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A review of nonlinear flexural-torsional vibration of a cantilever beam

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Abstract

A beam is an elongated member, usually slender, intended to resist lateral loads by bending (Cook, 1999). Structures such as antennas, helicopter rotor blades, aircraft wings, towers and high rise buildings are examples of beams. These beam-like structures are typically subjected to dynamic loads. Therefore, the vibration of beams is of particular interest to the engineer. The paper reviews the derivation by Crespo da Silva and Glyn (1978) for the nonlinear flexural-flexural-torsional vibration of a cantilever beam. Also the numerical algorithm used to solve the equation of motion for the planar vibration of the beam subjected to harmonic excitation at the base.

Introduction

For beams undergoing small displacements, linear beam theory can be used to calculate the natural frequencies, mode shapes, and the response for a given excitation. However, when the displacements are large, linear beam theory fails to accurately describe the dynamic characteristics of the system. Highly flexible beams, typically found in aerospace applications, may experience large displacements. These large displacements cause geometric and other nonlinearities to be significant. The nonlinearities couple the (linearly uncoupled) modes of vibration and can lead to modal interactions where energy is transferred between modes (Nayfeh, 1993). This investigation focuses in the study of the time response of a highly flexible cantilever beam, subjected to harmonic excitation at the base.

Literature Review

Crespo da Silva and Glyn (1978) derived a set of integro-differential equations describing the nonlinear flexural-flexural-torsional vibration of Inextensional beams. Their mathematical model includes nonlinear effects up to order three, such as the curvature and inertia effects. Crespo da Silva and Glyn used their model to investigate the non-planar oscillations of a cantilever beam (1979), and the out of plane vibration of a clamped-clamped/sliding beam subjected to planar excitation with support asymmetry (1979). Anderson, et al. (1992) conducted experiments on a flexible cantilever beam subjected to harmonic excitation along the axis of the beam. Their investigation demonstrated that energy from a high-frequency excitation can be transferred to a low-frequency mode of a structure through two mechanisms: a combination resonance and a resonance due to modulations of the amplitudes and phases of the high-frequency modes. Anderson, et al. (1992) also
Investigated the response of a flexible cantilever beam subjected to random base excitation. Their results demonstrate that energy from a high frequency excitation can be transferred to a low-frequency mode of the beam. Nayfeh and Nayfeh (1993) investigated the interaction between high and low frequency modes in a two degree of freedom nonlinear system. Nayfeh and Arafat (1998) studied the nonlinear flexural responses of cantilever beams to combination parametric and sub combination resonances. Malatkar and Nayfeh (2003) performed an experimental and theoretical study of the response of a flexible cantilever beam to an external harmonic excitation near the third natural frequency of the beam. Their investigation reveals the response of the beam consists of amplitude and phase modulated high frequency component, and a low frequency

component. Moreover, the modulation frequency of the high frequency component is equal to the low frequency component (Malatkar, 2003). Kim, et al. (2006) investigated the non-planar response of a circular cantilever beam subjected to base harmonic excitation. Their results show that the inertia nonlinear effect dominates the response of high frequency modes.

Equations of Motion

The approach employed by Crespo da Silva and Glynn (1978) is used to derive the equations of motion for the flexural-flexural-torsional vibrations of a cantilever beam. The equations of motion are derived using the extended Hamilton's principle. These equations are then simplified to include nonlinear terms up to order three. The chapter concludes with the derivation of the order three equations for the forced planar flexural vibration of the beam. This equation will be used in subsequent chapters.

Dynamic System and Assumptions

The dynamic system (Figure 1) consists of a vertically mounted cantilever beam of length *l*, mass per unit length *m*, made of isotropic material. The beam is assumed to be a nonlinear elastic structure. A nonlinear elastic structure undergoes large deformations, but small strains (Malatkar, 2003). The beam is also assumed to be an Euler-Bernoulli beam. Hence the effects of rotary inertia, shear deformation and warping are neglected.

