Sciencia Acta Xaveriana
An International Science Journal
ISSN. 0976-1152

Volume 3
No. 1
pp. 106-114
Apr 2012

Generalized Pebbling Numbers of Some Graphs

A. Lourdusamy[1] C. Muthulakshmi@Sasikala[2]

[1] Department of Mathematics, St. Xavier's College (Autonomous), Palayamkottai - 627 002, India lourdugnanam@hotmail.com

[2] Department of Mathematics, Sri Paramakalyani College, Alwarkurichi - 627 412, India

Abstract : The generalized pebbling number of a graph G, $f_{gl}(G)$, is the least positive integer n such that however n pebbles are placed on the vertices of G, we can move a pebble to any vertex by a sequence of moves, each move taking p pebbles off one vertex and placing one on an adjacent vertex. In this paper, we determine the generalized pebbling number of wheel W_n, fan F_n and complete r-partite graph.

Key words : Graph, wheel, fan, complete r-partite graph.

1 Introduction

Let G be a simple connected graph. The pebbling number of G is the smallest number f(G) such that however these f(G) pebbles are placed on the vertices of G, we can move a pebble to any vertex by a sequence of moves, each move taking two pebbles off one vertex and placing one on an adjacent vertex [2]. Suppose n pebbles are distributed on to the vertices of a graph G, a generalized p pebbling step [u,v] consists of removing p pebbles from a vertex u, and then placing one pebble on an adjacent vertex v, for any p = 2. Is it possible to move a pebble to a root vertex r, if we can repeatedly apply generalized p pebbling steps? It is answered in the affirmative by Chung in [1]. The generalized pebbling number of a vertex v in a graph G is the smallest number $f_{gl}(v,G)$ with the property that from every placement of $f_{gl}(v,G)$ pebbles on G, it is possible to move a pebble to v by a sequence of pebbling moves where a pebbling move consists of removing p pebbles from a vertex and placing one pebble on an adjacent vertex. The generalized pebbling number of the graph G, denoted by $f_{gl}(G)$, is the maximum $f_{gl}(G)$ over all vertices v in G.

Again the generalized t-pebbling number of a vertex v in a graph G is the smallest number $f_{glt}(v,G)$ with the property that from every placement of $f_{glt}(v,G\}$ pebbles on G, it is possible to move t pebbles to v by a sequence of pebbling moves where a pebbling move consists of the removal of p pebbles from a vertex and the placement of one of these pebbles on an adjacent vertex. The generalized t-pebbling number of the graph G, denoted by $f_{glt}(G)$ is the maximum $f_{glt}(v,G)$ over all vertices v of G. Throughout this paper G denotes a simple connected graph with vertex set V(G) and edge set E(G).

$\lfloor x \rfloor$ denote the largest integer less than or equal to x and $\lceil x \rceil$ denote the smallest integer greater than or equal to x.

2 Known Results

We find the following results with regard to the generalized pebbling numbers of graph in [2] and their generalized t-pebbling numbers in [3].

Theorem 2.1. For a complete graph K_n, $f_{gl}(K_n)=(p-1)n-(p-2)$ where $p \geq 2$.

Theorem 2.2. For a path of length n, $f_{gl}(P_n)=p^n$ where $p \geq 2$.

Theorem 2.3. For a star $K_{1,n}$, $f_{gl}(K_{1,n})=(p-1)n+(p^2-2p+2)$ if n>1 and $p \geq 2$.

Theorem 2.4. The generalized t-pebbling number for a path of length n is $f_{glt}(P_n)=tp^n$.

Theorem 2.5. The generalized t-pebbling number of a complete graph on n vertices where $n \geq 3$, $p \geq 2$ is $f_{glt}(K_n)=pt+(p-1)(n-2)$.

Theorem 2.6. The generalized t-pebbling number for a star $K_{1,n}$ where n>1 is $f_{glt}(K_{1,n})=p^2t+(p-1)(n-2)$ where $p \geq 2$.

