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Study of numerical algorithm used to solve the equation of motion for the planar flexural forced vibration of the cantilever beam

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Abstract

This paper develops the numerical algorithm used to solve the equation of motion for the planar flexural forced vibration of the cantilever beam. The partial differential equation is first discretized in the spatial coordinate using Galerkin's weighted residual method. Then, the equation is discretized in the time domain using the Newmark technique and a numerical algorithm is used to calculate the nonlinear response of the beam.

Introduction

Finite Element Model

The equation of motion for the nonlinear planar flexural forced vibration of a cantilever beam was derived in the previous chapter. The equation of motion for the transverse displacement in the *y* direction is given by

$$\rho A\ddot{v} + c_v\dot{v} + EIv^{iv} = \rho Ag[v''(s-l) + v'] - EI\{v'(v'v'')'\}' - \frac{1}{2}\rho A\{v'\int_l^s \frac{\partial^2}{\partial t^2}[\int_0^s v'^2\,ds]ds\}' + \rho Aa_b\cos\Omega t \qquad (1.1)$$

This equation can be written in the form

$$\rho A\ddot{v} + c_v\dot{v} + EIv^{iv} - \rho Agf_3 + EI\{v'f_1\}' + \frac{1}{2}\rho A\{v'f_2\}' - F = 0 \qquad (1.2)$$

With the functions f_1, f_2, f_3 , and F given by

$$f_1 = (v'v'')', \; f_2 = \frac{\partial^2}{\partial t^2}\int_l^s\int_0^s v'^2\,dsds, \; f_3 = (s-l)v'' + v', \quad F = \rho Aa_b\cos\Omega t \qquad (1.3)$$

The functions f_1, f_2, and f_3 originate from the curvature, inertial, and gravitational nonlinear effects, respectively. The function F is the force associated with the transverse displacement exciting the base of the beam. Equation (1.2) is a nonlinear integro-differential equation, for which a closed form solution is not available. Therefore, an approximate solution is sought by discretizing (1.2), first in the spatial coordinate using Galerkin's weighted residuals method, and then in the time domain using the Newmark technique. The discretization in the spatial coordinate is carried out in three steps: (1) mesh generation and function approximation, (2) element equation, and (3) assembly and implementation of boundary conditions. These steps are discussed in detail in the remaining of this section. The discretization in the time domain is the focus of section

Mesh Generation and Function Approximation

Figure 3.1 shows the cantilever beam divided into *N* cubic Hermite elements, each of length *h*.

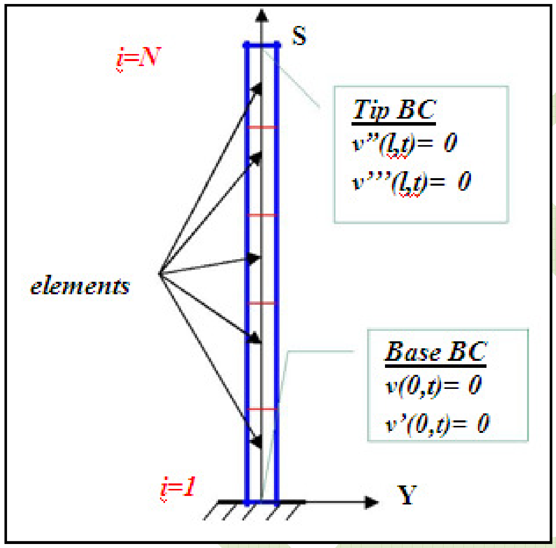

Figure 1: Cantilever beam divided into N elements

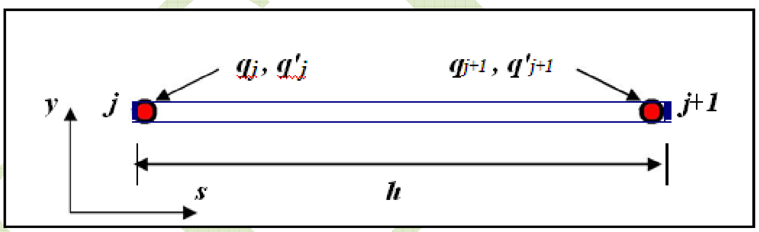

Figure 2: Typical cubic Hermite beam element

The typical cubic Hermite element (Figure 2) has two nodes with two degrees of freedom per node (Zienkiewicz, 1977), namely translation (q_j) and slope (q'_j). The displacement of any point inside the element is approximated as

$$\tilde{v}^e(s,t) = \sum_{j=1}^{4} \psi_j(s) q_j^e(t) \qquad (1.4)$$

The shape functions $\psi_j(s)$ are given by (Reddy, 1993).

