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BUCKLING OF A COLUMN WITH TEMPERATURE DEPENDENT MATERIAL PROPERTIES

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ABSTRACT

Buckling of a column with temperature dependent material properties is investigated. Euler-Bernoulli theory of thin beams is used to derive the element matrices by means of the minimum potential energy principle. Temperature dependency of material properties is taken into account in the formulation. The column is divided into finite elements with the axial degrees of freedom defined at the outer fiber of the column. Column elements have simpler derivations and compact element matrices than those of classical beam-bending element. Some illustrative examples are presented to show the convergence of numerical results obtained by the use of new elements. The results are compared with those of the classical beam-bending element and analytical solution. The new element converges to the analytical results as powerful as the classical beam-bending element. The temperature effects on the buckling loads of the column with temperature dependent material properties are also examined.

Key Words : Buckling, Column, Finite element method, Stability

MALZEME ÖZELLİKLERİ SICAKLIĞA BAĞLI BİR KOLONUN BURKULMASI

ÖZET

Malzeme özellikleri sıcaklığa bağlı bir kolonun burkulması araştırılmaktadır. İnce kirişler için Euler-Bernoulli teorisi, minimum potansiyel enerji prensibi vasıtasıyla eleman matrislerini çıkarmak için kullanılmaktadır. Formülasyonda malzeme özelliklerinin sıcaklığa bağımlılığı hesaba katılmaktadır. Kolon, kolonun en dış lifinde tarif edilen eksenel yönde serbestlik derecesine sahip sonlu elemanlara bölünmektedir. Kolon elemanları, klasik kiriş eğilme elemanından daha basit olarak çıkarılmaktadır ve daha küçük eleman matrislerine sahiptir. Yeni elemanlar kullanarak elde edilen sayısal sonuçların yakınsamasını göstermek için bazı örnekler sunulmaktadır. Sonuçlar hem klasik sonlu eleman hem de kesin sonuçlarla karşılaştırılmaktadır. Yeni eleman, analitik sonuçlara klasik kiriş eğilme elemanı kadar güçlü bir şekilde yakınsamaktadır. Yine malzeme özellikleri sıcaklığa bağlı olan bir kolonun burkulma yüklerine sıcaklık etkileri araştırılmaktadır.

Anahtar Kelimeler : Burkulma, Kolon, Sonlu eleman yöntemi, Kararlılık

1. INTRODUCTION

A column is one of the basic structural elements. Euler gave analytical solutions for the column buckling first, and since then many researchers have focused their attention on the concepts of buckling and stability of the columns. Finite element method is one of the most common methods in the numerical buckling analysis of structures (Weaver and Johnston, 1984). Coulter and Miller (1986) analyzed the free vibrations and buckling of elastic Euler-Bernoulli beams subjected to non-uniform axial forces by the use of various types of beam finite elements. Ali and Sridharan (1988) developed a new formulation to study the interactive buckling of thin-walled columns having arbitrary cross-sections. Sakiyama (1986) studied the elastic buckling of tapered columns numerically. Recently, Goda et all.

(1992) developed a finite element code to investigate the dynamic lateral buckling of thin walled T-shaped beam subjected to impulsive load. Vaziri and Xie (1992) proposed a new numerical model for analyzing the buckling of columns with variably distributed axial loads. More recently, Helwig and Yura (1999) investigated torsional buckling of column. Wu (1998), Smithpardo and Aristizabalochoa (1999) studied postbuckling behavior of column.

In the present paper new finite elements for the column buckling are introduced with the axial degrees of freedom (DOF). Euler-Bernoulli theory of thin beams is used to derive the element matrices by means of the minimum potential energy principle. Temperature dependency of the column material is taken into account in the derivation. The elements are tested by the buckling analysis of columns with different boundary conditions, and results are compared with those of the classical beam-bending element and analytical solution. The temperature effects on the stability characteristics of the column are also examined considering the temperature dependency of the material properties of the column.