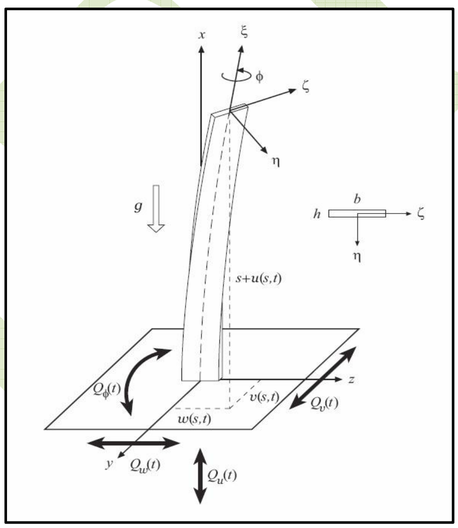

Figure 1 vertically mounted cantilever beam

Furthermore, the beam is idealized as an Inextensional beam, i.e., there is no stretching of the neutral axis. Beams with one end fixed and the other end free can be assumed to be Inextensional (Nayfeh and Pai, 2003). Two coordinate systems are used to derive the equations of motion. The *x,y,* and *z* axes define the inertial coordinate system with orthogonal unit vectors $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, and $\boldsymbol{e}_z$. The ξ,η and ζ axes define the local coordinate system with orthogonal unit vectors $\boldsymbol{e}_\xi$, $\boldsymbol{e}_\eta$ and $\boldsymbol{e}_\zeta$. The origin of this system is the centroid of the cross section at arc length *s*. The ζ and η axes are aligned with the principal axes of the cross section, while the ξ axis is aligned to the neutral axis of the beam. When the *x* and ξ axes are aligned with each other, the beam is in the un deformed position. The beam is subjected to generalized forces Qu, Qv, Qw, and Qφ , which cause it to deform as shown in Figure 2.1. Since there is no warping of the cross section, any change of shape of the beam is due to rigid body motion. At any time, the deformed position of the beam consists of translation and rotation, which can be described in terms of the axial displacement u(s,t); the flexural displacements v(s,t) and w(s,t); and the torsional angle φ(s,t).

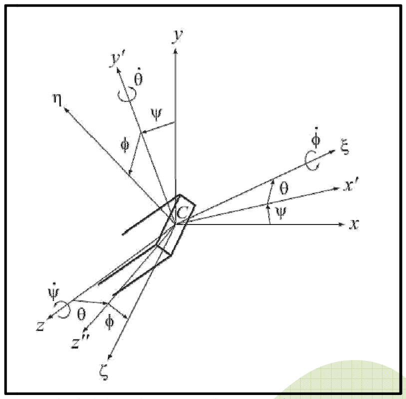

Figure 2 Rigid body rotations of beam cross section

Euler Angles

In the unreformed configuration, the inertial coordinate system *xyz* is aligned with the $\xi\eta\zeta$ system. When the two coordinate systems are no longer aligned with each other, some deformation has taken place. The transformation that the beam undergoes to get to the deformed state can be described in terms of three counterclockwise rigid body rotations. The Euler angles ψ, θ, and φ are used to describe these rotations Figure 2 shows the order in which the rotations are performed. First the *xyz* system is rotated about the *z* axis, then about *y′*, the new position of the *y* axis, and finally about the ξ axis. This order of rotation yields a set of equations of motion amenable to a study of moderately large amplitude flexural-torsional oscillations by perturbation techniques (Crespo da Silva, 1978). The unit vectors of the $\xi\eta\zeta$ coordinate system are related to unit vectors of the *xyz* coordinate system through a transformation matrix [T]. The transformation matrix [T] is the product of three transformation matrices, one for each rigid body rotation.

$$\begin{bmatrix} \boldsymbol{e}_\xi \\ \boldsymbol{e}_\eta \\ \boldsymbol{e}_\zeta \end{bmatrix} = [T] \begin{bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \\ \boldsymbol{e}_z \end{bmatrix} = [T_\phi][T_\theta][T_\psi] \begin{bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \\ \boldsymbol{e}_z \end{bmatrix} \tag{1.1}$$

The three individual transformation matrices and the transformation matrix [T] are then

$$[T_\psi] = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; [T_\theta] = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}, \; [T_\phi] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \tag{1.2}$$

$$[T] = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ -\sin\psi\cos\phi + \sin\phi\sin\theta\cos\psi & \cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi & \sin\phi\cos\theta \\ \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi & -\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi & \cos\phi\cos\theta \end{bmatrix} \tag{1.3}$$

The absolute angular velocity of the local coordinate system $\xi\eta\zeta$ is obtained from Figure 2

The over dot indicates differentiation with respect to time.

$$\omega(s,t) = \dot{\psi}\boldsymbol{e}_z + \dot{\theta}\boldsymbol{e}_{y'} + \dot{\phi}\boldsymbol{e}_\xi \qquad (1.4)$$

The expressions for $\boldsymbol{e}_z$, and $\boldsymbol{e}_{y'}$ are easily obtained from the transformation matrices in (1.2) and (1.3).