$$\psi_1 = 1 - 3\left(\frac{s}{h}\right)^2 + 2\left(\frac{s}{h}\right)^3, \quad \psi_2 = h\left[\left(\frac{s}{h}\right) - 2\left(\frac{s}{h}\right)^2 + \left(\frac{s}{h}\right)^3\right],$$

$$\psi_3 = 3\left(\frac{s}{h}\right)^2 - 2\left(\frac{s}{h}\right)^3, \quad \psi_4 = h\left[-\left(\frac{s}{h}\right)^2 + \left(\frac{s}{h}\right)^3\right] \qquad (1.5)$$

The vector q^e_j in (1.4) is the element nodal displacement vector. For the remaining of this derivation, the superscript e is dropped for the sake of simplicity. The numerical solution of the partial differential equation in (1.2) is a piecewise cubic polynomial comprised of the sum of the approximated displacement

$$\tilde{v}^e(s,t)\text{, i.e., } v \cong \sum_e \tilde{v}^e .$$

Element Equation

In order to obtain the element equation, the approximated displacement in (1.4) is substituted into the partial differential equation (1.2). When this is done, the left hand side is no longer equal to zero, but to a quantity R_x called the residual.

$$\rho A\sum_{j=1}^{4}\psi_j\ddot{q}_j + c_v\sum_{j=1}^{4}\psi_j\dot{q}_j + EI\sum_{j=1}^{4}\psi_j^{iv}q_j - F - \rho Agf_3 + EI[(\sum_{j=1}^{4}\psi'_j q_j)f_1]' + \frac{1}{2}\rho A[(\sum_{j=1}^{4}\psi'_j q_j)f_2]' = R_x \tag{1.6}$$

The weighted residual WR is defined using Galerkin's method. In Galerkin's method, the shape function ψ_i is used as the weighting function. The weighted residual is forced to be zero over the element. Therefore WR is given by

$$WR = \int_0^h \psi_i R_x\, ds = 0 \tag{1.7}$$

Multiplying both sides of (1.6) by the shape function ψ_i and integrating over the length of the element results in

$$[\rho A\sum_{j=1}^{4}\int_0^h \psi_i\psi_j ds]\ddot{q}_j + [c_v\sum_{j=1}^{4}\int_0^h \psi_i\psi_j ds]\dot{q}_j + [EI\sum_{j=1}^{4}\int_0^h \psi_i\psi_j^{iv} ds]q_j - \int_0^h \psi_i F ds - \rho Ag\int_0^h \psi_i f_3 ds + EI\int_0^h \psi_i[(\sum_{j=1}^{4}\psi'_j q_j)f_1]' ds + \frac{1}{2}\rho A\int_0^h \psi_i[(\sum_{j=1}^{4}\psi'_j q_j)f_2]' ds = 0 \tag{1.8}$$

Finally, several terms in equation (1.8) are integrated by parts to obtain the weak form of the element equation

$$[\rho A\sum_j\int_0^h \psi_i\psi_j ds]\ddot{q}_j + [c_v\sum_j\int_0^h \psi_i\psi_j ds]\dot{q}_j + [EI\sum_j\int_0^h \psi_i''\psi_j'' ds]q_j - \int_0^h \psi_i F ds - \rho Ag\int_0^h \psi_i f_3 ds - [EI\sum_j\int_0^h \psi_i'\psi_j' f_1 ds]q_j - [\frac{1}{2}\rho A\sum_j\int_0^h \psi_i'\psi_j' f_2 ds]q_j + [\psi_i\sum_i\psi_j' q_j EIf_1]_0^h + [\psi_i\sum_i\psi_j' q_j f_2]_0^h + [\psi_i EI\sum_i\psi_j''' q_j - \psi_i' EI\sum_i\psi_j'' q_j]_0^h = 0 \tag{1.9}$$

This can be simplified to

$$M^e_{ij}\ddot{q}_j + c^e_{ij}\dot{q}_j + K^e_{ij}q_j - kc^e_{ij}q_j - ki^e_{ij}q_j - F^e_i - g^e_i + b^e_i = 0 \tag{1.10}$$

with the matrices, M^e_{ij} , c^e_{ij}, and K^e_{ij} given by

$$M_{ij}^{e} = \rho A \int^{h} \psi_i \psi_j ds, \quad c_{ij}^{e} = c_v \int^{h} \psi_i \psi_j ds, \quad K_{ij}^{e} = EI \int^{h} \psi''_i \psi''_j ds \qquad (1.11)$$