2. GOVERNING EQUATIONS AND CONCEPT FOR THE NEW ELEMENTS

Consider a perfectly straight elastic column with temperature dependent material properties (See Figure 1). The column is subjected to an end axial force P that is applied along its centroidal axis and a temperature load T that has a variation along the centroidal axis. The strain-displacement equation of the column can be written as

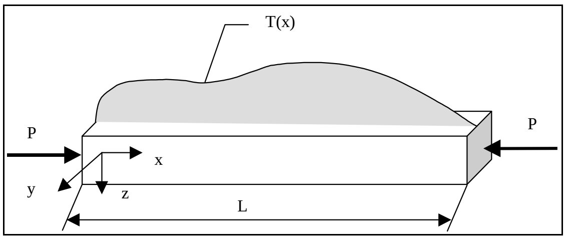


Figure 1. Elastic column subjected to buckling load

$$\varepsilon_x = \frac{du}{dx} - z\frac{d^2w}{dx^2} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 - \alpha T \tag{1}$$

where u and w are displacements along the x and w axis; α is the thermal expansion coefficient depending on temperature; and T is the temperature. Strain energy of a linear elastic column can be written as

$$U = \frac{1}{2}\int_V E\varepsilon_x{}^2 dV \tag{2}$$

Then, substituting Eq. (1) into (2) we obtain,

$$U = \frac{1}{2}\int_l\int_A\Bigg[\left(\frac{du}{dx}\right)^2 + z^2\left(\frac{d^2w}{dx^2}\right)^2 + \frac{1}{4}\left(\frac{dw}{dx}\right)^4 + \alpha^2T^2 - 2z\frac{du}{dx}\frac{d^2w}{dx^2} + \frac{du}{dx}\left(\frac{dw}{dx}\right)^2 - 2\alpha T\frac{du}{dx}$$
$$- z\frac{d^2w}{dx^2}\left(\frac{dw}{dx}\right)^2 + 2\alpha Tz\frac{d^2w}{dx^2} - \alpha T\left(\frac{dw}{dx}\right)^2\Bigg]EdAdx \tag{3}$$

where E is the Young modulus, which is also a function of temperature. Since the temperature has a variation along the column axis (no variation over the cross-section), the Young modulus and the thermal expansion coefficient of the column material will depend on only the coordinate x. Performing the integration over the cross-section of the column and disregarding the fourth order term, we obtain

$$U = \frac{1}{2}\int_l\left[EA\left(\frac{du}{dx}\right)^2 + EI\left(\frac{d^2w}{dx^2}\right)^2 + EA\frac{du}{dx}\left(\frac{dw}{dx}\right)^2 + F_T - 2P_T\frac{du}{dx} - P_T\left(\frac{dw}{dx}\right)^2\right]dx \tag{4}$$

where A and I are the cross-sectional area and the moment of inertia of the column about centroidal axis, respectively. The thermal load terms in Eq. (4) can be obtained performing the integration over the cross-sectional area of the column as,

$$F_T = \int_A E\alpha^2 T^2 dA = EA\alpha^2 T^2 \qquad P_T = \int_A E\alpha T dA = EA\alpha T \tag{5}$$

The potential energy of the axial load P and total potential energy of the column becomes

$$\Omega = P\int_0^L \frac{du}{dx}dx \tag{6}$$

$$\Pi = U + \Omega \tag{7}$$

Now, we can investigate the buckling of the column from the undeflected configuration as

$$\begin{aligned} u &\rightarrow u_0 + u_1 \\ w &\rightarrow w_0 + w_1 \end{aligned} \tag{8}$$

where u_0 and w_0 denote the undeflected configuration; u_1 and w_1 are infinitesimally small increments. Minimization of the total potential energy needs the first variation of the total potential energy be zero. Taking into account the column is initially straight (i. e. $w_0 = 0$), the first variations becomes

$$\delta\Pi = \int_0^L \left[EA\frac{du_0}{dx}\frac{du_1}{dx} + (P - P_T)\frac{du_1}{dx} \right] dx = 0 \tag{9}$$

Axial displacement of the undeflected column is obtained from Eq. (9) as

$$u_0 = -\frac{(P - P_T)}{EA}x \tag{10}$$

Using Eq. (10) the second variation of the total potential energy of the column is obtained as follows

$$\delta^2\Pi = \int_0^L \left[EA\left(\frac{du_1}{dx}\right)^2 + EI\left(\frac{d^2 w_1}{dx^2}\right)^2 - P\left(\frac{dw_1}{dx}\right)^2 \right] dx \tag{11}$$

