# KNOWING MATHEMATICS-FOR-TEACHING: THE CASE OF PLANNING LEARNING ACTIVITIES

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## Abstract

*Traditionally mathematical subject matter knowledge for teachers was defined with reference to the academic discipline of mathematics. Since the 1990s, however, a reconceptualization of what Shulman has called content knowledge for teachers has been suggested. In mathematics education, it has been conceptualized by some researchers through the notion of mathematics-for-teaching: the mathematics teachers need to know derives from the practice of teaching mathematics at the school level. The study reported upon in this article inquired into the mathematics-for-teaching used by elementary school teachers during joint lesson planning sessions. Implications for teacher education are discussed.*
**Key words**: *teaching mathematics, mathematics-for-teaching, teachers' subject matter knowledge, mathematics teacher education.*

## Introduction

Traditionally, mathematical subject matter knowledge as it is relevant to teaching was understood as the mathematics as defined by the discipline of mathematics. In the early 1990s a reconceptualization of the notion of subject matter knowledge as it is relevant to teaching was undertaken, and its implications are still being researched. At the core of the reconceptualization lies the idea that what is relevant for a teacher to know about mathematics as a subject matter has to be connected to the teaching practice the teacher engages in. Claiming that a teacher's teaching practice goes beyond practices based on a transmission view of teaching and learning (demonstrating the solving of problems of a particular type and having students practice such problems on their own), teacher's mathematical subject matter knowledge – so the argument goes – *has to go beyond* understanding the mathematics as it is laid out in a traditional mathematics textbook. The concept of the mathematical subject matter knowledge a teacher needs to know for (reform-based) mathematics teaching is *qualitatively* different from the knowing of mathematics in other contexts, like the use of mathematics in engineering. Some central research questions that develop through this reconceptualization then are "What qualitatively different mathematical subject matter knowledge looks like? "How it is used by teachers?", and "How can teachers acquire it?" Of course, any answers to these questions will have to be appropriate to the respective level of education (preschool, primary school, etc.). The study reports on in this paper addresses aspects of the first and second of these three questions. In responding to the two questions, this paper focuses on: (1) *the quality* of this mathematics-for-teaching, in particular on whether this mathematics-for-teaching is indeed *qualitatively* different from corresponding discipline-based mathematics; (2) *what* mathematics-for-teaching teachers would use / need in their planning of learning activities.

## Theoretical Framework

There seems to be no doubt that subject matter knowledge is important in the education of teachers (see, for instance, Grossman, Wilson, & Shulman, 1989; Kennedy, 1990; Ma, 1999). After-degree preservice teacher education programs make a degree in a teachable field of study or at least sufficient coursework in that field an admission requirement; concurrent preservice teacher education programs require the sufficient coursework in a teaching relevant subject matter as part of the program. For certified teachers, literature on professional development strongly suggests the importance of content knowledge for teachers for improving student learning (see, for instance, Darling-Hammond & Ball, 1999). But the crucial question is how 'subject matter knowledge' needs to be understood here?

Traditionally, the notion of a *knowledge base for teaching* is referenced to Shulman's (1986, 1987) categorization of teacher knowledge:

> At minimum, [the knowledge categories] would include: content knowledge; general pedagogical knowledge; curriculum knowledge; pedagogical content knowledge; knowledge of learners and their characteristics; knowledge of educational contexts; knowledge of educational ends, purposes, and values, and their philosophical and historical grounds (Shulman, 1987, p. 8)

The study focused on what Shulman has called 'content knowledge', which Shulman himself explicates as the content knowledge of the respective discipline and the historical and philosophical scholarship of the nature of the knowledge in that discipline (Shulman, 1987).

How the notion of mathematical content knowledge is *conceptualized* makes a crucial difference in the implications that derive from the importance of mathematical content knowledge as part of a teacher's knowledge base. In particular in mathematics education research a shift has taken place in the *conceptualization* of the notion of (mathematical) content knowledge. In the early 1990s a new conceptualizing of mathematical content knowledge started to emerge (Ball, 1991; Kennedy, 1990). At the core of this reconceptualization is the idea that mathematical content knowledge *as it is relevant for teachers* derives from *the practice of teaching mathematics*; that means that the central question for investigating the mathematical content knowledge (for teaching mathematics) is what teachers need to know about mathematics *in order to teach mathematics well* (Ball & Bass, 2002; Ball, Hill & Bass, 2005). Adopting a term coined by Davis and Simmt (2006), let us refer to the mathematical content knowledge *for teaching* as "mathematics-for-teaching". (The term is appropriate, although Davis and Simm's approach to the concept is somewhat different from the one taken by Ball and her collaborators.)