We will now proceed to compute the genearlized pebbling numbers of wheel graph W_n, Fan graph F_n and complete r-partite graph.

3. Computation of genearlized pebbling number

Definition 3.1. We define the wheel graph denoted by W_n to be the graph with $V(W_n)= \{h,v_1,v_2, \ldots ,v_n\}$ where h is called the hub of W_n and $E(W_n)=E(C_n) \cup \{hv_1, hv_2, \ldots , hv_n\}$ where C_n denotes the cycle graph on n vertices.

Theorem 3.2. For $n \geq 4$, the generalized pebbling number of the wheel graph W_n is $f_{gl}(W_n)=(p-1)+(p^2-2p+1)$ where $p \geq 2$.

Proof: By Theorem 2.1, $f_{gl}(h,W_n)=p+(p-1)(n-1)$. Let us now find the generalized pebbling number of v_1. If we place p-2 pebbles at v_n, (p^2-1) pebbles at $v_{\lceil \frac{n}{2} \rceil}$ and p-1 pebbles at $W_n-\{v_1,v_n, v_{\lceil \frac{n}{2} \rceil}\}$ then we cannot move a pebble to v_1.

So $f_{gl}(v_1,W_n) > (p^2-1)+(p-1)(n-3)+(p-2)$

$> (p-1)n+(p^2-2p+1)$.

Let us now prove that $(p-1)n+(p^2-2p+1)$ pebbles are sufficient to put a pebble on v_1. Assume that v_1 has zero pebbles. Now v_1 is adjacent with h,v_2,v_n. Hence in the given distribution, any one of h,v_2,v_n, receives p pebbles, then a pebble can be moved to v_1. Also any one of the vertices $\{v_3,v_4, \ldots ,v_{n-1}\}$ receives at least p^2 pebbles then a pebble can be moved to v_1 through h. Let $q_i=pm_i+r_i$ where $0 \leq r\_i \leq p-1$ be the number of pebbles on v_i for i=2 to n. Let a be the number of pebbles on h. Suppose $a \geq p$, then from h, we can move a pebble to v_1. Suppose $a < p$, then let $b = p-a > 0$. Let us transfer the pebbles from v_i (i = 2 to n) to h.

Let m = $\sum_{i=2}^{n} m_i$. After this transfer, the number of pebbles on h is b+m. If b+m $\geq$ p, then we can put a pebble on v_1. So we assume that b+m < p. Therefore p-b-m > 0.

Let s = p-b-m. In order to place p-b-m pebbles on h we are in need of p(p-b-m) pebbles on C_n.
Consider $(p-1)n+(p^2-2p+1)-b-pm-p^2+pb+pm=(p-1)n+b(p-1)+(1-2p)$.
Since $p \geq 2$, $n \geq 4$, $b > 0$ we get $(p-1)n+b(p-1)+(1-2p) \geq 2$.

From h, a pebble can be moved to v_1. If h has zero pebbles, v_2 and v_n have at most (p-2) pebbles each and no vertex of $\{v_3, v_4, \ldots , v_{n-1}\}$ has p^2 pebbles and assume $n-3 \geq p$, then there will be at least p pebbles each, then we can move p pebbles to h and so we are done.

Let us assume n-3 < p.

Consider $(p-1)n+(p^2-2p+2)-2(p-2)=(p-1)n+(p^2-4p+6)$.

Now, $p^2+(n-4)p-(n-6)$ pebbles are distributed on to C_n. Using p^2 pebbles we can move a pebble to v_1. □

Definition 3.3. A fan graph denoted by F_n is a path P_n plus an extra vertex connected to all vertices of the path P_n. A fan graph with vertices $v_1, v_2, \ldots ,v_n,v_{n+1}$ in order means the fan graph F_n whose vertices of the path P_n are $v_1,v_2, \ldots ,v_n$ in order and whose extra vertex is v_{n+1}.

Theorem 3.4. The generalized pebbling number of the fan graph F_n is $f_{gl}(F_n)=(p-1)n+(p^2-2p+1)$.