For inextensional buckling, the second variation becomes

$$\delta^2\Pi = \int_0^L \left[EI\left(\frac{d^2 w_1}{dx^2}\right)^2 - P\left(\frac{dw_1}{dx}\right)^2 \right] dx \tag{12}$$

The material properties in Eq. (12) are dependent on temperature, and their dependency largely changes with increasing temperature. The influence of temperature-dependent material properties on the thermomechanical behavior at elevated temperature and/or high gradient temperature is quite significant.

In this study, it is assumed that the material properties of the column have linear variation with the temperature as follows,

$$E(T) = E_0 + E_1 T \text{ and } \alpha_T(T) = \alpha_0 + \alpha_1 T \tag{13}$$

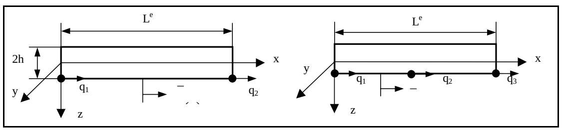
By the use of the rotation defined as $\phi(x) = dw_1/dx$ in Eq. (12), we obtain,

$$\delta^2\Pi = \int_0^L \left[EI\left(\frac{d\phi}{dx}\right)^2 - P\phi^2 \right] dx \tag{14}$$

From Eq. (14) we see that the second variation of the total potential energy depends on only the rotation. The rotation causes the displacements at any distance from the neutral axis and they vary linearly with the distance from the neutral axis. By the use of this approximation of the thin beam theory, we may introduce a column element with the axial degrees of freedom (DOF) defined at the outer fiber of the column. The new element has the simpler formulation and more compact element matrices than those of the classical beam-bending element. It has a little difficulty to apply the boundary conditions.

3. COLUMN-BUCKLING ELEMENTS

The geometry of the elements used is shown in Figure 2. The column is assumed to sustain only axial and flexural deformation; shear deformation is disregarded. The bending strains and displacement field due to the flexural behavior is given as follows:



(a) (b)

Figure 2. Geometry of the elements with (a) 2-DOF and (b) 3-DOF

$$\phi = \frac{\bar{u}}{h} \tag{15}$$

$$\frac{d\phi}{dx} = \frac{1}{h}\frac{d\bar{u}}{dx} \tag{16}$$

where $\bar{u}(x)$ is the axial displacement of the fiber at z = h. Substituting Eqs. (15) and (16) into Eq. (14), the second variation of the total potential energy becomes,

$$\delta^2\Pi = \int_0^L \left\{ \frac{EI}{h^2}\left(\frac{d\bar{u}}{dx}\right)^2 - \frac{P}{h^2}\bar{u}^2 \right\} dx \qquad (17)$$

Now, some column elements may be defined by considering the axial degrees of freedom on the outer fiber. The displacement field in a column element can be approximated by the use of interpolation functions and unknown nodal displacements, so that

$$\bar{u}^e(x) = N.q \qquad (18)$$

where N and q denotes the interpolation function matrix and nodal displacement vector of the elements. Now, two kinds of column elements are introduced: the elements with two nodes and three nodes. The interpolation function matrices and the nodal displacements vectors are obtained for the element with two nodes and for the element with three nodes as follows:

$$\mathbf{N} = \left[1 - \frac{x}{L^e} \quad \frac{x}{L^e}\right] \qquad q^T = [q_1 \quad q_2]^T \qquad (19)$$

$$\mathbf{N} = \left[1 - \frac{3}{L^e}x + \frac{2}{L^{e^2}}x^2 \quad \frac{4}{L^e}x - \frac{4}{L^{e^2}}x^2 \quad -\frac{1}{L^e}x + \frac{2}{L^{e^2}}x^2\right] \qquad q^T = [q_1 \quad q_2 \quad q_3]^T \qquad (20)$$