$$\boldsymbol{e}_z = -\sin\theta\boldsymbol{e}_\xi + \sin\phi\cos\theta\boldsymbol{e}_\eta + \cos\phi\cos\theta\boldsymbol{e}_\zeta$$

$$\boldsymbol{e}_{y'} = \cos\phi\boldsymbol{e}_\eta - \sin\phi\boldsymbol{e}_\zeta \qquad (1.5) \,\&\, (1.6)$$

Substituting (1.5) and (1.6) into (1.4) yields

$$\omega(s,t) = (\dot{\phi} - \dot{\psi}\sin\theta)\boldsymbol{e}_\xi + (\dot{\psi}\sin\phi\cos\theta + \dot{\theta}\cos\phi)\boldsymbol{e}_\eta + (\dot{\psi}\cos\phi\cos\theta - \dot{\theta}\sin\phi)\boldsymbol{e}_\zeta \qquad (1.7)$$

According to Kirchhoff's kinetic analogue (Love, 1944), the curvature components can be obtained from the angular velocity components by replacing the time derivatives in (1.7) with spatial derivatives. Hence the curvature vector is given by

$$\rho(s,t) = (\varphi' - \psi'\sin\theta)\boldsymbol{e}_\xi + (\psi'\sin\varphi\cos\theta + \theta'\cos\varphi)\boldsymbol{e}_\eta + (\psi'\cos\varphi\cos\theta - \theta'\sin\varphi)\boldsymbol{e}_\zeta \qquad (1.8)$$

Inextensional Beam

Figure 2 shows a segment of the neutral axis of the cantilever beam. Segment *CD* is in the undeformed configuration while C^*D^* is in the deformed configuration. The strain at point *C* is given by

$$e = \frac{ds^* - ds}{ds} = \sqrt{(1+u')^2 + (v')^2 + (w')^2} - 1 \qquad (1.9)$$

Since the beam is assumed to be Inextensional, the strain along the neutral axis is zero. Therefore (1.9) becomes.

$$1 = (1+u')^2 + (v')^2 + (w')^2 \qquad (1.10)$$

Equation (1.10) is referred to as the *inextensionality constraint*.

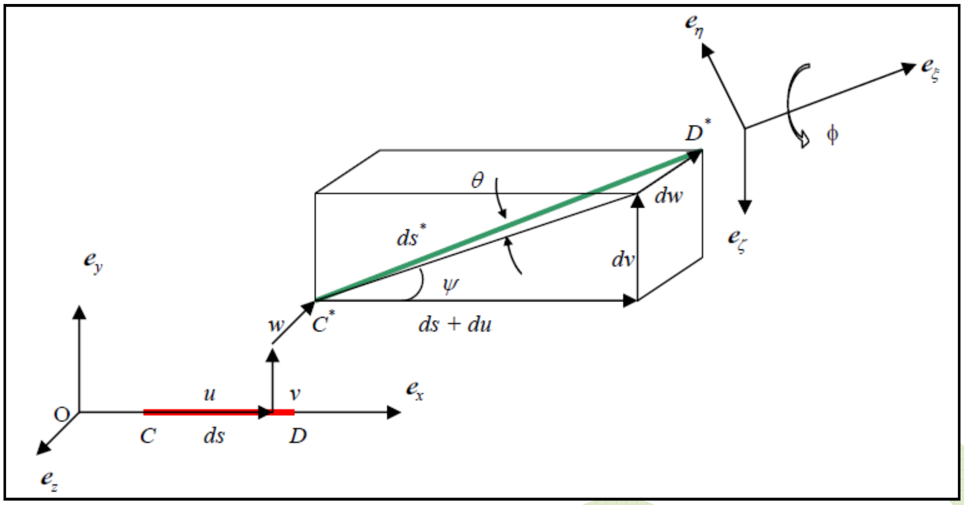

Figure 3: Deformation of a segment of the neutral axis

Figure 3 can be used to determine the expressions for the angles ψ and θ in terms of the spatial derivatives of the transverse displacements. These expressions will be helpful in the simplification of the equations of motion later on. The relationships for ψ and θ are then

$$\tan\psi = \frac{v'}{1+u'} \ ,\tan\theta = \frac{-w'}{(1+u')^2+(v')^2} \qquad (1.11)$$

Equation (1.11) indicates ψ and θ are dependent on the spatial derivatives of the displacement components *u, v,* and *w*. Therefore there are only four independent variables for this problem, namely *u, v, w,* and φ.

Strain-Curvature Relations

This section presents the derivation of the strain tensor components in terms of the curvature components. Figure 4 shows the beam cross section at arc length *s* for both the deformed and undeformed configurations.

