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Maximum Independent Set Cover Pebbling Number of a Binary Tree

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Abstract : A pebbling move is defined by removing two pebbles from some vertex and placing one pebble on an adjacent vertex. A graph is said to be cover pebbled if every vertex has a pebble on it after a series of pebbling moves. The maximum independent set cover pebbling number of a graph G is the minimum number, $\rho(G)$, of pebbles required so that any initial configuration of $\rho(G)$ pebbles can be transformed by a sequence of pebbling moves so that after the pebbling moves the set of vertices that contains pebbles form a maximum independent set S of G. In this paper, we determine the maximum independent set cover pebbling number of a binary tree.

Key words : graph pebbling, cover pebbling, maximum independent set cover pebbling, binary tree.

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1. Introduction

Given a graph G, distribute k pebbles on its vertices in some configuration, call it as C. Assume that G is connected in all cases. A pebbling move is defined by removing two pebbles from some vertex and placing one pebble on an adjacent vertex. [1] The *pebbling number* $\pi(G)$ is the minimum number of pebbles that are sufficient, so that for any initial configuration of $\pi(G)$ pebbles, it is possible to move a pebble to any root vertex v in G. [2] The *cover pebbling number* $\gamma(G)$ is defined as the minimum number of pebbles needed to place a pebble on every vertex of the graph using a sequence of pebbling moves, regardless of the initial configuration. A set S of vertices in a graph G is said to be an independent set (or an internally stable set) if no two vertices in the set S are adjacent. An independent set S is maximum if G has no independent set S' with $|S'| > |S|$.

We have introduced the concept maximum independent set cover pebbling number in [5]. The *maximum independent set cover pebbling number*, $\rho(G)$, of a graph G , to be the minimum number of pebbles that are placed on $V(G)$ such that after a sequence of pebbling moves, the set of vertices with pebbles forms a maximum independent set S of G, regardless of their initial configuration. In this paper, we determine the maximum independent set cover pebbling number $\rho(G)$ for a binary tree.

Notation 1.1: $f(a)$ denotes the number of pebbles placed at the vertex a . Also $f(G)$ denotes the number of pebbles on the graph G .

2. Maximum independent set cover pebbling number of a binary tree

Definition 2.1. [3] A complete binary tree, denoted by B_n, is a tree of height n, with 2^i vertices at distance i from the root. Each vertex of B_n has two “children”, except for the set of 2^n vertices that are distance n away from the root, none of which have children. The root will be denoted by R_n.

Obviously $\rho(B_0) = 1$, and $\rho(B_1) = 6$ since $\rho(P_3) = 6$ [6].

Theorem 2.2. For the binary tree B_2, $\rho(B_2) = 41$.

Proof: Clearly B_2 contains two B_1's as subtrees which are adjacent to the vertex R_2, where R_2 is the root vertex of B_2. Let B' be the right subtree with the vertices R', a_1, a_2 and B'' be the left subtree with the vertices R'', b_1, b_2 of the binary tree B_2 (as given in Figure 1). Put forty pebbles on the vertex a_2. Then we cannot cover the maximum independent set of B_2. Thus $\rho(B_2) \geq 41$.

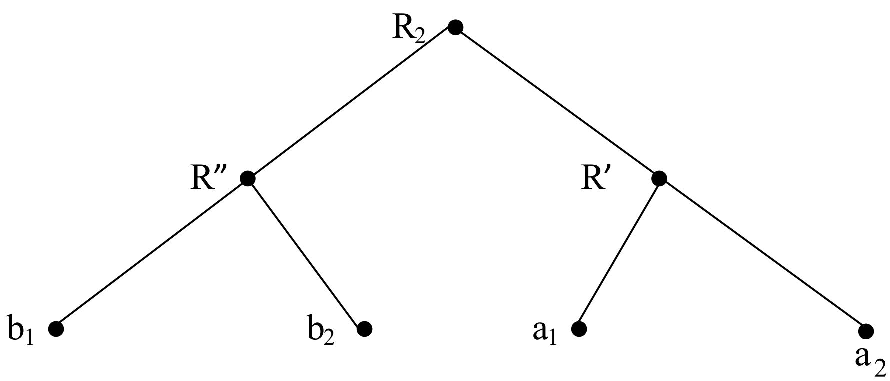

Figure 1. The Binary tree B_2

Now consider the distribution of forty one pebbles on the vertices of B_2. According to the distributions, we find the following three cases:

Case 1: Let $f(B') \geq 6$ and $f(B'') \geq 6$.