This shift in focus of how mathematical content knowledge for teaching is established is embedded into a shift of what it means to teach mathematics well. As long as the teaching of mathematics was understood as telling students the mathematical principles, facts and algorithms and then letting students practice those algorithms, mathematical content knowledge for teaching could be conceptualized as the discipline-defined content. However, the understanding of what it means to teach mathematics in schools has dramatically changed in North America, partially through the influential NCTM Standards (NCTM, 1989, 2000). Now, students are to explain their thinking, come up with their own solutions, come up with alternative solutions, justify their procedures and answers, discuss their approaches and solutions with others, and so on. That means for teachers that academic discipline-based mathematical content knowledge is not enough anymore. Now, proficiency in the practice of teaching mathematics *additionally* includes *understanding* connections between different mathematical concepts, *understanding* underlying mathematical concepts in a deeper way to be able to assess alternative approaches

and solutions, and so on. Ball and Bass (2002) characterize one central aspect of that difference as follows: "a powerful characteristic of mathematics is its capacity to *compress* information into abstract and highly usable forms", but for teaching mathematics "that mathematical knowledge needs to be *unpacked*" (p. 11).

Ball and her collaborators conceptualized the notion of teachers' required subject matter knowledge by looking at *what kind of mathematics is required of teachers in their practice of teaching mathematics* (Ball, 1991; Ball, Lubienski & Mewborn, 2001). Ball and Bass (2002) and Ball, Hill and Bass (2005) list different aspects of a mathematics teacher's teaching practice that impact what it means for a teacher to know 'mathematics-for-teaching': understanding students' thinking; responding to alternative ways of solving problems; explaining; planning learning activities; creating assessment tools; grading. For some of those aspects of mathematics teaching practice, Ball and her collaborators characterize the mathematics-for-teaching up to a point where they have provided validated instruments to assess the quality of a teacher's mathematical subject matter knowledge as it is relevant for those particular aspects of mathematics teaching practice. Less attention, however, has been given to the mathematics-for-teaching involved in lesson planning as part of a teacher's mathematics teaching practice. The study focuses on the mathematics-for-teaching as it is part of teachers' planning of mathematics lessons.

Recently, theorizing about what teachers need to know about mathematics to teach it well has shifted the conceptualization of 'mathematical content knowledge' as it is relevant for teaching even further. Davis and Simmt (2006) argue "that the mathematics teachers need to know is *qualitatively different* than the mathematics their students are expected to master" (pp. 315-316, emphasis added), and they coined the term "mathematics-for-teaching" to capture the mathematics relevant for teachers. The study sheds some light on what that qualitative difference looks like in some cases.

**Focus and Methodology of the Study**

The research was part of a larger project with different foci. For this part the focus is on the following two questions:

1.  What knowing of mathematics-for-teaching would teachers draw upon in their planning of their learning activities?
2.  How is that mathematics-for-teaching different from 'academic mathematics'?

For the study a sequence of six lesson planning sessions was organized with eight teachers of a K-5 school in a large city in Canada. For those sessions the teachers grouped themselves into three groups of two to three teachers for the purpose of designing mathematical playing activities to help their respective students develop mathematical conceptual understanding. The grouping was based on the curricular area the teachers planned to address with their respective activities. The teaching experience ranged from beginning teachers to veteran teachers with over ten years of teaching experience. The researcher participated in the planning sessions as participant observer (Anderson-Levitt, 2006), joining one of the three groups at any given time over the six sessions. We met once a week, leaving time for the teachers to implement (part of) their ideas between the meetings, which some teachers did. Others could work on their project only during our sessions and, thus, were able to implement their project only shortly before or after we completed the sessions sequence.