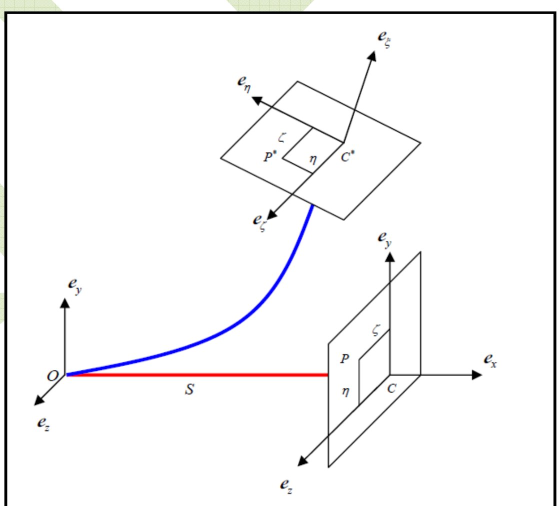

Figure 4: Initial and deformed positions of an arbitrary point P

An arbitrary point *P* in the undeformed beam cross section moves to point P^* in the deformed cross section.

The position vectors for points P and P^* are defined from Figure 4 as

$$\vec{r}_P = \overrightarrow{OC} + \overrightarrow{CP} = se_x + \eta e_y + \zeta e_z$$

$$\vec{r}_{P^*} = \overrightarrow{OC^*} + \overrightarrow{C^*P^*} = (s+u)e_x + ve_y + we_z + \eta e_\eta + \zeta e_\zeta \qquad (1.12) \,\&\, (1.13)$$

The distance differentials for points P and P^* are given by

$$d\vec{r}_P = dse_x + d\eta e_y + d\zeta e_z$$

$$d\vec{r}_{P^*} = dse_\xi + \eta de_\eta + d\eta e_\eta + \varsigma de_\zeta + d\zeta e_\zeta \qquad (1.14) \,\&\, (1.15)$$

With the first term in (1.15) given by

$$dse_\xi = (1+u')dse_x + v'dse_y + w'dse_z = C^*D^* \qquad 1.16)$$

This is obtained directly from Figure 3. Equations (1.14) and (1.15) are used to obtain

$$d\vec{r}_{P*} \cdot d\vec{r}_{P*} - d\vec{r}_P \cdot d\vec{r}_P = 2(\zeta\rho_\eta - \eta\rho_\zeta)ds^2 - 2\zeta\rho_\xi ds d\eta + 2\eta\rho_\xi ds d\zeta + H.O.T. \qquad 1.17)$$

Where H.O.T. stands for higher order terms.
The difference of the squared distance differentials is related to the Green's strain tensor by (Mase, 1970)

$$d\vec{r}_{P*} \cdot d\vec{r}_{P*} - d\vec{r}_P \cdot d\vec{r}_P = 2[ds \quad d\eta \quad d\zeta] \cdot [\varepsilon_{ij}] \cdot [ds \quad d\eta \quad d\zeta]^T \qquad (1.18)$$

The components of the strain tensor in terms of the curvature are found by expanding the right hand side of (1.18) and comparing it to the right hand side of (1.17).

$$\varepsilon_{11} = \zeta\rho_\eta - \eta\rho_\zeta \,,\; \gamma_{12} = 2\varepsilon_{12} = -\zeta\rho_\xi \,,\; \gamma_{13} = 2\varepsilon_{13} = \eta\rho_\xi \,,\; \varepsilon_{22} = \varepsilon_{23} = \varepsilon_{33} = 0 \qquad (1.19)$$

Lagrangian of Motion

The Lagrangian of motion is defined as

$$\mathcal{L} = T - V = \int_0^l \ell \, ds \qquad (1.20)$$

where T is the kinetic energy, V the potential energy, l the length of the beam and ℓ the Lagrangian density.

The kinetic energy has two components, one due to translation and one due to rotation. The kinetic energy due to translation is given by

$$T_{tr} = \frac{1}{2} m \int_0^l (\dot{u}^2 + \dot{v}^2 + \dot{w}^2) ds \qquad (1.21)$$

The kinetic energy due to rotation is given by

$$T_{rot} = \frac{1}{2}\int_0^l [\omega_\xi \quad \omega_\eta \quad \omega_\zeta][J][\omega_\xi \quad \omega_\eta \quad \omega_\zeta]^T ds \tag{1.22}$$

where [J] is the distributed inertia matrix. Because the local coordinate system coincides with the principal axes of the beam, the product moments of inertia are zero (Budynas, 1999). Therefore, the inertia matrix [J] is given by

$$[J] = \begin{bmatrix} J_\xi & 0 & 0 \\ 0 & J_\eta & 0 \\ 0 & 0 & J_\zeta \end{bmatrix} \tag{1.23}$$