These matrices are the element mass, damping and stiffness matrices, respectively. Equation (1.10) is written in indicial notation. Therefore, repeated indices denote summation. At this point it is convenient to introduce the matrix naming convention used in the remaining of the chapter. Matrices in capital letters are linear matrices, while matrices in lower case letters are nonlinear matrices. For instance, in (1.11) the matrices M_{ij}^{e} and, K_{ij}^{e} are linear matrices, while the matrix c_{ij}^{e} is a nonlinear matrix For a cubic Hermite beam element the matrices M_{ij}^{e} and, K_{ij}^{e} are given by (Reddy, 1993)

$$M_{ij}^{e} = \frac{\rho A h}{420}\begin{bmatrix} 156 & 22h & 54 & -13h \\ 22h & 4h^2 & 13h & -3h^2 \\ 54 & 13h & 156 & -22h \\ -13h & -3h^2 & -22h & 4h^2 \end{bmatrix} \qquad (1.12)$$

$$K_{ij}^{e} = \frac{EI}{h^3}\begin{bmatrix} 12 & 6h & -12 & 6h \\ 6h & 4h^2 & -6h & 2h^2 \\ -12 & -6h & 12h & -6h \\ 6h & 2h^2 & -6h & 4h^2 \end{bmatrix} \qquad (1.13)$$

where ρ, A, E, I, and h are the density, cross sectional area, Young's modulus, area moment of inertia and length of the element, respectively. Equation (1.10) has two additional stiffness matrices kc_{ij}^{e} , and ki_{ij}^{e} , resulting from the nonlinear effects in (1.2). These matrices are given by

$$kc_{ij}^{e} = EI \int_0^h \psi'_i \psi'_j f_1 \, ds, \qquad ki_{ij}^{e} = \frac{1}{2} \rho A \int_0^h \psi'_i \psi'_j f_2 \, ds \qquad (1.14)$$

The matrix kc_{ij}^{e} represents the curvature nonlinearity, while ki_{ij}^{e} represents the inertia nonlinearity.

The vectors F_i^{e} and g_i^{e} are the force and gravitational effect vectors, respectively and are defined as

$$F_i^{e} = \int^{h} \psi_i F ds, \qquad g_i^{e} = \rho A g \int^{h} \psi_i f_3 ds \qquad (1.15)$$

The vector b_i^e is the combination of the boundary terms in (1.9).

$$b_i^e = [\psi_i v'(v'\mathcal{V} + v''\mathcal{M})]_0^h + [\psi_i v' \mathcal{F}_a]_0^h + [\psi_i \mathcal{V} - \psi_i{}' \mathcal{M}]_0^h \tag{1.16}$$

The quantities $\mathcal{V}$ and $\mathcal{M}$ in (3.16) are the transverse shear force and bending moment of the beam. For a beam, the bending moment and shear force are given by (Rao, 1990)

$$\mathcal{M} = EIv'' , \quad \mathcal{V} = (EIv'')' \tag{1.17}$$

The force $\mathcal{F}_a$ in (1.16) can be interpreted as part of the axial force required to maintain the inextensionality constraint. The origin of $\mathcal{F}_a$ is understood upon examination of the order two expressions for the Lagrange multiplier. The Lagrange multiplier is interpreted as the axial force required maintaining the inextensionality constraint (Malatkar, 2003). Recall from Chapter 2, the order two expression for the Lagrange multiplier is

$$\lambda = -D_\zeta v'''v' - D_\eta w'''w' - \frac{1}{2} m \int_l^s \frac{\partial^2}{\partial t^2}\left[\int_0^s [(v')^2 + (w')^2] ds\right] ds - \int_l^s Q_u ds \tag{1.18}$$

For planar motion of the cantilever beam (1.18) becomes

$$\lambda = -EIv'''v' - \frac{1}{2}\rho A \int_l^s \frac{\partial^2}{\partial t^2}\left[\int_0^s v'^2 ds\right] ds - \rho A g(s-l) \tag{1.19}$$

This can be written as

$$\lambda = -\mathcal{F}_e - \mathcal{F}_a - \mathcal{W}_b \tag{1.20}$$

with $\mathcal{F}_e$, $\mathcal{F}_a$, and $\mathcal{W}_b$ given by

$$\mathcal{F}_e = EIv'''v' , \quad \mathcal{F}_a = \frac{1}{2}\rho A \int_l^s \frac{\partial^2}{\partial t^2}\left[\int_0^s v'^2 ds\right] ds , \quad \mathcal{W}_b = \rho A g(s-l) \tag{1.21}$$

From (1.20) it is clear the Lagrange multiplier λ is the combination of three forces, namely the elastic force ($\mathcal{F}_e$), the inertial force ($\mathcal{F}_a$) and the weight of the beam above point s along the neutral axis ($\mathcal{W}_b$). Hence, $\mathcal{F}_a$ is indeed part of the axial force required to maintain the inextensionality constraint.