If f $f(R_2) \geq 1$, then clearly we can cover the maximum independent set of B_2. So assume that $f(R_2) = 0$. Without loss of generality, let $f(B') \geq 21$. So either the path a_1R' or the path a_2R' contains eleven pebbles or more, say a_1R'. We can move a pebble to R_2 using at most four pebbles from a_1R'. Then $f(B') \geq 6$ and hence we are done, since ρ $(B_1) = 6$.

Case 2: Let $f(B') \leq 5$ and $f(B'') \leq 5$.

This implies that $f(R_2) \geq 31$. We move six pebbles each to the vertices R′ and R″. Then $f(R_2) \geq 7$ and hence we are done.

Case 3: Let $f(B') \geq 6$ and $f(B'') \leq 5$.

Clearly we are done if $f(B'')+f(R_2) \geq 9$, since $<\{V(B'') \cup \{R_2\}\}> \cong K_{1,3}$ and $\rho(K_{1,3}) = 9$ [5]. So assume that $f(B'')+f(R_2) \leq 8$. This implies that $f(B') \geq 33$ pebbles. If the vertices a_1, a_2, and R′ contain 5 pebbles then we can move a pebble to the vertex R_2 at a cost of four (at most) pebbles. Since we have at least 27 extra pebbles on B′, either the path a_1R' (or) a_2R' receives at least four pebbles or both a_1 and a_2 receive two or more pebbles. If $f(B'') \geq 1$ or $f(R_2) \geq 2$ then we are done. So assume that $f(B'') = 0$ and $f(R_2) \leq 1$. Thus B′ contains forty pebbles. Now we can send eight pebbles to R_2 at a cost of thirty two (at most) pebbles from the vertices of B′. We cover the maximum independent set of B″ using the eight pebbles at R_2. If $f(R_2) = 1$ then clearly we are done. Otherwise $f(B') \geq 9$ and we are done since $<\{V(B'') \cup \{R_2\}\}> \cong K_{1,3}$ and $\rho(K_{1,3}) = 9$.

Therefore, $\rho(B_2) \leq 41$.

Theorem 2.3. For the binary tree B_3, $\rho(B_3) = 313$.

Proof: Let B′ be the right subtree of height two with the root vertex R′ and B″ be the left subtree of height two with the root vertex R″ of the binary tree B_3. Consider the distribution of 312 pebbles on the vertex $v \in V(B')$ where degree(v)=1. Then we cannot cover the maximum independent set of B_3. Thus $\rho(B_3) \geq 313$.

Now consider the distribution of 313 pebbles on the vertices of B_3. According to the configurations, we find the following three cases:

Case 1: Let $f(B') \geq 41$ and $f(B'') \geq 41$.

Clearly we are done if $f(R_3) = 0, 2$, or $f(R_3) \geq 4$. So, assume that $f(R_3) = 1$ or 3. Without loss of generality, let $f(B') \geq 155$. We have to move a pebble to R_3, to cover the maximum independent set of B_3. Anyone of the 4-paths leading from the root R_3

of B_3 to the bottom row of B′ contains at least eight pebbles. So we can move a pebble to R_3 using at most eight pebbles. Now we move $\frac{f(R_3)+1}{2}$ pebbles to R′ from R_3. Then f(B′) ≥ 41 and f(B″) ≥ 41 and hence we are done, since B′≅ B_2 and B″≅ B_2.

Case 2: Let f(B′) ≤ 40 and f(B″) ≤ 40.

This implies that, $f(R_3) \geq 233$. Using 164 of these pebbles from R_3, we can move 41 pebbles each to the root R′ of B′ and R″ of B″. Then the remaining number of pebbles in R_3 is at least five. If the remaining pebbles in R_3 are even then we move $\frac{f(R_3)}{2}$ pebbles to R′. Otherwise, we do the following pebbling moves to obtain even number of pebbles in R_3. We move two pebbles from R_3 to R′ and then move one pebble from R′ to R_3. Thus the remaining number of pebbles in R_3 is even then we move $\frac{f(R_3)-3}{2}$ pebbles to R′. Therefore f(B′) ≥ 41 and f(B″) ≥ 41 and hence we are done.