This paper draws upon the analysis of data collected from one of the three groups. This group involved three teachers: Anita (grade 5), Jody (grade 3) and Norma (grade 4), all three are in their first few years of teaching (all names are pseudonyms). This group worked on

game activities for the curriculum area of whole number operations with a particular focus on multiplication. The data for this aspect of the study consists of transcribed recordings of the planning sessions. The data were coded for information on the mathematical knowledge that teachers drew on when they were discussing their planning of mathematical learning activities. Particular attention was paid to the specificities of the mathematics drawn upon by the teachers during the planning activities. Then the quality of this mathematical knowledge was analyzed to see whether and in what way it is qualitatively different (mathematics-for-teaching) from the content knowledge of the academic discipline of mathematics.

## Results of Research

This section is divided into two parts, each responding to one of the two research questions outlined above. The first part reports on *what* mathematics the three teachers drew upon in the planning sessions. The data will show not just the richness of the mathematics that the teachers involved drew upon over a short period of time of joint planning, but it will also show the relevance of mathematics (mathematical content knowledge) for the planning of a teaching of mathematics. The second part uses a selection of this mathematics that the teachers drew upon in their planning to illustrate the *qualitative difference* of (some aspects of) this mathematics (mathematics-for-teaching) from the academic subject-specific content.

*Mathematics in Planning Mathematical Activities*

What knowing of mathematics did the teachers draw upon in their planning of mathematical learning activities to help students with their conceptual understanding of whole number operations? This section describes what was identified as ten different aspects of mathematics around the notion of multiplication that the three teachers drew upon in two of their planning sessions and that is known in a way that is specific to their role as teachers of mathematics. Following is a description of this kind of mathematics and the data upon which the interpretation is based.

(MT1)   *Multiplication as grouping equal-size groups*

> *Anita*: If you can show me multiples of two, because it is very visual at this point. Really, I am not even concerned about them saying "well we don't say times but two groups of three"; like that language comes; but even before that they need to be able to show it and see what that is. Although the language part is important so maybe there could be that part. You could say "These two groups of something two or multiples of something" but I am wondering if there is a way that they can show it. (SG, pp. 7-8)

> *Jody*: Multiplication, just being able to know what that means, like visually, and I guess symbolically grouping numbers. (SG5, p. 8)

(MT2)   *Multiplication as repeated addition*

> *Jody*: so I am not really interested if they know their facts or not. I am interested to see if they know what that means. So we played some games last year and, you know, its more starting understanding it as repeated addition. I think that's the biggest thing. (SG5, p. 3)

(MT3)   *Relationship between multiplication and division*

> *Anita*: I want this game to somehow transition them into division, because they are not comfortable with it. I steer them in that direction and they will not do it independently, and

they shy away from it, and they seem to freak out. But even though they are doing it already, like when they are playing, if they are playing a game like this and they have the number twenty-seven. They already know that it can be divided nine times. They know that, and yet when you ask them, they have no idea what the answer is. They just don't know that they are doing it, but if they are playing a game like this, maybe we can relate it to division. You know, they are doing the multiplication already. It seems logical that they would just be able to do division. (SG5, p. 9)

*Jody*: Teaching multiplication and then division, I mean, I kind of did that last year and than I realized, you know that probably doesn't make a lot of sense, because they are doing division as they are doing multiplication, and they are so connected, the same as addition and subtraction. (SG5, p. 10)

(MT4)   *Equal distributing as a central part of division*:

*Anita*: Well, kids are doing division when they are four years old, because they are obsessed with fairness. Everyone has to get the same amount of everything, so they already know what it [division] is. (SG5, p. 11)

(MT5)   *Arbitrariness of symbols for number operations in mathematics*:

*Anita*: But then the other thing that I was thinking of was, well, suits could be instead operations, like a heart could be like you have to divide, but a spade would be you have to multiply or something. (SG5, p. 4)

Anita is here referring to using playing cards as the context for her designing a playing activity for her students ("suits", "heart", "spade"). Anita draws on the idea of the arbitrariness of mathematical symbols to fit her concern for number operations *into the problem context of card playing* that she has created in her planning of meaningful learning activities for her students.

(MT6)   *Arbitrariness of order in the tripartite relationship of multiplication*:

At one point the three teachers discussed a game similar to scrabble but with numbers rather than with letters, where the numbers were to be connected through one operation and the equal sign. Probing the ideas the teachers explored, I asked the following question that received the following responses:

*Q*: So let's say there is a [two] and a twelve. Would you allow a six following that?
*Charlotte*: Following our thinking that we were, yes.
*Jody*: As long as you [the students] can explain it.
(SG5, p. 22)

Here the teachers demonstrate their knowing of the arbitrariness / conventionality of the order in a "multiplication fact family" like "2,6,12" ($2 \times 6 = 12$) by not insisting on the conventional order but only on the understanding of *how* the three numbers relate to each other as members of a multiplication fact family.