Assembly and Implementation of Boundary Conditions

Assembly of the N element equations yields the global finite element equation

$$M_{ij}\ddot{q}_j + c_{ij}\dot{q}_j + k_{ij}q_j = f_i + b_i \tag{1.22}$$

Which is a system of $2(N+1)$ ordinary differential equations, i.e., one for each nodal degree of freedom. The solution of this system is the vector q_j, which contains the nodal displacements and nodal rotations in the global coordinates S and Y (Figure 3.1). The global linear mass matrix M_{ij} is calculated using (1.12). The nonlinear damping matrix is calculated using proportional damping (Cook, 1995). Therefore, is approximated as a linear combination of the mass and nonlinear stiffness matrices.

$$c_{ij} = \alpha_1 M_{ij} + \alpha_2 k_{ij} \quad (1.23)$$

The nonlinear stiffness matrix k_{ij} is the combination of the linear stiffness matrix, calculated using (1.13), and the two nonlinear stiffness matrices kc_{ij} and ki_{ij} , calculated with (3.14).

$$k_{ij} = K_{ij} - kc_{ij} - ki_{ij} \quad (1.24)$$

The nonlinear force vector f_i is the combination of the linear force vector and the gravitational effect vector, both calculated with (3.15).

$$f_i = F_i + g_i \quad (1.25)$$

The boundary vector b_i is defined using the element boundary vector given by (1.16). The internal reactions $\mathcal{V}$, $\mathcal{M}$, and $\mathcal{F}_a$ in (1.16) cancel out upon assembly for all nodes except for the first and last nodes. Therefore, the global boundary vector has non zero elements only at the fixed and free ends of the beam. The boundary conditions of the problem are used to evaluate b_i. From the previous chapter the boundary conditions are

$$\begin{aligned} v(0,t) &= 0, \qquad v'(0,t) = 0 \\ v''(l,t) &= 0, \qquad v'''(l,t) = 0 \end{aligned} \quad (1.26)$$

The elements of the boundary vector for the fixed end are given by

$$\begin{aligned} b_1 &= \psi_1 v'(v'\mathcal{V} + v''\mathcal{M}) + \psi_1 v'\mathcal{F}_a + \psi_1 \mathcal{V} - \psi_1{}'\mathcal{M} \\ b_2 &= \psi_2 v'(v'\mathcal{V} + v''\mathcal{M}) + \psi_2 v'\mathcal{F}_a + \psi_2 \mathcal{V} - \psi_2{}'\mathcal{M} \end{aligned} \quad (1.27)$$

For the fixed end, v' is zero according to the boundary conditions in (1.26). Therefore, the first two terms of b_1 and b_2 vanish and (1.27) becomes

$$b_1 = \psi_1 \mathcal{V} - \psi_1{}'\mathcal{M}, \quad b_2 = \psi_2 \mathcal{V} - \psi_2{}'\mathcal{M} \quad (1.28)$$

The elements of the boundary vector corresponding to the free end are

$$\begin{aligned} b_{2(N+1)-1} &= \psi_3 v'(v'\mathcal{V} + v''\mathcal{M}) - \psi_3{}'\mathcal{M} + \psi_3 \mathcal{V} + \psi_3 v'\mathcal{F}_a \\ b_{2(N+1)} &= \psi_4 v'(v'\mathcal{V} + v''\mathcal{M}) - \psi_4{}'\mathcal{M} + \psi_4 \mathcal{V} + \psi_4 v'\mathcal{F}_a \end{aligned} \quad (1.29)$$

For the free end, both v'' and v''' are zero from the boundary conditions. As a result, both the sheer force and bending moment are zero according to (1.17), causing the first three terms in (1.29) to vanish. Also, the inertial force $\mathcal{F}_a$ is zero at the free end, according to (3.21). Therefore, both $b_{2(N+1)-1}$ and $b_{2(N+1)}$ are zero.

$$b_{2(N+1)-1} = b_{2(N+1)} = 0 \quad (1.30)$$

Since the displacement and rotation at the fixed end are both known from the boundary conditions, the first two equations in (1.22) do not need to be included as part of the system of equations to be solved. These equations are saved for post processing of the solution. Substituting the boundary vector into (1.22), and saving the first two equations of the system for post processing yields

$$M^r_{ij}\ddot{q}_j + c^r_{ij}\dot{q}_j + k^r_{ij}q_j = f^r_i \quad (1.31)$$

The superscript r in (1.31) stands for reduced, since the first two equations are eliminated.