Case 3: Let f(B′) ≥ 41 and f(B″) ≤ 40.

Clearly the remaining 232 pebbles are somewhere in the graph B′ $\cup$ $\{R_3\}$ to cover the maximum independent set of B″. If $f(R_3) \geq 34$ then we can move seventeen pebbles to the root R″ of B″ and hence we are done. So assume that $f(R_3) \leq 33$. This implies that $f(B'')+f(R_3) \leq 73$. Thus f(B′) ≥ 240. Note that, if B′ contains 13 pebbles then we can move a pebble to the root R_3 of B_3 at a cost of at most eight pebbles from B′. Also note that we should not decrease the least possibility of the total pebbles in B′. Thus we can send twenty four pebbles to the root R_3. Clearly we are done if $f(R_3) \geq 10$ or f(B″) ≥ 6. So assume that $f(R_3) \leq 9$ and f(B″) ≤ 5. This implies that f(B′) ≥ 299. So we can move 32 pebbles to the root R_3 of B_3. Clearly we are done if $f(R_3) \geq 2$ or f(B″) ≥ 1. Otherwise, f(B′) ≥ 312. So we can move 33 pebbles to R_3. If $f(R_3) = 1$ then clearly we are done. Otherwise we can move exactly thirty four pebbles to R_3 while retaining forty one pebbles on B′. Thus we are done.

Therefore, $\rho(B_3) \leq 313.$ ■

Theorem 2.4. For the binary tree B_4, $\rho(B_4) = 2505.$

Proof: Let B′ be the right subtree of height three with the root vertex R′ and B″ be the left subtree of height three with the root vertex R″ of the binary tree B_4. Let $v \in V$ (B′) such that degree(v) = 1 and v is the rightmost vertex of B′.

Consider the distribution of 2504 pebbles on the vertex v. Then we cannot cover the maximum independent set of B_4. Thus $\rho(B_4) \geq 2505$.

Now consider the distribution of 2505 pebbles on the vertices of B_4. According to the distributions, we find the following three cases:

Case 1 : Let f(B′) ≥ 313 and f(B″) ≥ 313.

If $f(R_4) \geq 1$ then we can cover the maximum independent set of B_4, since $B' \cong B_3$ and $B'' \cong B_3$. So assume that $f(R_4) = 0$. Without loss of generality, let f(B′) ≥ 1253. So any one of the 8-paths leading from the root R_4 of B_4 to the bottom row of B′ contains thirty two pebbles or more. So we can move a pebble to R_4 using at most thirty two pebbles from B′. Then f(B′) ≥ 313 and f(B″) ≥ 313 and hence we are done.

Case 2 : Let f(B′) ≤ 312 and f(B″) ≤ 312.

This implies that $f(R_4) \geq 1881$. Using 1252 of these pebbles from the vertex R_4, we can move 313 pebbles each to the root R′ of B′ and R″ of B″. Then $f(R_4) \geq 629$ and hence we are done.

Case 3 : Let f(B′) ≥ 313 and f(B″) ≤ 312.

Clearly the remaining 1880 pebbles are somewhere in the graph $B' \cup \{R_4\}$ to cover the maximum independent set of B″. If $f(R_4) \geq 137$ then clearly we are done. So assume that $f(B'')+f(R_4) \leq 447$. Thus f(B′) ≥ 2058. So we can move a pebble to the root R_4 of B_4 at a cost of at most sixteen pebbles whenever 33 pebbles are in B′. Also note that we should not decrease the least possibility of the total pebbles in B′. Thus we can send 109 pebbles to the root R_4. Clearly we are done if $f(R_4) \geq 28$ or f(B″) ≥ 14. So assume that $f(R_4) + f(B'') \leq 40$. Thus f(B′) ≥ 2465. So we can move 134 pebbles to the root R_4. If $f(R_4) \geq 3$ or f(B″) ≥ 1 then clearly we are done. Otherwise, we can send exactly 137 pebbles to R_4 while retaining 313 pebbles on B′ and hence we are done.

