Sciencia Acta Xaveriana
An International Science Journal
ISSN. 0976-1152

Volume 3
No. 1
pp. 87-106
Apr 2012

Pebbling Number for Jahangir Graph $J_{2,m}$ $(3 \leq m \leq 7)$

A.Lourdusamy[1], S. Samuel Jeyaseelan[2] and T.Mathivanan[3]

[1] Department of Mathematics, St. Xavier 's College (Autonomous), Palayamkottai - 627 002, India.E-mail: lourdugnanam@hotmail.com

[2] Department of Mathematics, Loyola College (Autonomous), Chennai - 600 002, India. E-mail: samjeya@yahoo.com

[3] Research scholar, Department of Mathematics, St. Xavier 's College (Autonomous), Palayamkottai-627 002, India. E-mail: tahit_van_man@yahoo.com

Abstract : Given a configuration of pebbles on the vertices of a connected graph G, a pebbling move (or pebbling step) is defined as the removal of two pebbles off a vertex and placing one pebble on an adjacent vertex. The pebbling number, f(G), of a graph G is the least number m such that, however m pebbles are placed on the vertices of G, we can move a pebble to any vertex by a sequence of pebbling moves. In this paper, we determine f(G) for Jahangir graph $J_{2,m}$ $(3 \leq m \leq 7)$.

Key words : pebbling, Jahangir graph.

2009 Mathematics Subject Classification: 05C99.

1 Introduction

One recent development in graph theory, suggested by Lagarias and Saks, called pebbling, has been the subject of much research. It was first introduced into the literature by Chung [1], and has been developed by many others including Hulbert, who published a survey of pebbling results in [3]. There have been many developments since Hulbert's survey appeared.

Given a graph G, distribute k pebbles (indistinguishable markers) on its vertices in some configuration C. Specifically, a configuration on a graph G is a function from V(G) to $N \cup \{0\}$ representing an arrangement of pebbles on G. For our purposes, we will always assume that G is connected.

A pebbling move (or pebbling step) is defined as the removal of two pebbles from some vertex and the placement of one of these pebbles on an adjacent vertex. Define the pebbling number, f(G), to be the minimum number of pebbles such that regardless of their initial configuration, it is possible to move to any root vertex v, a pebble by a sequence of pebbling moves. Implicit in this definition is the fact that if after moving to vertex v one desires to move to another root vertex, the pebbles reset to their original configuration.

Fact 1.1 [5,6]. For any vertex v of a graph G, $f(v,G) \geq n$ where $n=|V(G)|$.

Proof. Consider the configuration $C:V-\{v\}\rightarrow N$ defined by $C(w)=1$ for all $w \in V-\{v\}$. Then the size $|C|$ of the configuration is n-1. In this configuration we cannot move a pebble to v. Hence $f(v,G) \geq |V(G)|$.

Fact 1.2 [5]. The pebbling number of a graph G satisfies $f(G) \geq \max \{2^{diam(G)}, |V(G)|\}$.

Proof. Let $w \in V(G)$ be a vertex at a distance diam(G) from the target vertex v. Place $2^{diam(G)}-1$ pebbles at w. clearly we cannot move any pebble to v. Thus $f(G) \geq \max \{2^{diam(G)}, |V(G)|\}$.

There are few other interesting results in the pebbling number of graphs. Hulbert [3] has written an excellent survey article on pebbling. We earnestly request the interested readers to refer to it for further study.

We also request the readers to read [2] in which Moews has studied the pebbling number of product of trees.

We now proceed to determine the pebbling number for $J_{2,m}$ $(3 \leq m \leq 7)$.

2 Pebbling Number of Jahangir Graph $J_{2,m}$ $(3 \leq m \leq 7)$

Definition 2.1 [4] Jahangir graph $J_{n,m}$ for $m \geq 3$ is a graph on $nm + 1$ vertices, that is, a graph consisting of a cycle C_{nm} with one additional vertex which is adjacent to m vertices of C_{nm} at distance n to each other on C_{nm}.