(MT7)   *The relationship between multiplication and the dimensions of geometrical objects*

*Anita*: Yes, and that's why for this, because I was not evil with them, I did have other options, and one of them was using the colour tiles to make a raise, but I told them you can make a raise that are two-dimensional and if you want, you can try to make a three-dimensional array but I am not going to tell you how to do that, just try to figure it out. What do you do if it is

three-dimensional? How can you turn that into a sentence? But they, I mean most people went to this maybe because it was novel, and then there was a few kids making three-dimensional shapes and trying to make sentences for it. (SG7, p. 16)

(MT8)    *Short-cut notions for concatenations of bi-term relationships*
         "5÷5×5÷5×5 =" is short for a chain of bi-term relations of the types of "5÷5=*a*" and
         "*a*×5=*b*" and "*b*÷5=*c*" and "*c*×5 ="

         *Anita*: Yeah, he was trying to get something for his turn because I didn't give them . . . There is no limited amount of numbers they can use but its one equation per turn, so he tried to have all across the whole board horizontally one equation that would take up the whole thing, but he just did five times five divided by five times five divided by five on and on and on, but then trying to get the answer, which he said was five, but because of the number of fives that he had, it was actually one, but he didn't understand why five divided by five isn't five [If Anita remembers the answer as being one correctly, the chain of fives has to have had started with "five divided by five".]
         *Norma*: Then you should have had manipulatives.
         *Anita*: And when they have those questions. So that was a question, yeah, okay, let's get our manipulatives and figure out why five divided by five is not five. And then he understood it. (SG7, p. 3)

(MT9)    *Unique number relationships for multiplication and division*
         For a whole-number triplet <*x, y, z*> it is possible to uniquely determine whether the operation between *x* and *y* (whole numbers) resulting in *z* is multiplication or division, which is true in all cases but <1, 1, 1>.
         The background for the following excerpt from the transcript is that Anita had used in her class the scrabble-like game where students were to link three numbers with two numbers multiplying to the third number. Students were asked to include arrows on their game board to indicate which two numbers multiplied or divided result in the third number (e.g., "3, 4→12"). Anita raises the concern that when playing the game some students were *"just mindlessly throwing out numbers because we were playing the game and they put something that didn't quite work. They put for example two times three is fourteen or something incorrect."* (SG7, p. 12) The following excerpt follows that concern with suggestions on how to extend the activity by including a process of checking the numbers on the board.

         *Anita*: I want them to record their work maybe I should have them do more than just drawing arrows so maybe . . .
         *Charlotte*: Well, what if you gave, you know, once they had done a grid, give that grid … like, you did one, you and Thomas did a grid, then you would give your grid to [Jodie] and I, and we would have to write to follow the arrows, right, and write their thought process, right.
         *Anita*: That is cool. That would be really good.
         *Charlotte*: Because then we would have to see three five fifteen right if that's the way they drew the arrows. We would have to say "ok, that means three times five is fifteen". We would have to figure out how those arrows went and what that meant, and what the other group was thinking. That would be an interesting challenge and get at a lot of things. It would get at whether or not yours was correct thinking, but it would also elevate it to another level in terms of abstractness.
         (SG7, pp. 12-13)

(MT10)   *Hierarchy between mathematical concepts, here: between multiplication and fractions*

         *Anita*: "Yeah, but I do even thought they say, like, "be realistic about the curriculum". I do think I will not be able to finish everything on time. Then what is it going to look like, you

know, what if they do not actually get to do everything? But I think I just right now maybe could just make sure they know those two things, multiplication and division, and then move on, once that is reached; and especially if I am going to do fractions, because if they cannot divide, how are we going to do fractions? (SG7, p. 24)