Thus $\rho(B_4) \leq 2505$. ■

Theorem 2.5. For the binary tree B_n ($n \geq 3$), the maximum independent set cover pebbling number is given by,

$$\rho(B_n) = \sum_{k=0}^{\left\lceil \frac{n-1}{2} \right\rceil} 2^{n-2k-1} 2^{2n-2k} + \sum_{i=0}^{\left\lceil \frac{n-1}{2} \right\rceil} \left(2^{2i} + \sum_{j=1}^{n-2i-1} 2^{j-1} 2^{2i+2j} \right) + \gamma_n$$

$$= S_{1,n} + S_{2,n} + S_{3,n}, \text{ (say)},$$

where $S_{i,n}$ denotes the i^{th} term of the above sum and $\gamma_n = 2^n$ if n is even and $\gamma_n = 0$ otherwise.

Proof: Let B′ be the right subtree of height n-1 with the root vertex R′ and B″ be the left subtree of height n-1 with the root vertex R″ of the binary tree B_4. Let $v \in V$ (B′) such that degree(v) = 1 and v is the rightmost vertex of B′.

Note that the maximum independent set of B_n is the maximum independent set of B′ plus the maximum independent set of B″, if n is odd. The vertex R_n is also included if n is even. The maximum independent set of a subtree contains the vertices starting from the bottom row vertices and then every vertex of every second row. If n is odd then this process ends at the root of the subtree. If n is even then this process ends at the below row of the root vertex of that subtree.

First consider the left subtree B″. To cover the vertices of the bottom row of B″, we need $2^{n-1} 2^{2n}$ pebbles from v, since bottom row of B″ contains 2^{n-1} vertices and that are all at 2n distance from v. Similarly, we need $2^{n-3} 2^{2n-2}$ pebbles to cover the vertices of second row from the bottom row. Thus we need $\sum_{k=0}^{\left\lceil \frac{n-1}{2} \right\rceil} 2^{n-2k-1} 2^{2n-2k}$ pebbles to cover the maximum independent set of B″ from the vertex v.

A similar work can be done to cover the maximum independent set of B′ from v, using $\sum_{i=0}^{\left\lceil \frac{n-1}{2} \right\rceil} \left(2^{2i} + \sum_{j=1}^{n-2i-1} 2^{j-1} 2^{2i+2j} \right)$ pebbles. So we cover the maximum independent set of B_n if n is odd. Suppose n is even then we have to cover the R_n also. Since $d(v, R_n) = n$, we need 2^n pebbles from v to cover the vertex R_n. Thus,

$$\rho(B_n) \geq \sum_{k=0}^{\lceil \frac{n-1}{2} \rceil} 2^{n-2k-1} 2^{2n-2k} + \sum_{i=0}^{\lceil \frac{n-1}{2} \rceil} \left(2^{2i} + \sum_{j=1}^{n-2i-1} 2^{j-1} 2^{2i+2j} \right) + \gamma_n.$$

where $\gamma_n = 0$ if n is odd and $\gamma_n = 2^n$ if n is even.

Now consider the distribution of $\rho(B_n)$ pebbles on the vertices of B_n, where $n \geq 3$. We prove the upper bound by induction on n. By Theorem 2.3 and Theorem 2.4, the result is true for n = 3 and n = 4 respectively. Assume that the result is true for B_{n-1}. According to the distributions, we find the following cases:

Case 1: Let $f(B') < \rho(B_{n-1})$ and $f(B'') < \rho(B_{n-1})$.

If we prove that $f(R_n) \geq 4\rho(B_{n-1})+5$ then we are done. Since $f(B') + f(B'') \leq 2\rho(B_{n-1})-2$, we get $f(R_n) \geq \rho(B_n) - 2\rho(B_{n-1}) + 2$. So it is enough to prove that

$$\rho(B_n) \geq 6\,\rho(B_{n-1}) + 3. \text{ - - - - - - (1)}$$

First note that, $\rho(B_n) \geq 2^{3n-1}$,- - - - (2) by considering only the k=0 term of $S_{1,n}$.