Example: Figure1 shows Jahangir graph $J_{2,8}$. The figure $J_{2,8}$, appears on Jahangir's tomb in his mausoleum. It lies in 5 kilometer north- west of Lahore, Pakistan, across the River Ravi.

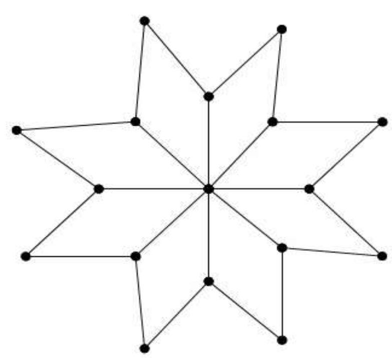
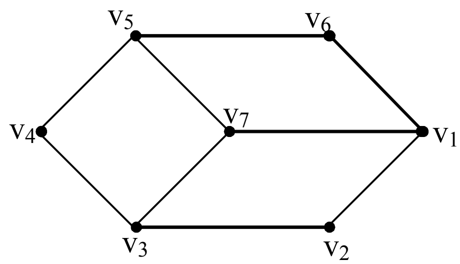


Figure 1: $J_{2,8}$

Remark 2.2 Let v_{2m+1} be the label of the center vertex and $v_1, v_2, \ldots, v_{2m}$ be the label of the vertices that are incident clockwise on cycle C_{2m} so that $\deg(v_1) = 3$.

Theorem 2.3 For the Jahangir graph $J_{2,3}$, $f(J_{2,3}) = 8$.



$J_{2,3}$

Proof Put seven pebbles at v_4. Clearly we cannot move a pebble to v_1, since $d(v_4, v_1) = 3$. Thus $f(J_{2,3}) \geq 8$.

Now, consider a distribution of eight pebbles on the vertices of $J_{2,3}$. Consider the sets $S_1 = \{v_1, v_3, v_5\}$ and $S_2 = \{v_2, v_4, v_6\}$.

Case (i) Suppose we cannot move a pebble to the vertex v_7.

The vertices of S_1 must contain at most one pebble each. The vertices of S_2 contain at most three pebbles each. If any vertex of S_1 contains a pebble, then the neighbors of that vertex contain at most one pebble each. Otherwise, a pebble can be moved to v_7. Also, note that, if a vertex of S_2 contains two or three pebbles then the neighbors of that vertex contain zero pebbles and also no other vertex of S_2 contain two or three pebbles. Otherwise, we can move a pebble to v_7. So, in any such distributions, the vertices of $J_{2,3}$ contain at most six pebbles so that a pebble could not be moved to v_7- a contradiction to the total number of pebbles placed over the vertices of $J_{2,3}$. Thus, we can always move a pebble to v_7. So, our assumption (Case(i)) is wrong.

Case (ii) Suppose we cannot move a pebble to a vertex of S_1. Without loss of generality, let v_1 be the target vertex.

Clearly, the neighbors v_2, v_6, v_7 of v_1 contain at most one pebble each. The vertices v_3 , v_5 contain at most three pebbles each and the vertex v_4 contains at most seven pebbles. Also, note that, both v_3 and v_5 cannot contain two or three pebbles each (otherwise, a pebble can be moved to v_1 through v_7) and if either v_3 or v_5 contains two or three pebbles then v_4 contains at most three pebbles.

Suppose v_7 has a pebble on it, then the vertices v_3 and v_5 contain at most one pebble each. This implies that, the vertex v_4 contains at least three pebbles on it. If any one of the vertex v_3 or v_5 has a pebble on it, then we can easily move a pebble to v_1. Otherwise, v_4 contains at least four pebbles on it and so we can move a pebble to v_1 through v_7. So, assume that v_7 has zero pebbles on it.

Suppose v_2 has a pebble on it. Then, the path v_4v_5 contains at least five pebbles. Clearly, we can move a pebble to v_1. So, assume that v_2 has zero pebbles on it. In a similar way, we may assume that,v_6 has zero pebbles on it.