The column is discretized by finite elements and the displacement field given by Eq. (18) is used for the elements. Then, the total potential energy of the column can be obtained by simply summing the potential energy of the elements as follows:

$$\delta^2\Pi = \sum_e q^T \left[\frac{I^e}{h^{e^2}} \int_e E^e(T) \left(\frac{dN}{dx}\right)^T \left(\frac{dN}{dx}\right) dx \right] q - P \sum_e q^T \left[\frac{1}{h^{e^2}} \int_e N^T N dx \right] q \qquad (21)$$

The first term given in the corner bracket is the element stiffness matrix, k, and the second one is the element geometric matrix, g. The temperature field within an element can be written in terms of nodal temperatures, T_1^e and T_2^e, as follows

$$T^e(x) = T_1^e + \frac{T_2^e - T_1^e}{L^e} x \qquad (22)$$

Then, the variation of the Young modulus and thermal expansion coefficient within the element can be written for an element as

$$E^e(x) = E_0 + E_1 T^e(x) \qquad (23)$$

Substituting Eqs. (22) and (23) into Eq. (21), the stiffness and geometric matrices of the 2-DOF element can be expressed as,

$$k = \frac{I^e}{h^{e^2}} \int_e \left[E_0 + E_1 \left(T_1^e + \frac{T_2^e - T_1^e}{L^e} x \right) \right] \left(\frac{dN}{dx}\right)^T \left(\frac{dN}{dx}\right) dx \qquad (24)$$

Performing the integration, the element stiffness matrix can be obtained as

$$k = \frac{I^e}{L^e h^{e^2}} \left(E_0 + E_1 \frac{T_1^e + T_2^e}{2} \right) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad (25)$$

The geometrical matrix of the column element with 2-DOF can be written as

$$g = \frac{L^e}{6h^{e^2}} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad (26)$$

For the element with 3-DOF, the stiffness and geometric matrices are obtained by the use of the interpolation functions given by Eq. (20), as follows

$$k = \frac{1}{3L^e h^{e^2}} \left(E_0 + E_1 \left(\frac{T_1^e}{6} + \frac{2T_2^e}{3} + \frac{T_3^e}{6} \right) \right) \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \qquad g = \frac{L^e}{30h^{e^2}} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} \qquad (27)$$

$$[K - PG]Q = 0 \tag{28}$$

where K is global stiffness matrix; and G is global geometric matrix; Q is the global displacements of the column.

Various boundary conditions for the column are considered: (i) a column clamped at one end and free at the other, (ii) a column with simply supported at both ends, and (iii) a column clamped at both ends. Boundary conditions for the element with 2-DOF can be approximated by the use of Eq. (18). The rotation is zero for a clamped edge. If the column is clamped at x = 0, the boundary condition can be expressed in the local degrees of freedom considering Eq. (15), so that

$$q_1 = 0 \tag{29}$$

It is obvious that q_2 becomes zero if the column is clamped at x=L. Since bending moment is zero for a simply supported edge, the derivative of rotation with respect to x becomes zero at that edge. For example, the simply supported end conditions can be approximated by

$$\frac{d\bar{u}}{dx} = \frac{dN}{dx} q = 0 \tag{30}$$

Substituting the interpolation functions Eq. (19) into Eq. (30), the conditions that will be imposed on the matrix equation can be written in the local degrees freedom as

$$q_1 = q_2 \tag{31}$$

The boundary conditions for the element with 3-DOF are derived as explained in detail above. For a clamped end Eq. (29) is valid again ($q_3 = 0$ if the column is clamped at x = L). For a simply supported end, the condition will be

either $q_1 = \frac{4}{3} q_2 - \frac{1}{3} q_3$ or

$$q_3 = \frac{4}{3} q_2 - \frac{1}{3} q_1 \tag{32}$$

according to the location of the support being at left or at right.

4. NUMERICAL STUDIES

The columns with different end conditions are discretized by the use of new elements. And then the stability characteristics of the columns are obtained numerically. The results are compared with those of the classical beam bending element (i. e. displacement and rotation degrees of freedom are considered at each node) and the exact solutions. Exact value of the buckling load of a prismatic beam (Brush and Almroth, 1975) are given by

$$P_{cr} = \beta \frac{EI}{L^2} \tag{33}$$

where β is the instability coefficient for prismatic beams with various boundary conditions. The temperature effects are not considered in the comparison studies.