Also, $S_{1,n-1} = \sum_{k=0}^{\lceil \frac{n-2}{2} \rceil} 2^{n-2k-2} 2^{2n-2k-2} = \frac{2^{3n}}{16} \sum_{k=0}^{\lceil \frac{n-2}{2} \rceil} 2^{-4k}$

$$S_{1,n-1} \leq \frac{2^{3n}}{15} \Rightarrow 6(S_{1,n-1}) \leq \frac{6(2^{3n})}{15} \text{ - - - - - (3)}$$

and $S_{2,n-1} = \sum_{i=0}^{\lceil \frac{n-2}{2} \rceil} \left(2^{2i} + \sum_{j=1}^{n-2i-2} 2^{j-1} 2^{2i+2j} \right)$

$$= \sum_{i=0}^{\lceil \frac{n-2}{2} \rceil} 2^{2i} + \sum_{i=0}^{\lceil \frac{n-2}{2} \rceil} 2^{2i-1} \sum_{j=1}^{n-2i-2} 2^{3j}$$

$$\leq \frac{2^{n+1}}{3} + \frac{1}{2} \sum_{i=0}^{\lceil \frac{n-2}{2} \rceil} 2^{2i} \frac{8(8^{n-2i-2})}{7}$$

$$\le \frac{2^{n+1}}{3} + \frac{2^{2n}}{112} \sum_{i=0}^{\left\lceil \frac{n-2}{2} \right\rceil} 2^{-4i}$$

$$S_{2,n-1} \le \frac{2^{n+1}}{3} + \frac{2(2^{2n})}{221}$$

$$6(S_{2,n-1}) \le 4(2^n) + \frac{12(2^{2n})}{221} \quad \text{- - - - - (4)}$$

and

$$6(S_{3,n-1}) \le 6(2^{n-1}) = 3(2^n) \quad \text{- - - - - (5)}$$

Equations (2) through (5) show that (1) holds if,

$$2^{3n-1} \ge \frac{6(2^{3n})}{15} + 4(2^n) + \frac{12(2^{3n})}{221} + 3(2^n) + 3$$

Or if,

$$\frac{1}{2} - \frac{6}{15} - \frac{12}{221} \ge \frac{7}{2^{2n}} + \frac{3}{2^{2n}}$$

which holds for $n \ge 5$. Of course the fact that $\rho(B_n) \ge 6\,\rho(B_{n-1}) + 3$ holds for $n = 3,4$ as well.

Case 2: Let $f(B') \ge \rho(B_{n-1})$ and $f(B'') \ge \rho(B_{n-1})$.

Let n is odd.

If $f(R_n) = 0, 2$ or $f(R_n) \ge 4$ then clearly we are done. So assume that $f(R_n) = 1$ or 3. Without loss of generality, let $f(B') \ge \left\lceil \frac{\rho(B_n) - 3}{2} \right\rceil$. Anyone of the 2^{n-1}-path leading from the root R_n to the bottom vertices of B′ contains at least 2^n pebbles and hence we can move a pebble to R_n using (at most) 2^n pebbles. This is always possible since, $\frac{\rho(B_n) - 3}{2(2^{n-1})} \ge \frac{2^{3n-1} - 3}{2^n} \ge \frac{2^{2n}}{2} - \frac{3}{2^n} \ge 2^n$. Now we move $\frac{f(R_n)+1}{2}$ pebbles to R′ from R_n. Then $f(B') \ge \rho(B_{n-1})$ and $f(B'') \ge \rho(B_{n-1})$ and hence we are done, since $B' \cong B_{n-1}$ and $B'' \cong B_{n-1}$.

Let n is even.

If $f(R_n) \geq 1$ then clearly we are done. So assume that $f(R_n) = 0$. Without loss of generality, let $f(B') \geq \left\lceil \frac{\rho(B_n)}{2} \right\rceil$. Anyone of the 2^{n-1}-paths leading from the root R_n to the bottom vertices of B' contains at least 2^n pebbles (since, $\frac{\rho(B_n)}{2(2^{n-1})} \geq \frac{2^{3n-1}}{2^n} \geq 2^n$) and hence we can move a pebble to R_n using (at most) 2^n pebbles. Then $f(B') \geq \rho(B_{n-1})$ and $f(B'') \geq \rho(B_{n-1})$ and hence we are done, since $B' \cong B_{n-1}$ and $B'' \cong B_{n-1}$.