Newmark Technique

In this section, the linear global finite element equation of motion is used to illustrate the Newmark technique. The linear equation of motion is given by

$$M_{ij}\ddot{Q}_j + C_{ij}\dot{Q}_j + K_{ij}Q_j = F_i \quad (1.33)$$

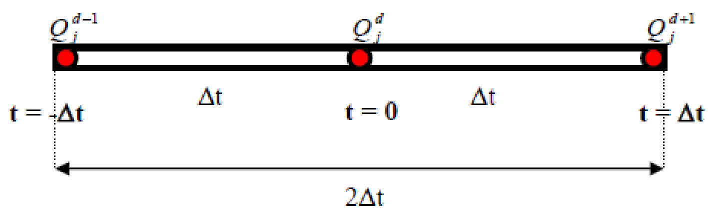

Figure 3: Interval for discretization in the time domain

And is obtained by omitting the nonlinear matrices kc_{ij} and ki_{ij} , as well as the gravitational effect vector g_i in (1.31). The matrices M_{ij} , C_{ij} , K_{ij} and the vector F_i are reduced matrices since the first two rows and columns are eliminated. However, the superscript r is dropped for simplicity. Equation (1.32) is discretized within the time interval [- t, t], where t is the time step (Figure 1.3) and t is an arbitrary time. This interval is divided in two segments of length t each. Dividing the interval in this manner creates three discrete time points (Figure 3). For each one of these *time nodes* [5] there is a displacement vector associated to it. The displacement vector for time t is Q_j^{d+1} , while the displacement vectors for times *0* and - t are Q_j^{d} and Q_j^{d-1} , respectively.

The displacement vector at any time inside the interval in Figure 3 is approximated by [6]

$$Q_j = \Phi_{d-1}Q_j^{d-1} + \Phi_d Q_j^{d} + \Phi_{d+1}Q_j^{d+1} = \Phi_k Q_j^{k} \quad (1.33)$$

where Φ_{d-1}, Φ_d, and Φ_{d+1} are the shape functions given by (Zienkiewicz, 1977)

$$\Phi_{d-1} = \frac{-\nu}{2}(1-\nu)\,,\ \ \Phi_d = (1-\nu)(1+\nu)\,,\ \ \Phi_{d+1} = \frac{\nu}{2}(1+\nu) \quad (1.34)$$

The dimensionless time coordinate ν in (1.34) is defined as

$$\nu = \frac{t}{\Delta t} \quad (1.35)$$

The displacement vector Q_j in (1.33) is a quadratic polynomial in t. The force inside the interval [- t, t] is interpolated in a way similar to the displacement vector (Zienkiewicz, 1977). Therefore the force F_i is given by

$$F_i = \Phi_{d-1}F_i^{d-1} + \Phi_d F_i^{d} + \Phi_{d+1}F_i^{d+1} = \Phi_k F_i^{k} \quad (1.36)$$

Substitution of (1.33) and (1.36) into (1.32) yields the residual R_t.

$$M_{ij}\ddot{\Phi}_k Q_j^k + C_{ij}\dot{\Phi}_k Q_j^k + K_{ij}\Phi_k Q_j^k - \Phi_k F_i^k = R_t \quad (1.37)$$

with *k* ranging from *d-1* to *d+1*. The weighted residual method is applied by multiplying (1.37) by a weighting function *ω(t)* and integrating from - *t* to *t*. Equation (1.37) becomes then

$$\int_{-\Delta t}^{\Delta t} \omega(t)[M_{ij}\ddot{\Phi}_k Q_j^k + C_{ij}\dot{\Phi}_k Q_j^k + K_{ij}\Phi_k Q_j^k - \Phi_k F_i^k]dt = 0 \qquad (1.38)$$

Substituting the shape functions (1.34) into (1.38) results in

$$\begin{aligned}&\{M_{ij} + \gamma\Delta t C_{ij} + \beta\Delta t^2 K_{ij}\}Q_j^{d+1} + \{-2M_{ij} + (1-2\gamma)\Delta t C_{ij} \\ &+ (0.5 - 2\beta + \gamma)\Delta t^2 K_{ij}\}Q_j^d + \{M_{ij} - (1-\gamma)\Delta t C_{ij} + (0.5 + \beta - \gamma)\Delta t^2 K_{ij}\}Q_j^{d-1} \\ &- \Delta t^2\{\beta F_i^{d+1} + (0.5 - 2\beta + \gamma)F_i^d + (0.5 + \beta - \gamma)F_i^{d-1}\} = 0\end{aligned} \qquad (1.39)$$

where the quantities γ and β are given by

$$\gamma = \frac{\int_{-1}^{1} \omega(v)(v + \frac{1}{2})dv}{\int_{-1}^{1} \omega(v)dv}, \quad \beta = \frac{\frac{1}{2}\int_{-1}^{1} \omega(v)(1 + v)v dv}{\int_{-1}^{1} \omega(v)dv} \qquad (1.40)$$