It is observed that convergence is slow for the columns with the simply supported ends when the column is discretized by the new elements with equal length. The undesired result is due to weakness in the approximation to the boundary conditions by the use of elements with equal length. If the elements used at the boundary are chosen smaller than those of the interior, a better convergence can be obtained. The variation of the relative error with length ratio of elements (interior element length to boundary element length) is shown in Figure 3 and 4 for five and ten elements models of the column, respectively. The variations show that the relative error decreases as the length ratio increases. Therefore, the element length ratio 25 is selected for a better convergence in the calculations of a column with simply supported boundary conditions. For a clamped-clamped boundary condition, the lowest eigenvalue gives the trivial solution and is ignored in the calculations.

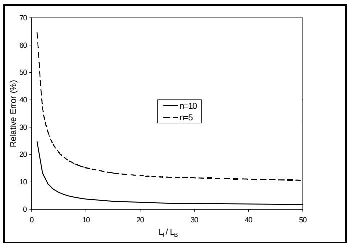


Figure 3. Improvement of convergence for 2-DOF model

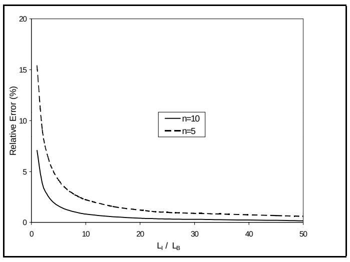


Figure 4. Improvement of convergence for 3-DOF model

The results of the convergence studies are shown in Tables 1-3 for the new elements and classical element. The results obtained by the use of the new elements, especially for the clamped cases, are in good agreement with the classical element.

Table 1. Convergence and Comparison of the Instability Coefficients β (For a Clamped-Free Column, Exact Value of β is 2.467)

Number of Element	Present (2 DOF)	Present (3 DOF)	Classical Element
1	3.000 (1)*	2.486 (2)	2.486 (2)
2	2.597 (2)	2.469 (4)	2.469 (4)
3	2.524 (3)	2.468 (6)	2.468 (6)
4	2.499 (4)	2.468 (8)	2.468 (8)
5	2.488 (5)	2.467 (10)	2.467 (10)
10	2.472 (10)	2.467 (20)	2.467 (20)

* : Numbers in the parenthesis show the total number of DOF

Table 2. Convergence and Comparison of the Instability Coefficients β (For a Simply Supported - Simply Supported Column, Exact Value of β is 9.870)

Number of element	Present (2 DOF)	Present (3 DOF)	Classical Element
3	12.497 (2)*	10.207 (5)	9.885 (6)
4	12.245 (3)	10.058 (7)	9.875 (8)
5	11.030 (4)	9.969 (9)	9.872 (10)
6	10.571 (5)	9.939 (11)	9.871 (12)
10	10.094 (9)	9.902 (19)	9.870 (20)

* : Umbers in the parenthesis show the total number of DOF

Table 3. Convergence and Comparison of the Instability Coefficients β (For a Clamped-Clamped Column, Exact Value of β is 39.478)

Number of element	Present (2 DOF)	Present (3 DOF)	Classical Element
3	54.000 (2)*	40.343 (5)	40.343 (4)
4	48.000 (3)	39.775 (7)	39.775 (6)
5	44.888 (4)	39.605 (9)	39.605 (8)
6	43.200 (5)	39.541 (11)	39.541 (10)
7	42.193 (6)	39.513 (13)	39.513 (12)
10	40.794 (9)	39.487 (19)	39.487 (18)

* : Numbers in the parenthesis show the total number of DOF

In the classical element shape functions corresponding to rotation degrees of freedom are quadratic. The new element with 3-DOF uses quadratic shape functions for axial degrees of freedom defined at the outer fiber of the beam, yielding quadratic variation of the rotation along the beam. Since boundary conditions for clamped-free and clamped-clamped cases are satisfied exactly by the use of the new elements, the results of classical element and the new element with 3-DOF would be similar for those cases as given in Tables 1 and 3. However, there is some difference for simply supported case due to the approximate satisfaction of boundary condition. As expected, the use of higher order elements in discretization gives better accuracy.