MT1 to MT10 all describe mathematics that the teachers know in a way that is relevant to their role as teachers of mathematics. First, the knowledge expressed in MT1 to MT10 concerns mathematical ideas rather than pedagogical or curricular ideas, which makes this knowledge *mathematical* knowledge. Furthermore, knowing the mathematics in the way described in MT1 to MT10 is specifically relevant to a teacher's role as *teacher* of mathematics. MT1, MT2, MT3 and MT7 each express a different aspect of the concept of multiplication. Since teachers are charged to help students with *conceptual understanding* of multiplication, they need to know that multiplication can be understood in multiple ways, namely as grouping equal-size groups (MT1), as repeated addition (MT2), through its relationship with division (MT3), and as a way of relating dimensions of rectangular shape or object (MT7). Equal distributing (MT4) is a concrete way of solving partition division problems – thus, a *mathematical* idea – which is of specific importance to the *teaching* of division of elementary school students because they need to make meaning of mathematical concepts by linking them to concrete experiences. MT5, MT6 and MT8 each deal with aspects of the mathematical language and represent as such *mathematical* ideas, namely that the symbols used for number operations are arbitrary (MT5), that the order in which we come to arrange the expression of a multiplication statement is also arbitrary (MT6), and that a concatenation of binary relationship with a common link can be written in an abbreviated format (MT8). As with natural languages, students need to learn about these features of the mathematical language, which makes MT5, MT6, and MT8 knowledge relevant for *teaching*. Such understanding, for instance, will allow students to understand that the Pythagorean Theorem can be expressed in the form "$a^2 + b^2 = c^2$" but also in the form "$p^2 + c^2 = m^2$". MT9 represents as well a *mathematical* idea, even if it does not come up usually in mathematical thinking because the idea is linked to an unconventional way of writing multiplication and division statements, namely without the operation and the equal sign. This unconventional way of writing mathematical statements came about because of the teaching context that the teachers wanted to create for their students; which makes MT9 knowledge relevant to the teachers' mathematics *teaching* practice. MT10 expresses the *mathematical* idea that multiplication as a number operation and fractions as a type of number symbols are interlinked (fractions can be multiplied). This is not a specific curricular or pedagogical idea. What makes this mathematical idea relevant to *teaching* is that it is central to mathematics teaching for understanding: a teacher of mathematics needs to understand which aspect of what mathematical idea is a *prerequisite* for other mathematical ideas.

The study shows that the *richness* of the mathematics that the teachers drew upon in their planning sessions was extensive and intensive. Its intensity can be gleaned from a comparison with data from Davis and Simmt (2006). In one of their in-service sessions with 24 Kindergarten to high school teachers Davis and Simmt inquired into the teachers' knowing of multiplication by, first, prompting them with the question "What is multiplication?" and then follow-up with the question "And what else?". The teachers came up with ten different ways of characterizing multiplication. In the planning sessions of the teachers involved in this study four such characterizations of multiplication (MT1, MT2, MT7; and to some degree MT3) were drawn upon in two one-hour planning sessions – and were *unprompted*. This illustrates the intensity of the mathematics that teachers drew upon. The extensity of the richness of that mathematics-for-teaching can be gleaned from the different aspect of the concept of multiplication that the teachers brought into play. We could see not just features of the notion of multiplication, as just illustrated, but also distributive division (MT4), arbitrariness in the mathematical language

(MT5 and MT6), transitivity of equality (MT8), uniqueness features of number families (MT9), and hierarchical structure of mathematical concepts (MT10). The teachers drew from this range of mathematical knowledge within only two one-hour lesson planning sessions.

### The Distinct Quality of Mathematics-for-Teaching

As referenced above, Ball and her collaborators have characterized the mathematics that teachers need to know for teaching mathematics as being different from the mathematics to be known for other practices, and Davis and Simmt have talked about mathematics-for-teaching having a "distinct quality" than the mathematics that their students are expected to master. In the previous section some of the mathematics that teachers drew upon during lesson planning sessions were identified, and it was argued how these mathematical ideas are relevant to their role as teachers. In this section at least some of these mathematical ideas will be identified as being known by teachers in a way that is not just relevant to the professional practice but also *qualitatively different* from the mathematics students are to know mathematics and qualitatively different from what is commonly understood to be the content of mathematics, thus, making these mathematical ideas (mathematics-for-teaching) as distinct from mathematics used in other professional contexts. Important to keep in mind here is that the difference in quality of what is known is generally referring to *how* something is known. In will make the case for three of the ten examples of the teachers' mathematical knowledge identified above.