With the elements along the diagonal defined by

$$J_\xi = \iint_A \rho(\eta^2 + \zeta^2)d\eta d\zeta\,,\quad J_\eta = \iint_A \rho\zeta^2 d\eta d\zeta\,,\quad J_\zeta = \iint_A \rho\eta^2 d\eta d\zeta \tag{1.24}$$

Substituting (1.24) into (1.22) and adding (1.21) to the resulting expression yields the expression for the total kinetic energy of the system.

$$T = \frac{1}{2}\int_0^l [m(\dot{u}^2 + \dot{v}^2 + \dot{w}^2) + J_\xi\omega_\xi^2 + J_\eta\omega_\eta^2 + J_\zeta\omega_\zeta^2] \quad ds \tag{1.25}$$

The potential energy is equal to the strain energy of the beam, which is calculated using the strain tensor components in (1.19). Therefore, the total strain energy is given by

$$U = \frac{1}{2}\int_0^l \{\iint_A (\sigma_{11}\varepsilon_{11} + \sigma_{12}\gamma_{12} + \sigma_{13}\gamma_{13})d\eta d\zeta\}ds \tag{1.26}$$

For this derivation, a linear relationship between the stress and the strain is assumed. Therefore, Hooke's law can be used to relate the stress to the strain.

$$\sigma_{11} \approx E\varepsilon_{11}\,,\ \sigma_{12} \approx G\gamma_{12}\,,\ \sigma_{13} \approx G\gamma_{13} \tag{1.27}$$

Substituting the strain tensor components from the previous section into (1.26) and (1.27), and noting that the cross section is symmetric about the η and ζ axes, the strain energy is written as

$$U = \frac{1}{2}\int_0^l (D_\xi \rho_\xi^{\,2} + D_\eta \rho_\eta^{\,2} + D_\zeta \rho_\zeta^{\,2})\, ds \tag{1.28}$$

where D_ξ, D_η, and D_ζ are the torsional and flexural rigidities, respectively. The potential energy is then

$$V = \frac{1}{2}\int_0^l (D_\xi \rho_\xi^{\,2} + D_\eta \rho_\eta^{\,2} + D_\zeta \rho_\zeta^{\,2})\, ds \tag{1.29}$$

Equations (1.25) and (1.29) are substituted into (1.20) to obtain the final expression for the Lagrangian. The inextensionality constraint in (1.10) must be maintained during the variational process. To enforce this constraint, equation (1.10) is attached to the Lagrangian density by using a Lagrange multiplier $\lambda(s,t)$. The Lagrangian density is then

$$\ell = \frac{1}{2}m(\dot{u}^2 + \dot{v}^2 + \dot{w}^2) + \frac{1}{2}(J_\xi \omega_\xi^{\,2} + J_\eta \omega_\eta^{\,2} + J_\zeta \omega_\zeta^{\,2})$$
$$- \frac{1}{2}(D_\xi \rho_\xi^{\,2} + D_\eta \rho_\eta^{\,2} + D_\zeta \rho_\zeta^{\,2}) + \frac{1}{2}\lambda[1 - (1+u')^2 - (v')^2 - (w')^2] \tag{1.30}$$

The Lagrangian resulting from substituting (1.30) into (1.20) is an example of an augmented functional. Variation problems dealing with finding the external of an augmented functional are known as *isoperimetric problems* (Török, 2000).

Extended Hamilton's Principle

The cantilever beam is subjected to non conservative forces such as viscous damping and the generalized forces Q_α ($\alpha = u, v, w,$ and φ). Therefore, the extended Hamilton's principle (Meirovitch, 2001) is used to derive the equations of motion. The extended Hamilton's principle can be stated as follows

$$\delta I = \int_{t_1}^{t_2} (\delta \mathcal{L} + \delta W_{NC}) = 0 \tag{1.31}$$

where $\delta\mathcal{L}$ is the virtual change in mechanical energy, and δW_{NC} the virtual work due to non conservative forces. The virtual work due to non conservative forces is given by

$$\delta W_{NC} = \int_0^l (Q^*_{\ u}\delta u + Q^*_{\ v}\delta v + Q^*_{\ w}\delta w + Q^*_{\ \phi}\delta\phi)ds \tag{1.32}$$

where $Q^*_{\ \alpha}$ ($\alpha = u, v, w,$ and φ) stands for the generalized non conservative forces defined as

$$Q^{*}_{\alpha} = Q_{\alpha} - c_{\alpha}\alpha \tag{1.33}$$

The variation of the Lagrangian is given by

$$\delta\mathcal{L} = \int_0^l \delta\ell ds \tag{1.34}$$

The Lagrangian density is a functional of *13* variables, namely *u,v,w,*φ, their time and space derivatives, and the Lagrange multiplier λ. Therefore, the variation of the Lagrangian density is given by

$$\delta\ell = \sum_{i=1}^{13} \frac{\partial \ell}{\partial x_i}\delta x_i \tag{1.35}$$