Influence of the temperature dependency of the material properties is studied on the buckling of the beam with the following parameters:

$$E_0 = 20121186 \quad N/cm^2,\ E_1 = -5981\ N/cm^2\ {}^\circ C, \tag{34}$$

Three different temperature loadings have been considered:

(i) $T(x) = 100\ {}^\circ C$ (constant temperature along the column)

$$\text{(ii)}\ T(x) = \left[400 - 300\frac{x}{L}\right]\ {}^\circ C,\ \text{(linear variation)} \tag{35}$$

$$\text{(iii)}\ T(x) = \left[400 - 600\frac{x}{L} + 300\left(\frac{x}{L}\right)^2\right]\ {}^\circ C,\ \text{(quadratic variation)}$$

Buckling load depending on temperature loading will be

$$P_{cr}(T) = \beta\frac{E(T)I}{L^2} \quad \text{or} \quad P_{cr}(T) = \beta_T\frac{E_0 I}{L^2} \tag{36}$$

The calculations are carried out for a beam with clamped-free ends. Results obtained by using of beam elements with 2-DOF are shown in Table 4. The buckling loads are lowered considerably by the thermal effect.

Table 4. The Instability Coefficients β Without/With Temperature Loading

Number of element	β (T=0)	β_T (i)	β_T (ii)	β_T (iii)
1	3.000	2.911	2.777	2.777
2	2.597	2.519	2.362	2.391
3	2.524	2.449	2.292	2.325
4	2.499	2.425	2.269	2.303
5	2.488	2.414	2.258	2.293
10	2.472	2.399	2.243	2.279

5. CONCLUSIONS

New finite elements with axial degrees of freedom are proposed for the buckling analysis of columns with temperature dependent material properties. Euler-Bernoulli theory of thin beams is used to derive the element matrices by means of the minimum potential energy principle. The temperature effects are also examined on the column buckling, and temperature dependency of material properties is taken into account in the formulation. The new column elements have simple derivations and compact element matrices than the classical beam-bending element. The boundary condition for a simply supported edge is satisfied approximately. Various end conditions of columns are considered in the examples. The results are compared with both the classical and exact ones. The new element is as powerful as the classical one, yet it has more compact matrices.

The temperature effects on the buckling loads of the column with temperature dependent material properties are studied by the use of new element. The temperature loads affects the elastic modulus of column. The buckling loads are lowered by the thermal effect.

6. REFERENCES

Ali, M. A. and Sridharan, S. 1988. Versatile Model for Interactive Buckling of Columns and Beam-Columns. Int. J. Solids and Structures 24, 481-496.

Brush, D. O. and Almroth, B. O. 1975. Buckling of Bars, Plates and Shells, McGraw-Hill, USA.

Coulter, B. A. and Miller, R. E. 1986. Vibration and Buckling of Beam-Columns Subjected to Non-Uniform Axial Loads. Int. J. Num. Meth. Eng. 23, 1739-1755.

Goda, K., Watanabe O., Motoyama, K. 1992. “Lateral Buckling of a Thin-walled Beam Subjected to Impulsive Load” **Proceedings of the Second International Offshore and Polar Engineering Conference**, Published by International Society of Offshore and Polar Engineers (ISOPE), USA, 558-567.

Helwig, T.A. and Yura, J.A. 1999. Torsional Bracing of Columns. J. Struct. Eng. ASCE 125(5), 547-555.

Sakiyama, T. 1986. A Method of Analyzing the Elastic Buckling of Tapered Columns. Comp. and Struct. 23, 119-120.

Smithpardo, J. P. and Aristizabalochoa, J. D. 1999. Buckling Reversals of Axially Restrained Imperfect Beam-column. J. Eng. Mech. ASCE, 125 (4), 401-409.

Vaziri, H. H. and Xie, J. 1992. Buckling of Columns under Variably Distributed Axial Loads. Comp. and Struct. 45, 505-509.

Weaver, W. and Johnston, P. R. 1984. Finite Elements For Structural Analysis, Prentice-Hall, New Jersey.

Wu, B. 1998. Secondary Buckling of an Elastic Column With Spring-Supports at Clamped Ends. Archive of Applied Mechanics 68 (5), 342-351.