset$.

A set of vertices A is called *module* if every vertex in A has the same neighborhood outside of A. A bipartite graph whose every module is of size 1 will be called *prime*. It is not hard to see that any bipartite graph G has a unique (up to isomorphism) maximal prime induced subgraph that can be obtained by choosing exactly one vertex in each module of G. Lozin in [8] proved the following Lemma.

Lemma 1 *If H is a maximal prime induced subgraph of a graph G, then $i\mu(G) = i\mu(H)$.*

Suppose now α is a N-node. As motioned above, α induces an extended path EP_k or its bi-complement or an extended cycle EC_k or its bi-complement. Clearly, in this case, the maximal prime induced subgraph of α is a path P_k or its bi-complement (if α induces an extended path EP_k or its bi-complement) or a cycle C_k or its bi-complement (if α induces an extended cycle EC_k or its bi-complement). The following simple Lemma is proved in [8].

Lemma 2 $|i\mu(P_k)| = \lfloor (k+1)/3 \rfloor$, $|i\mu(C_k)| = \lfloor k/3 \rfloor$. Let $k \geq 7$ then $\left|i\mu\left(\bar{P}_k^{bip}\right)\right| = \left|i\mu\left(\bar{C}_k^{bip}\right)\right| = 2$.

By Lemma 2, the set $\{v_{3i-2}v_{3i-1}: 1 \leq i \leq \lfloor (k+1)/3 \rfloor\}$ is a maximum induced matching of the path $P_k = v_1 v_2 \ldots v_k$, the set $\{v_{3i-2}v_{3i-1}: 1 \leq i \leq \lfloor k/3 \rfloor\}$ is a maximum induced matching of the cycle $C_k = v_1 v_2 \ldots v_k$, and the set $\{v_1 v_4, v_2 v_5\}$ is a maximum induced matching of $\bar{P}_k^{bip}$ or $\bar{C}_k^{bip}$.

Let's discus the cases when α is a S-node or a $K+S$-node.

Lemma 3 *Let α be a $K+S$ node. Then*

a) If every child of α is a vertex then $|i\mu(\alpha)| \leq 1$

b) Else $i\mu(\alpha) = i\mu(\alpha_j)$ where α_j is the child of α which satisfies $|i\mu(\alpha_j)| = max\{|i\mu(\alpha_i)|:\ \alpha_i \text{ is not a vertex}\}$

Proof By Observation 1 the father of a leaf is either a N-node, a P'-node or a $K+S$-node. The validity of Lemma deduces directly from Property 3 by remarking that there is no $2K_2$ of α can share vertices with two different children. □

Lemma 4 *Assume that α is a S-node.*

a) If any child α_i of α satisfies that $|i\mu(\alpha_i)| \leq 1$ then $|i\mu(\alpha)| = 2$

b) Else $i\mu(\alpha) = i\mu(\alpha_j)$ where α_j is the child of α which satisfies $|i\mu(\alpha_j)| = max\{|i\mu(\alpha_i)|:\ 1 \leq i \leq k\}$.

Proof The Lemma can be deduced from the following Claim :

Claim the cardinality of maximum induced matching of α which shares vertices between different children is 2

Proof Let X denotes to such induced matching. Every child of α contains two nonadjacent vertices of different color, otherwise α would contain a universal vertex and hence, by Observation 1, α is a $K+S$-node, a contradiction. Let v_1, v_2 be two nonadjacent vertices of a child say α_1 of α such that v_1, v_2 are of different color, and v_3, v_4 be two non adjacent vertices of a child say α_2 distinct of α_1 and v_3, v_4 are of different color, then the set $\{v_1, v_2, v_3, v_4\}$ induces a $2K_2$. Therefore, $X \geq 2$. Since α is a S-node, any vertex v of a child distinct of α_1 and α_2 is adjacent to the two vertices of $\{v_1, v_2, v_3, v_4\}$ whose have the same color, so $X = 2$.

The proof of Lemma 4 shows that any child of a S-node contains two nonadjacent vertices of different color. The following Lemma allow us to find easily these two vertices when any child α_i of α satisfies that $|i\mu(\alpha_i)| \leq 1$.

Lemma 4 *if $|i\mu(\alpha)| \leq 1$ then one of the following is hold:*

a) α is a vertex.

b) α is a P' node.

c) α is a $K+S$-node and every child of α is a vertex.