Equation (1.11) is used to obtain the variations of ψ and θ

$$\delta\psi = \frac{\partial\psi}{\partial u'}\delta u' + \frac{\partial\psi}{\partial v'}\delta v' = \frac{-v'\delta u' + (1+u')\delta v'}{(1+u')^2 + (v')^2} \tag{1.36}$$

$$\delta\theta = \frac{\partial\theta}{\partial u'}\delta u' + \frac{\partial\theta}{\partial v'}\delta v' + \frac{\partial\theta}{\partial w'}\delta w' = \frac{w'[(1+u')\delta u' + v'\delta v']}{\sqrt{(1+u')^2 + (v')^2}} - \sqrt{(1+u')^2 + (v')^2}\,\delta w' \tag{1.37}$$

The preceding two equations are used to compute the variation of the Lagrangian density given by (1.35). Taking the variation of the Lagrangian density and integrating by parts several times results in

$$\begin{aligned}\int_{t_1}^{t_2} \{ & \int_0^l (-m\ddot{u} + Q^{*}_{u} + G'_{u})\delta u ds + \int_0^l (-m\ddot{v} + Q^{*}_{v} + G'_{v})\delta v ds + \\ & \int_0^l (-m\ddot{w} + Q^{*}_{w} + G'_{w})\delta w ds + \int_0^l (Q^{*}_{\phi} - A_{\phi})\delta\phi ds + \\ & \left[-G_u\delta u - G_v\delta v - G_w\delta w + H_u\delta u' + H_v\delta v' + H_w\delta w' + \frac{\partial\ell}{\partial\phi'}\delta\phi \right]_{s=0}^{l} \}dt = 0\end{aligned} \tag{1.38}$$

with Gu, Gv, and Gw given by

$$G_u = A_{\psi}\frac{\partial\psi}{\partial u'} + A_{\theta}\frac{\partial\theta}{\partial u'} + \lambda(1+u'),\quad G_v = A_{\psi}\frac{\partial\psi}{\partial v'} + A_{\theta}\frac{\partial\theta}{\partial v'} + \lambda v',\quad G_w = A_{\theta}\frac{\partial\theta}{\partial w'} + \lambda w' \tag{1.39}$$

And Aψ, Aθ, Aφ, Hu, Hv, and Hw given by

$$A_\alpha = \frac{\partial^2 \ell}{\partial t \partial \dot{\alpha}} + \frac{\partial^2 \ell}{\partial s \partial \alpha'} - \frac{\partial \ell}{\partial \alpha} \qquad (\alpha = \psi, \theta, \phi)$$
$$H_\alpha = \frac{\partial \ell}{\partial \psi'}\frac{\partial \psi}{\partial \alpha'} + \frac{\partial \ell}{\partial \theta'}\frac{\partial \theta}{\partial \alpha'} \qquad (\alpha = u, v, w) \tag{1.40}$$

Equation (2.38) is valid for any arbitrary δu, δv, δw and $\delta \varphi$, implying that the individual integrands be equal to zero (Török, 2000). Thus the equations and boundary conditions for the flexural-flexural-torsional vibrations of a cantilever beam are given by

$$m\ddot{u} - Q^*_u = G'_u, \quad m\ddot{v} - Q^*_v = G'_v, \quad m\ddot{w} - Q^*_w = G'_w, \quad Q^*_\phi = A_\phi$$
$$\left[-G_u \delta u - G_v \delta v - G_w \delta w + H_u \delta u' + H_v \delta v' + H_w \delta w' + \frac{\partial \ell}{\partial \phi'}\delta \phi \right]_{s=0}^{l} = 0 \tag{1.41}$$

Order Three Equations of Motion

The equations of motion derived in the previous section are simplified to include nonlinear effects up to order three. This is accomplished by expanding each term in the equations into a Taylor series and discarding terms of order greater than three. The simplification is necessary to enable the use of the equations to study the motion of the beam via numerical techniques. Previous mathematical models for the non linear vibration of cantilever beams included nonlinearities up to order two (Crespo da Silva, 1978). This model is more complete since it incorporates nonlinearities up to order three. The simplification process begins by obtaining the order three Taylor series expansions of u', ψ, and θ. These are derived using the Taylor series expansion of *arctan(x)*