Notice the variable of integration in (1.40) has been changed from *t* to the dimensionless time coordinate v (1.35).Equation (1.39) can be simplified to

$$A1_{ij}Q_j^{d+1} + A2_{ij}Q_j^d + A3_{ij}Q_j^{d-1} - \mathcal{F}_i = 0 \qquad (1.41)$$

where the matrices $A1_{ij}$, $A2_{ij}$, $A3_{ij}$ and the vector $\mathcal{F}_i$ are defined as follows

$$\begin{aligned}A1_{ij} &= M_{ij} + \gamma\Delta t C_{ij} + \beta\Delta t^2 K_{ij} \\ A2_{ij} &= -2M_{ij} + (1-2\gamma)\Delta t C_{ij} + (0.5 - 2\beta + \gamma)\Delta t^2 K_{ij}\end{aligned} \qquad (1.42)$$

$$\begin{aligned}A3_{ij} &= M_{ij} - (1-\gamma)\Delta t C_{ij} + (0.5 + \beta - \gamma)\Delta t^2 K_{ij} \\ \mathcal{F}_i &= \Delta t^2\{\beta F_i^{d+1} + (0.5 - 2\beta + \gamma)F_i^d + (0.5 + \beta - \gamma)F_i^{d-1}\}\end{aligned} \qquad (1.43)$$

Equation (1.41) is used to solve for the displacement vector Q_j^{d+1} in terms of the displacement vectors Q_j^d and Q_j^{d-1}.

$$Q_j^{d+1} = A1_{ij}^{-1} A2_{il} Q_l^d - A1_{ij}^{-1} A3_{il} Q_l^{d-1} + \Delta t^2 A1_{ij}^{-1}\mathcal{F}_i \qquad (1.44)$$

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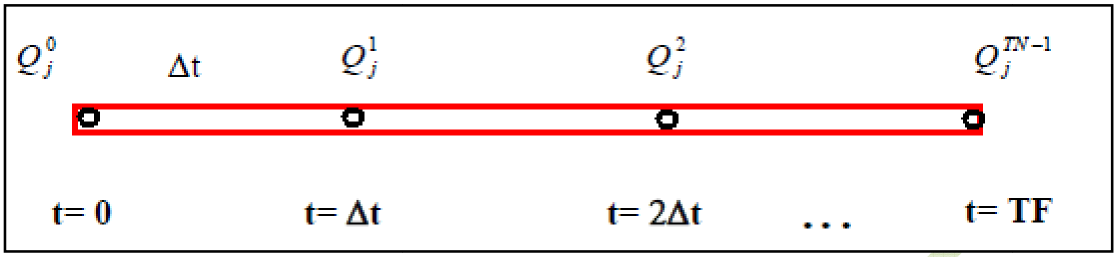

Figure 4: Time nodes in [0,TF]

The use of (1.44) to calculate the time history of the displacement vector Q_j in the interval [0, TF] is illustrated next. Here *TF* is an arbitrary time. The time interval [0,TF] has *TN* time nodes with *TN* given by (Figure 3.4)

$$TN = \frac{TF}{t} + 1 \qquad (1.45)$$

The first time node corresponds to *d=1*, the second to *d=2,* and so on. The displacement vector for the second time node in [0, TF] is simply $Q^1{}_j$. In order to calculate $Q^1{}_j$, the vectors $Q^0{}_j$ and $Q^-{}_j{}^1$ must be prescribed. This is done by using the initial conditions of the problem. For this problem it is assumed the beam starts from rest, which means the displacement vectors $Q^0{}_j$ and $Q^-{}_j{}^1$ are equal to the zero vector.

$$Q^0{}_j = Q^-{}_j{}^1 = 0 \qquad (1.46)$$

Substituting *d=0* along with (1.46) into (1.44) yields

$$Q_j^1 = \Delta t^2 A1_{ij}^{-1} \mathcal{F}_i \qquad (1.47)$$

To calculate $Q^2{}_j$, the displacement vector for the next time node. In this manner (1.44) is used to calculate the displacement vectors for all time nodes in [0, TF]. The quantities γ and β in (1.40) vary depending on the choice for weighting function ω (v). For this problem, the values $\gamma = 0.5$ and $\beta = 0.25$ are used. This corresponds to an average acceleration scheme (Zienkiewicz,). These values of γ and β ensure the computation of the time history of the displacement vector using (1.44) is unconditionally stable, i.e., independent of the size of *t* (Bathe).