Case 3: Let $f(B') \geq \rho(B_{n-1})$ and $f(B'') < \rho(B_{n-1})$.

The remaining $\rho(B_n) - 2\rho(B_{n-1}) + 1$ pebbles are in somewhere of the graph B_n to cover the maximum independent set of B_n. Our strategy is to move all extraneous pebbles to the root R_n of B_n from the vertices of B' so that we can cover the maximum independent set of B'' and also the vertex R_n if needed. Note that any pebbles in B'' can substitute for at least one pebble on the root. Clearly, placing all the $\rho(B_n)$ pebbles on B' is the worst case configuration. Indeed, if pebbles are placed on the other vertices of B_n, then moving all those pebbles which are not in B', to the rightmost vertex of B' would require more pebbles to cover the maximum independent set of B_n. Also, note that, we can send at least one pebble to the root R_n of B_n if $f(B_{n-1}) \geq \rho(B_{n-1}) + 2^n$. This is always possible, since $\rho(B_{n-1}) \geq 2^{3n-4}$. We have $\rho(B_n) - \rho(B_{n-1})$ pebbles in B' to cover the maximum independent set of B'' and also R_n if needed.

Let compute $\rho(B_n) - \rho(B_{n-1})$

y_n

$$\geq \begin{cases} \frac{7}{8}\sum_{k=0}^{\left\lceil \frac{n-2}{2} \right\rceil} 2^{3n-4k-1} + 0 + \frac{1}{8}\left(\sum_{i=0}^{\left\lceil \frac{n-2}{2} \right\rceil} 2^{3n-4i-1}\right) + 2^n, \textit{if n is even} \\ \frac{7}{8}\sum_{k=0}^{\left\lceil \frac{n-2}{2} \right\rceil} 2^{3n-4k-1} + 2^{n+1} + 2^{n-1} + \frac{1}{8}\left(\sum_{i=0}^{\left\lceil \frac{n-2}{2} \right\rceil} 2^{3n-4i-1} + 2^{n+1}\right) - 2^{n-1}, \textit{if n is odd} \end{cases}$$

$$\geq \begin{cases} \sum_{k=0}^{\lceil \frac{n-2}{2} \rceil} 2^{2n-4k-1} + 2^n, if\ n\ is\ even \\ \sum_{k=0}^{\lceil \frac{n-2}{2} \rceil} 2^{2n-4k-1} + 2^{n+1}, if\ n\ is\ odd \end{cases}$$

So we can send $\sum_{k=0}^{\lceil \frac{n-2}{2} \rceil} 2^{2n-4k-1} + \begin{cases} 1, & if\ n\ is\ even \\ 2, & if\ n\ is\ odd \end{cases}$ pebbles to the root R_n of B_n from B′.

Subcase (a): n is odd.

Using the $\sum_{k=0}^{\lceil \frac{n-2}{2} \rceil} 2^{2n-4k-1}$ pebbles from R_n, we can cover the maximum independent set of B″, except the root R″ of B″. But R″ is also covered by using the remaining two pebbles from R_n. Hence we are done.

Subcase (b): n is even.

Using the $\sum_{k=0}^{\lceil \frac{n-2}{2} \rceil} 2^{2n-4k-1}$ pebbles from R_n of B_n, we can cover the maximum independent set of B″. But R_n is also covered since $f(R_n) \geq 1$. Hence we are done.

Thus the upper bound follows.

Therefore $\rho(B_n)$ is as desired.

Note 2.6: we can reformulate the maximum independent set cover pebbling number of B_n, if we know the value of $\rho(B_{n-1})$ where $n \geq 3$. That is,

$$\rho(B_n) = \rho(B_{n-1}) + \begin{cases} 2^{n+1}\left(\frac{2^{2n+2}-1}{15}\right), & if\ n\ is\ odd \\ 2^{n}\left(\frac{2^{2n+2}+7}{15}\right), & if\ n\ is\ even \end{cases}.$$

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