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \ldots \tag{1.42}$$

which is combined with (1.10), and (1.11) to get

$$u' = [1 - (v')^2 - (w')^2]^{1/2} - 1 = -\frac{1}{2}[(v')^2 + (w')^2] + \ldots \tag{1.43}$$

$$\psi = \tan^{-1}\frac{v'}{1+u'} = \tan^{-1}\{v'[1-(v')^2-(w')^2]^{-1/2}\} = v'[1 + \frac{1}{6}(v')^2 + \frac{1}{2}(w')^2] + \ldots \tag{1.44}$$

$$\theta = \tan^{-1}\frac{-w'}{[(1+u')^2 + (v')^2]^{1/2}} = \tan^{-1}\{-w'[1-(w')^2]^{-1/2}\} = -w'[1 + \frac{1}{6}(w')^2] + \ldots \tag{1.45}$$

The order three expansions for the angle of twist is obtained from the twisting curvature ρ_ξ. The third order expansion for the twisting curvature is given by

$$\rho_\xi = \varphi' + v''w' \tag{1.46}$$

Equation (1.46) is integrated over the length of the beam to obtain the order three expressions for the angle of twist.

$$\gamma \equiv \varphi + \int_0^s v''w'ds \tag{1.47}$$

Expanding the remaining terms in (2.41) and retaining terms up to order three yields

$$m\ddot{u} + c_u\dot{u} - Q_u = \{D_\xi\gamma'(w''v' - v''w') - (D_\eta - D_\zeta)[w'(v''\gamma)' + v'(w''\gamma)'] + D_\zeta v'''v' + D_\eta w'''w' + \lambda(1+u')\}' \tag{1.48}$$

$$m\ddot{v} + c_v\dot{v} - Q_v = \{-D_\xi\gamma'w''' + (D_\eta - D_\zeta)[(w''\gamma)' - (v''\gamma^2)' + w'''\int_0^s v'w''ds - D_\zeta[v''' + v'((v'')^2 + (w'')^2)] + \lambda v'\}' \tag{1.49}$$

$$m\ddot{w} + c_w\dot{w} - Q_w = \{D_\xi\gamma'v'' + (D_\eta - D_\zeta)[(v''\gamma)' + (w'\gamma^2)' - v'''\int_0^s w'v''ds] - D_\eta[w''' + w'((v'')^2 + (w'')^2)] + \lambda w'\}' \tag{1.50}$$

$$D_\xi\gamma'' = (D_\eta - D_\zeta)[\gamma((v'')^2 - (w'')^2) - v''w''] \tag{1.51}$$

Which are the order three equations of motion for the cantilever beam.

The boundary conditions for the fixed end are given by

$$\alpha(0,t) = 0 \qquad (\alpha = u, v, w, \gamma, v', w') \tag{1.52}$$

The natural boundary conditions obtained from (2.41-e) are used for the free end of the beam. Thus, the boundary conditions for the free end are given by

$$\alpha(l,t) = 0 \quad \left(\alpha = H_v - H_u \frac{v'}{1+u'}, H_w - H_u \frac{w'}{1+u'}, \gamma'\right) \tag{1.53}$$

$$G_\alpha(l,t) = 0 \quad (\alpha = u, v, w) \tag{1.54}$$

Equations (1.49) and (1.50) are further simplified by removing λ and γ using the Taylor series expansions of u, λ and γ. The expansion of u is obtained directly from (1.43) as

$$u = -\frac{1}{2}\int_0^s [(v')^2 + (w')^2]ds \tag{1.55}$$

Equations (1.55), (1.47), (1.48), (1.49), and (1.50) suggest u, λ and γ are of order two.

Equation (1.55) is substituted into (1.48) and only terms up to order two are kept. Integrating the resulting expression from l to s, produces the expression for the Lagrange multiplier λ in terms of the transverse displacements. Incidentally, the Lagrange multiplier is interpreted as an axial force necessary to maintain the inextensionality constraint (Malatkar, 2003).

$$\lambda = -D_\zeta v''' v' - D_\eta w''' w' - \frac{1}{2} m \int_l^s \frac{\partial^2}{\partial t^2}\left[\int_0^s [(v')^2 + (w')^2]ds\right]ds - \int_l^s Q_u ds \tag{1.56}$$