Numerical Algorithm

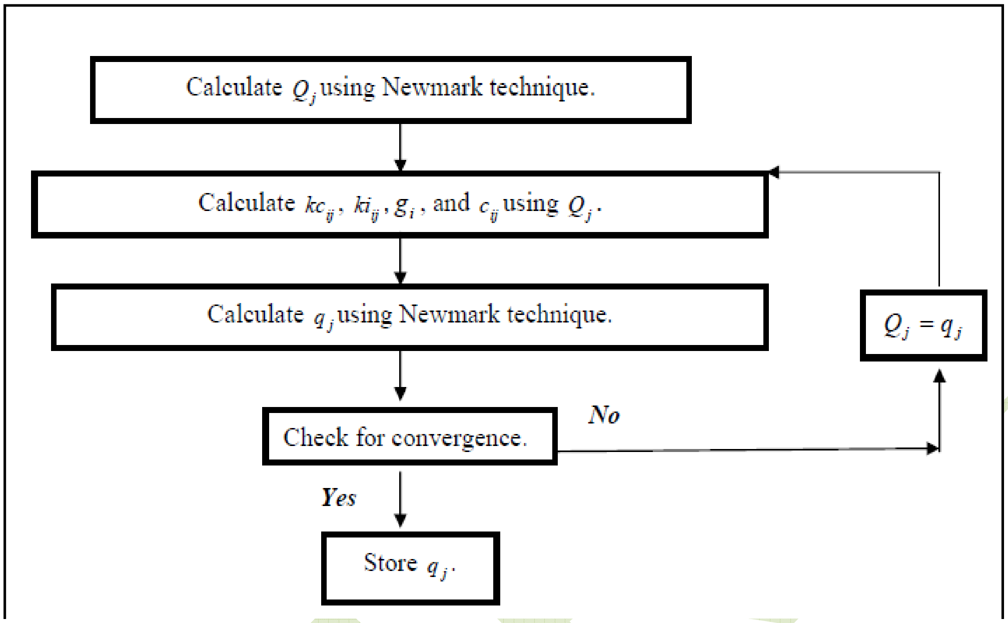

Figure 5: Algorithm used to calculate the time history of the displacement

Figure 5 illustrates the process used to calculate the time history of the nonlinear displacement vector q_j. The linear displacement vector Q_j is calculated first. This linear displacement vector is then used to calculate a first guess of q_j. Finally, the iterative process is used to obtain the nonlinear displacement vector q_j for time *t*. This algorithm is implemented in the Matlab® program *NLB* [7] in Appendix A

.Calculation of the Linear Displacement Q_j

The Newmark technique is used to calculate the linear displacement vector Q_j in the interval [0,TF], with $0 < t < TF$. This interval is divided in *TN* time nodes with *TN* defined by (1.45).

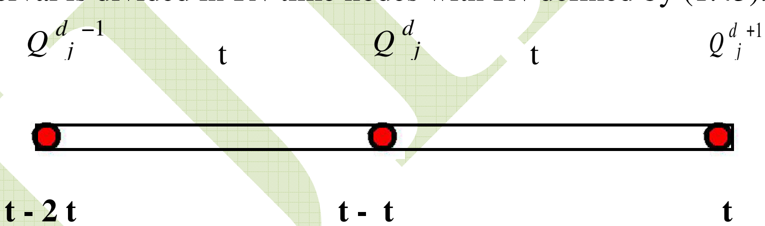

Figure 6: Displacement vectors used to calculate Q_j at time t

The linear displacement vector at time *t* is given by

$$
\begin{aligned}
A1_{ij} &= M_{ij} + \gamma\, t C_{ij} + \beta\, t^2 K_{ij} \\
A2_{ij} &= -2M_{ij} + (1-2\gamma)\, tC_{ij} + (0.5 - 2\beta + \gamma)\, t^2 K_{ij} \qquad (1.49) \\
A3_{ij} &= M_{ij} - (1-\gamma)\, tC_{ij} + (0.5 + \beta - \gamma)\, t^2 K_{ij}
\end{aligned}
$$

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The coefficients γ and β are taken as 0.5 and 0.25, respectively. This corresponds to the average acceleration scheme (Zienkiewicz). The matrix Cij is the linear damping matrix and is calculated as a linear combination of the mass and stiffness matrices (Cook). Thus C_{ij} is given by

$$C_{ij} = \alpha_1 M_{ij} + \alpha_2 K_{ij} \tag{1.50}$$

The coefficients α_1 and α_2 are obtained by solving the system

$$\xi_1 = \frac{\alpha_1}{2\omega_1} + \frac{\alpha_2 \omega_1}{2} \quad , \quad \xi_4 = \frac{\alpha_1}{2\omega_4} + \frac{\alpha_2 \omega_4}{2} \tag{1.51}$$

The focus of this investigation is the time response of the cantilever beam when the base is excited at a frequency close to the third natural frequency. Therefore, the first and fourth natural frequencies and modal damping ratios are used in (1.51). The force vector in (1.48) is calculated as

$$\mathcal{F}_i = \Delta t^2 \{\beta F_i^{d+1} + (0.5 - 2\beta + \gamma) F_i^d + (0.5 + \beta - \gamma) F_i^{d-1}\} \tag{1.52}$$

where F_i^{d+1}, F_i^d , and F_i^{d-1} are the linear force vectors for times t, $t - \Delta t$, and $t - 2\Delta t$, Respectively (Figure 6)