The first case of knowing mathematics in a qualitatively different way is expressed in MT6: knowing that the order in the tripartite relationship between the three numbers of a "multiplication number family" is arbitrary and can be changed. Part of knowing mathematics is knowing its conventions and part of 'doing mathematics' is using those convention. It is actually a characteristic of mathematics as an *academic* discipline to use conventions to the extreme in order to shorten the way in which ideas are expressed, as anyone who has ever opened an academic mathematics textbook can attest to. At the school level, the convention of order of operation is a prime example for the disciplinary approach to conventions. "Undoing" the convention of writing "2" and "3" together in 2×3=6 by allowing the three numbers to be written in *any* order – together with an "understanding" of how the numbers are to be combined – is an example of the "unpacked" knowing that Ball and Bass (2002) suggest is a characteristic for knowing mathematics for teaching purposes as distinct from knowing mathematics for other purposes. It is not part of knowing mathematics for other purposes than for teaching others to understand mathematical concepts to "undo" mathematical conventions and explore what mathematics could look like without such conventions. MT6 illustrates also an example of mathematical understanding that is distinct from the mathematics teachers want their students to know. While students work with "undone" conventions in the game the teachers have described, "undoing the convention" is *not* part of what students are to learn. Rather they engage with such "undoing" for *learning* purposes (and thus *teaching* purposes). First, they learn about the meaning of a mathematical convention by experiencing that and how it can be changed. Second, it is through the "undoing of the convention" that they engage more deeply with number relationships as they are defined through their multiplicative relationship; seeing the numbers 2, 12, 6 in *any* order should trigger in students their "multiplicative relationship": two and six multiply to 12. Again, the "undoing the convention" itself is *not* part of what students are to learn.

The mathematics expressed in MT5 is another example of mathematics known in a way that is specific to the practice of teaching and that is qualitatively different from the way in which the mathematics needs to be known in other contexts. In academic mathematics the arbitrariness is *used* but *not unpacked* as a feature of the mathematical language. A view in any academic mathematics book demonstrates actually that the arbitrariness is very much

*hidden*: more or less consistently "$x$", "$y$" and "$z$" are used as variables; the equation of a line in analytic geometry is most often described as "$y = mx + c$", and the Pythagorean Theorem is most often represented as "$a^2 + b^2 = c^2$". This, as well, illustrates the "compacting" in academic mathematics, Ball and Bass are talking about. In the teaching of mathematics, this "compacted use of the mathematical language" needs to be unpacked for the teaching of students. While a research mathematician, of course, knows about the conventional use of certain variables in certain contexts and would have no problems using other symbols, their actual *professional practice* hides these aspects of the mathematical language. On the other hand, as the data that gave rise to MT5 illustrate, a teacher of mathematics needs to make these conventions *explicit* to herself rather than just using them in her professional practice of teaching. The data also illustrate that the teacher was not thinking of unpacking the arbitrariness *as a teaching objective* for her students, but rather, she used here understanding of the arbitrariness to create a learning context that allows students to grapple with the concept of multiplication and division. This illustrates a qualitative difference between the ways in which a teacher of mathematics needs to know the conventions of the mathematical language and, for instance, the way in which a professional mathematician needs to know the conventions. The mathematician needs to know those conventions to *use* them to prove a theorem, etc. A teacher of mathematics, on the other hand, needs to know the contentions in a way that allows her to "unpack" them. She or he needs to know the convention in such a way that she recognizes when a student struggles with the Pythagorean Theorem because the student does not necessarily consider the arbitrariness of the symbols used in "$a^2 + b^2 = c^2$". She needs to know the convention in such a way that she can see advantages and disadvantages of existing conventions and advantages and disadvantages of alternative language conventions. As the data for MT5 demonstrate, teachers of mathematics would also need to know the conventions in such a way that they can break those conventions for the learning benefits of students.