Proof If α is a N-node or a S-node then by Lemma 2 and Lemma 4, $|i\mu(\alpha)| \geq 2$. So it is enough to prove that α cannot be a P-node. Suppose that α is a P-node. Then every child of α contains at least one edge, otherwise α would contain an isolated vertex and hence α would be a $K+S$-node. Therefore $|i\mu(\alpha)| \geq 2$, a contradiction. □

The above discussion leads us to the following algorithm

Algorithm Maximum Induced Matching

Input : A bipartite $Star_{123}$-free graph G and its canonical decomposition tree $T(G)$.

Output : A maximum induced matching $i\mu(G)$.

Let α be a node on a post order traversal of $T(G)$
if α is a vertex or a P'-node then $i\mu(G) = \emptyset$
else let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be the children of α
  if α is a N-node then for every child α_i of α, pick a vertex v_i, $1 \leq i \leq k$
    if α induces an EP_k then $i\mu(G) = \{v_{3i-2}v_{3i-1}: 1 \leq i \leq \lfloor (k+1)/3 \rfloor\}$
    if α induces an EC_k then $i\mu(G) = \{v_{3i-2}v_{3i-1}: 1 \leq i \leq \lfloor k/3 \rfloor\}$
    if α induces an $\overline{EP}_k^{bip}$ or an $\overline{EC}_k^{bip}$ then $i\mu(G) = \{v_1 v_4, v_2 v_5\}$
  else if α is a $K+S$-node then
    if every child of α is a vertex then
      if there is two adjacent vertices v_1, v_2 of α then
        $i\mu(G) = \{v_1 v_2\}$
      else $i\mu(G) = \emptyset$
      if α contains two nonadjacent vertices v_1, v_2 of different color then $I_\alpha = \{v_1, v_2\}$
    else $i\mu(G) = i\mu(\alpha_j)$ where α_j is the child of α satisfying
      $|i\mu(\alpha_j)| = max\{|i\mu(\alpha_i)|:\ \alpha_i \text{ is not a vertex}\}$
  else if α is a S-node then
    if for every $1 \leq i \leq k$, $|i\mu(\alpha_i)| \leq 1$ then
      let $I_{\alpha_1} = \{v_1, v_2\}$ and $I_{\alpha_2} = \{v_3, v_4\}$ such that v_1, v_3 are black vertices and v_2, v_4 are white, $i\mu(G) = \{v_1 v_4, v_2 v_3\}$
    else $i\mu(G) = i\mu(\alpha_j)$ where α_j is the child of α satisfying
      $|i\mu(\alpha_j)| = max\{|i\mu(\alpha_i)|:\ 1 \leq i \leq k\}$

else //α is a P-node// $i\mu(G) = \cup_{i=1}^{k} i\mu(\alpha_i)$

Complexity The number of operation performed in every node is proportional with the number of children of that node. Since the number of visited node is $O(n)$, this algorithm runs with $O(n)$ time complexity.

Figure 3 illustrates the computation of the maximum induced matching for the graph in Figure 2 using our algorithm. The set above every node represents the maximum induced matching of that node and the set under a node represents two nonadjacent vertices in this node.

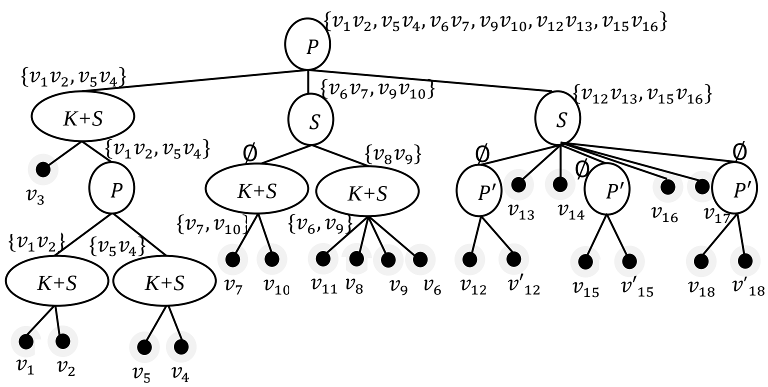

Figure. 3. The computation of the maximum induced matching for the graph in Figure 2.

IV. CONCLUSION

The maximum induced matching algorithm is computed in $O(n)$ time**,** given a canonical decomposition tree of a bipartite $Star_{123}$-free graph. The canonical decomposition of a bipartite $Star_{123}$-free graph can be done in $O(n+m)$ time where m is the number of edges [6]. Thus, the whole process is in $O(n+m)$ time.

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