Calculation of the Nonlinear Displacement q_j

The nonlinear displacement vector at an arbitrary time t in [0, TF] is given by

$$q_j^{d+1} = a1_{ij}^{-1} a2_{il} q_l^d - a1_{ijil}^{-1} a3 q_l^{d-1} + t^2 a1_{ij}^{-1} \mathcal{F}_i \tag{1.53}$$

where q_j^d, q_j^{d-1} are the nonlinear displacement vectors for times $t -\ t$, and $t - 2\ t$, Respectively (Figure 8).

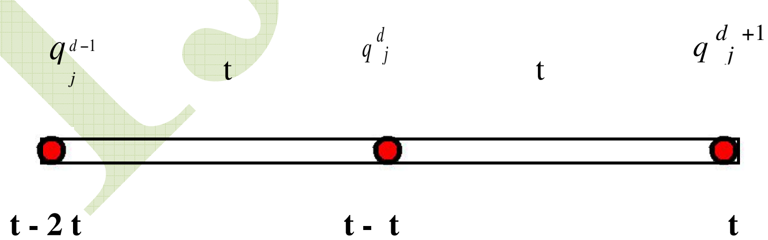

Figure 7: Displacement vectors used to calculate q_j at time t

The matrices $a1_{ij}$, $a2_{ij}$ and $a3_{ij}$ in (1.53) are calculated using the global linear mass matrix and the nonlinear stiffness matrix k_{ij} given by (1.24).

$$\begin{aligned} a1_{ij} &= M_{ij} + \gamma\, tc_{ij} + \beta\, t^2 k_{ij} \\ a2_{ij} &= -2M_{ij} + (1-2\gamma)\, tc_{ij} + (0.5-2\beta+\gamma)\, t^2 k_{ij} \\ a3_{ij} &= M_{ij} - (1-\gamma)\, tc_{ij} + (0.5+\beta-\gamma)\, t^2 k_{ij} \end{aligned} \tag{1.54}$$

The coefficients γ and β are taken as 0.5 and 0.25, respectively. This corresponds to the average acceleration scheme (Zienkiewicz) In order to obtain the nonlinear stiffness matrix k_{ij} , the functions f_1 and f_2 defined and must be calculated. These functions are used with (1.14) to compute the nonlinear stiffness matrices kc_{ij} and ki_{ij} , which are substituted into (1.24) to obtain k_{ij} . A detailed example of the procedure used to calculate ki_{ij} is included in Appendix D. The matrix kc_{ij} is calculated using a similar procedure. The matrix c_{ij} is the nonlinear damping matrix and is calculated as a linear combination of the mass and stiffness matrices (Cook)

$$c_{ij} = \alpha_1 M_{ij} + \alpha_2 k_{ij} \tag{1.55}$$

The coefficients α_1 and α_2 are the same above. The force vector in (1.53) is calculated as

$$\ell_i = t^2 \{\beta f_i^{d+1} + (0.5-2\beta+\gamma) f_i^d + (0.5+\beta-\gamma) f_i^{d-1}\} \tag{1.56}$$

where f_i^{d+1}, f_i^d , and f_i^{d-1} are the nonlinear force vectors for times t, t - Δt, and t - $2\Delta t$, Respectively

The nonlinear force vector f_i is simply the combination of the linear force vector and the gravitational effect vector (1.25). The gravitational effect vector is calculated using a procedure similar to the one illustrated

Iterative Procedure

The iterative procedure used to obtain the nonlinear displacement vector at any given time t is illustrated in Figure 5. Once vectors Q_j and q_j are obtained as discussed, the error θ is calculated

$$\theta = \sum_{m=1}^{k} q^m_j - Q^m_j$$

$$\theta \le TOL \ \ \text{for convergence} \tag{1.57}$$

Where k is the total number of elements in each vector.

The error θ is compared to a maximum allowed error TOL. Once $\theta \le TOL$, the solution is converged and the vector q_j is stored. However, if the error θ exceeds the maximum allowed error, q_j is assigned to Q_j and a new q_j is calculated (Figure 5). This procedure is repeated until convergence is achieved.

Conclusion

The vibration of a highly flexible cantilever beam is investigated in this paper the order three equations of motion, developed by Crespo da Silva and Glyn (1978), for the nonlinear flexural-flexural-torsional vibration of Inextensional beams can be used to investigate the time response of the beam subjected to harmonic excitation at the base. The equation for the planar flexural vibration of the beam can also be solved using the finite element method.

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