For the angle of twist, (1.51) is integrated twice and only terms up to order two are retained. The angle of twist is then

$$\gamma = -\frac{D_\eta - D_\zeta}{D_\xi}\int_0^s \int_l^s v'' w'' ds ds \tag{1.57}$$

Equation (1.57) indicates flexure induced torsion is a nonlinear phenomenon (Malatkar, 2003)

Substituting (1.56) and (1.57) into (1.49) and (1.50) yields the order three equations of motion for the flexural-flexural-torsional vibration of a cantilever beam.

$$m\ddot{v} + c_v \dot{v} + D_\zeta v^{iv} = Q_v + \{(D_\eta - D_\zeta)\left[w'' \int_l^s v'' w'' ds - w''' \int_0^s v'' w' ds\right] - \frac{(D_\eta - D_\zeta)^2}{D_\xi}(w'' \int_0^s \int_l^s v'' w'' ds ds)'\}' - D_\zeta \{v'(v'v'' + w'w'')'\}' \tag{1.58}$$

$$-\frac{1}{2}m\{v'\int_l^s \frac{\partial^2}{\partial t^2}[\int_0^s (v'^2+w'^2)ds]ds\}'-(v'\int_l^s Q_u ds)'$$

$$m\ddot{w}+c_w\dot{w}+D_\eta w^{iv} = Q_w - \{(D_\eta - D_\zeta)\left[v''\int_l^s v''w''ds - v'''\int_0^s w''v'ds\right] \quad (1.59)$$

$$+\frac{(D_\eta - D_\zeta)^2}{D_\xi}(v''\int_0^s\int_l^s v''w''dsds)'\}'-D_\eta\{w'(v'v''+w'w'')'\}'$$

$$-\frac{1}{2}m\{w'\int_l^s \frac{\partial^2}{\partial t^2}[\int_0^s (v'^2+w'^2)ds]ds\}'-(w'\int_l^s Q_u ds)'$$

The boundary conditions for (1.58) and (1.59) are given by

$$\begin{array}{llll} v(0,t) = 0, & w(0,t) = 0, & v'(0,t) = 0, & w'(0,t) = 0 \\ v''(l,t) = 0, & w''(l,t) = 0, & v'''(l,t) = 0, & w'''(l,t) = 0 \end{array} \quad (1.60)$$

The boundary conditions for the free end are derived from (1.53).

The equation of motion and boundary conditions for the forced planar flexural vibration of the beam is obtained from equations (1.58) and (1.59). For planar motion, equation (1.59) is dropped along with the w terms in (1.58). With these substitutions equation (1.58) becomes

$$m\ddot{v}+c_v\dot{v}+D_\zeta v^{iv} = Q_v - D_\zeta\{v'(v'v'')'\}'-\frac{1}{2}m\{v'\int_l^s \frac{\partial^2}{\partial t^2}[\int_0^s v'^2\, ds]ds\}'-(v'\int_l^s Q_u ds)'$$

For base excitation of the vertical beam, the transverse displacement v is given by

$$v = v_b + v_0 \cos(\Omega t) \quad (1.62)$$

where v_b is the transverse displacement of the beam relative to the base, v_0 the amplitude of the excitation at the base, and Ω the excitation frequency. Moreover, the generalized force Q_v is zero and the generalized force Q_u is the weight per unit length.

$$Q_u = mg, \quad Q_v = 0 \quad (1.63)$$

Substituting (1.62) and (1.63) into (1.61) yields the equation for the planar flexural forced vibration of the beam

$$\rho A\ddot{v}+c_v\dot{v}+EIv^{iv} = \rho Ag[v''(s-l)+v'] - EI\{v'(v'v'')'\}'$$
$$-\frac{1}{2}\rho A\{v'\int_l^s \frac{\partial^2}{\partial t^2}[\int_0^s v'^2\, ds]ds\}'+\rho Aa_b\cos\Omega t \quad (1.64)$$

where ρ, A, c_v, E, I, and a_b are the coefficient, Young's modulus, moment respectively. The first term on the right hand side of (1.64) arises from the effect of gravity on the beam. The second and third terms represent the curvature and inertia nonlinearity, respectively. Density, cross sectional area, viscous damping of inertia and acceleration applied at the base respectively. The first term on the right hand side of (1.64) arises from the effect of gravity on the beam. The second and third terms represent the curvature and inertia nonlinearity, respectively.

Conclusion

The equation of motion for the transverse nonlinear vibration of the beam, derived by Crespo da Silva and Glyn (1978), was solved numerically for the nonlinear displacement using a finite element model. This finite element model used Galerkin's weighted residuals method combined with the Newmark technique, and an iterative process, to obtain the nonlinear displacement of the beam

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