Proof: Fan graph F_n is the spanning subgraph of W_n so $f_{gl}(F_n) \leq f_{gl}(W_n)$.
Hence $f_{gl}(F_n)=(p-1)n+(p^2-2p+1)$.

Suppose that there are $(p-1)n+(p^2-2p+1)$ pebbles that are distributed on to the vertices of F_n where F_n is the fan graph with vertices $v_1, v_2, \ldots, v_n, v_{n+1}$ in order. First let the target vertex be v_{n+1}. By Theorem 2.1 $f_{gl}(v_{n+1}, F_n)=p+(p-1)(n-1)$. So if v_{n+1} has zero pebbles then there exists some v_i where $i \in \{1,2,3, \ldots, n\}$ with at least p pebbles, so we can move one pebble to v_{n+1} from v_i.

Next suppose the target vertex if v_k and assume v_k has zero pebbles where $k \in \{1,2,3, \ldots, n\}$. Suppose v_{n+1} receives at least p pebbles, then a pebble can be moved to v_k or if any one of the vertices of v_i where $i \in \{1,2, \ldots, n\}$ and $i \neq k$ receives p^2 pebbles then from v_i a pebble can be moved to v_k through v_{n+1}. Suppose v_{n+1} receives m where $1 \leq m \leq p-1$ pebbles and the vertices of P_n-$\{v_k\}$ receive at the most p^2-1 pebbles, using $p(p-2)$ pebbles, we can move $(p-2)$ pebbles to v_{n+1}, and the remaining $(p-1)n$ pebbles are also distributed on to the vertices of P_n. Hence there exists a vertex w with at least p pebbles. So a pebble can be moved to v_{n+1} from w. Now v_{n+1} receives at least p pebbles,and so a pebble can be moved to v_k from v_{n+1}. Suppose v_{n+1} has zero pebbles and all the vertices of P_n except v_k receive at the most p^2-1 pebbles. Then there must be at least one vertex v_j with at least p pebbles. If in addition, there are at least two vertice v_j and v_l with m pebbles in which $p \leq m \leq p^2-1$, then we can move at least $\left\lfloor \frac{p}{2} \right\rfloor$ pebbles from v_l to v_{n+1}. So, p pebbles can me moved to v_{n+1}. Hence a pebble can be moved to v_k. Otherwise, there is only one vertex v_j with at least p pebbles. Therefore all v_i in which $1 \leq i \leq n$ and $i \neq j, k$ have p-1 pebbles. Suppose $j < k$, then using the sequence of pebbling moves v_j-v_{j+1}-v_{j+2}- … -v_k we can move a pebble to v_k. Otherwise using the sequence of moves v_j- v_{j-1}- … -v_k, a pebble can be moved to v_k. Hence in all the cases $f_{gl}(v_k,F_n) \leq (p-1)n+(p^2-2p+1)$. □

Definition 3.5. A graph G=(V,E) is called an r-partite graph if V can be partitioned into r non-empty subsets $V_1,V_2, \ldots, V_r$ such that no edge of G joins vertices in the

$K_{s_1,s_2,\ldots,s_r}$.

same set. The sets $V_1, V_2, \ldots, V_r$ are called partite sets or vertex classes of G. If G is an r-partite graph having partite sets $V_1, V_2, \ldots, V_r$ such that every vertex of V_i is joined to every vertex of V_j where $1 \le i, j \le r$ and $i \ne j$, then G is called a complete r-partite graph. If $|V_i|=s_i$ for i=1,2, … ,r then we denote G by $K_{s_1, s_2, \ldots, s_r}$.

Notation 3.6. For $s_1 \ge s_2 \ge \ldots \ge s_r$, $s_1 > 1$ and if r=2, $s_2 > 1$, let $K_{s_1, s_2, \ldots, s_r}$ be the complete r-partite graph with $s_1, s_2, \ldots, s_r$ vertices in vertex classes $C_1, C_2, \ldots, C_r$ respectively. Let $n = \sum_{i=1}^{r} s_i$.