MT9 also provides a case of knowing mathematics that is qualitatively different from the way mathematics is known in other than teaching contexts and the mathematics of which is different from the mathematics teachers want their students to know. For the purpose of designing the game in the way the teachers discussed it the *mathematical* question arises whether it is possible to uniquely identify the operation (multiplication or division) for a given whole-number triplet like <3,12,4>. While this is a mathematical question, its relevance for the teachers derives from the specifics of their *planning practice*, in other words, from their role as teachers. On the other hand, it seems very unlikely (although not impossible) that this question is of any relevance to the practice of professional mathematicians or engineers. If that is an accurate assessment, then *how* teachers know number relationships (for instance, knowing that <3, 12, 4> represents a division statement) differs from how other professional practitioners need to know number relationships involving multiplication and division. The teachers in the study in their planning practice needed to know that for a whole-number triplet <*x, y, z*> it is possible to uniquely determine whether the operation between *x* and *y* (whole numbers) resulting in *z* is multiplication or division (with the exception of <1, 1, 1>). While this knowledge was important for the teachers' planning of their activities, it is not part of what students are to understand about the operations of multiplication and division. It is the teacher, who needs to understand that uniqueness in order to make the activity work in the first place, thus, the mathematics-for-teaching here is distinct from the mathematics students are to learn.

This section has demonstrated how rich and diverse the mathematical knowledge is that the teachers in the study drew upon when they approached the planning of their mathematics lessons, and how the way in which those teachers know, and need to know, mathematics is qualitatively different from the way in which mathematical ideas are and need to be known in other professions that involve mathematical knowledge.

## Implications for Teacher Education

The findings of the study have suggested that *how* teachers need to know mathematics is to a good part *qualitatively different* from the way in which other professional practitioners need to know mathematics exactly because teaching practice is different from other professional practices, like the practices a research mathematician or an engineer engage in. The three examples discussed in the previous section illustrate this difference in the way of knowing mathematics. For instance, while the arbitrariness of the symbols used in mathematics is known regardless for what purpose the mathematics is used, that this arbitrariness can be used to fit number operations into a context of playing cards is very much specific to teaching mathematics.

Recently, Watson (2008) has argued that the practice of engaging with mathematical issues, problems, etc. in the context of school mathematics is quite distinct to the practice of engagement with mathematics in the context of academic mathematics, "because it has different warrants, authorities, forms of reasoning, core activities, purposes and unifying concepts, and necessarily truncates mathematical activity in ways that are different from those of the discipline" (p. 3). Watson argues here that the practice of engaging with mathematics is different for students than it is for mathematicians. The claim addressed in this paper has a similar orientation, though the focus is different. This paper has inquired into the claim that the way in which *teachers* need to know *the content* of the mathematics they engage with for their teaching is *qualitatively different* from the way in which other professions engage with mathematical content *and* that that way is also different from how they want their students to know mathematics. The mathematical understanding reported on was classified as *mathematics-for-teaching* exactly because it was mathematical understanding that teachers drew upon in an activity – planning teaching activities for their students – that has been part of their *teaching practice*. Thus, this paper contributes to a growing perspective that there are different forms of engagement with mathematics *and* that *what* one engages in can also be qualitatively different.

Such a shift in perspective has implications for the education of elementary school teachers (and others). One such implication concerns the way in which there is a wide-spread conceptual and practical "division of labour" in the education of elementary school teachers for the teaching mathematics in the Canadian context, which applies also to other contexts, like the USA and Germany. Most teacher education programs in Canada offer so-called "methods-courses" for teaching mathematics, a notion that suggests that the course is about the "how" of teaching mathematics, suggesting that the "what" has been already established and mastered – because, how can you deal with the "how to teach mathematics" if you are not clear what the mathematics is that you want to know how to teach? Using Shulman's framework of domains of teacher knowledge, the problem can be described as follows: While "methods courses" as part of a teacher education program seem to be conceptualized as being concerned with *pedagogical* subject matter knowledge (the "how to" component), the case study presented here suggests that it is also the mathematical *content* knowledge that needs to be taught *in conjunction with* the pedagogical content knowledge exactly because the form that that content takes is shaped by the teaching of mathematics in schools (see also Bednarz & Proulx, 2009; Proulx & Simmt, in press). Having mathematical content *for teachers* being taught by academic mathematicians in departments of mathematics is in principle (notwithstanding specific qualifications of particular academic mathematicians) in conflict with the promotion of a notion of mathematics-for-teaching. The problem can also be looked at from the other side. In a number of Canadian universities, courses on "mathematics for (elementary school) teachers" are offered to help them understand the subject that they are asked to teach in their classes. The study and the theoretical framework into which this study and its findings are embedded suggest, however, that such a "division of labour" is counterproductive, since the mathematical content (the what) is shaped by the practice of teaching that content (the how).

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