Theorem 3.7. For G = $K_{s_1, s_2, \ldots, s_r}$ the generalized pebbling number is given by

$$f_{gl}(G) = \begin{cases} p^2 + (p-1)(s_1 - 2) & \text{if } p \ge n - s_1 \\ p + (p-1)(n-2) & \text{if } p < n - s_1 \end{cases}.$$

Proof :

Case i: Assume $s_1 < n - p$.

Let the target vertex of v of C_i for some i=1, 2, … , r. Without loss of generality, we assume that v has zero pebbles on it. If we place (p-1) pebbles each on (n-1) vertices of G other than v, a pebble cannot be moved to v. So $f_{gl}(v,G) \ge p+(p-1)(n-2)$.

Let us place p+(p-1)(n-2) pebbles on the vertices of G. If there is a vertex w of c_j ($j \ne i$) with at least p pebbles then a pebble can be moved to v. Otherwise, there is a vertex w_1 of C_k ($k \ne i$) with at most (p-1) pebbles. Then at least p+(p-1)(n-3) pebbles are distributed on to each of n-p-1 vertices of C_i. Since $s_i \le s_l < n-p$, using (p-1)p pebbles we can move at most (p-1) pebbles to w_1. So w_1 has at least p pebbles. Then from w_1 a pebble can be moved to v. Otherwise every vertex of G-C_i contains zero pebbles on it. Then either there exists a vertex w_2 of C_i with at least p^2 pebbles or all the vertices of C_i-{v} contains at most p^2-1 pebbles. So p pebbles can be moved to a vertex w_3 of C_j ($j \ne i$). From w_3 a pebble can be moved to the vertex v of C_i.

Hence in all cases $f_{gl}(v,G) \leq p+(p-1)(n-2)$.

Since v is arbitrary, $f_{gl}(G) \leq p+(p-1)(n-2)$.

Case ii: Assume $n-s_1 \leq p$.

Let us choose the vertex class C_1. Let $v \in C_1$ be our target vertex. Without loss of generality assume that vertex v has zero pebbles on it. Let us place p^2-1 pebbles on one of the s_1 vertices of C_1, and place (p-1) pebbles on each of the remaining s_1-2 vertices of C_1. Then (p-1) pebbles can be moved to the vertex w of C_k where $k \neq 1$. Now all the pebbled vertices in G receive (p-1) pebbles. Hence pebbling move is impossible. So $f_{gl} > (p^2-1)+(p-1)(s_1-2) \geq p^2+(p-1)(s_1-2)$.

Suppose $p^2+(p-1)(s_1-2)$ pebbles are placed on the vertices of G. Let the target vertex be v of C_k.

If there is a vertex in some C_j $(j \neq k)$ with at least p pebbles then a pebble can be placed on v using p pebbles.

If not, then every vertex of $G-C_k$ will contain either zero or at most (p-1) pebbles on it. If there is a vertex say w in some C_j $(j \neq k)$ with a pebble on it we use p pebbles from a vertex of C_k to put a pebble on w. Then from the remaining $p(p-1)+(p-1)(s_1-2)-1$ vertices we can put (p-1) pebbles on w and from w a pebble can be moved to v.

Otherwise every vertex of $G-C_k$ will have zero pebbles on it. Then all the $p^2+(p-1)(s_1-2)$ pebbles are distributed on the vertices of C_k. Then using p^2 pebbles a pebble can be moved to the vertex v of C_k.

Hence $f_{gl}(v,G) \leq (p-1)(s_1-2)+p^2$.

Therefore $f_{gl}(G) \leq p^2+(p-1)(s_1-2)$.

Conclusion

We have determined the generalized pebbling numbers of wheel graph, fan graph and complete r-partite graph. We leave it to the reader the computation of the generalized t-pebbling numbers of wheel graph, fan graph and complete r-partite graph.

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