Suppose v_3 has two or three pebbles on it. Then, v_4 contains at least four pebbles on it and so we can move a pebble to v_1 easily. So, assume v_3 contains at most one pebble on it. In a similar way, we may assume that,v_5 has at most one pebble on it. Then, v_4 contains at least six pebbles on it. If any one of the vertices v_3 or v_5 contains a pebble then we can move a pebble to v_1. Otherwise, v_4 contains eight pebbles on it and hence we can move a pebble to v_1. Thus, we can always move a pebble to v_1. So, our assumption (Case(ii)) is wrong.

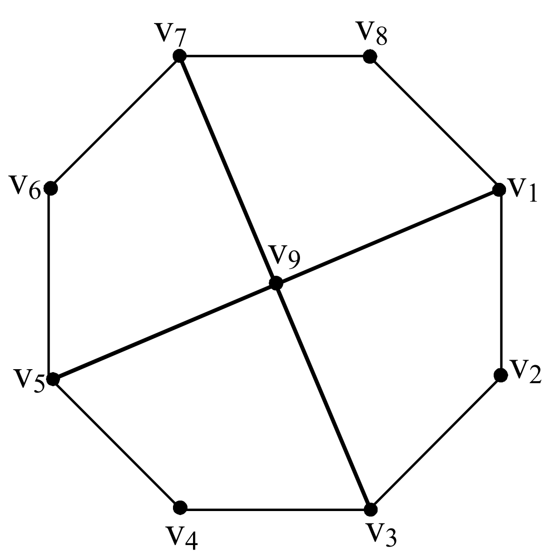
Case (iii) Suppose we cannot move a pebble to a vertex of S_2. Without loss of generality, let v_2 be the target vertex.

The neighbors v_1, and v_3 of v_2 contain at most one pebble each. The vertices v_4, v_6, and v_7 contain at most three pebbles each and the vertex v_5 contains at most seven pebbles. Suppose v_3 has a pebble on it. Then the path v_5v_6 contains at least four pebbles. If either v_4 or v_7 contains a pebble on it, clearly we can move a pebble to v_2. Otherwise, the path v_5v_6 contains at least five pebbles and so we can move a pebble to v_2 easily. So, assume that v_3 has zero pebbles on it. In a similar way, we may assume that, v_1 has zero pebbles on it.

Suppose v_7 has two or three pebbles on it. Then, clearly we can move a pebble to v_2, since v_5 contains at least four pebbles on it. So, assume that v_7 has at most one pebble on it. Suppose v_4 contains two or three pebbles on it. Then, the path v_5v_6 contains at least four pebbles and so we are done if v_7 contains a pebble. Otherwise, the path v_5v_6 contains at least five pebbles. Then, a pebble can easily be moved to v_2. So, assume that, v_4 contains at most one pebble. In a similar way, we may assume that, v_6 has at most one pebble on it. Then, v_5 contains at least five pebbles. If v_7 contains one pebble on it, then we are done if either v_4 or v_6 has one pebble on it. Otherwise, it is easy to move a pebble to v_2. So, assume that v_7 has zero pebbles on it. Then, v_5 contains at least six pebbles. In this case also it is easy to move a pebble to v_2. Thus, we can always move a pebble to v_2. So, our assumption (Case(iii)) is wrong. Thus, $f(J_{2,3}) \leq 8$.

Therefore, $f(J_{2,3}) = 8$.

Theorem 2.4 For the Jahangir graph $J_{2,4}$, $f(J_{2,4}) = 16$.



$J_{2,4}$

Proof Put fifteen pebbles at v_8 then we cannot move a pebble to v_4, since $d(v_4, v_8) = 4$. Thus $f(J_{2,4}) \geq 16$.

Now, consider a distribution of sixteen pebbles on the vertices of $J_{2,4}$. Consider the sets $S_1 = \{v_1, v_3, v_5, v_7\}$ and $S_2 = \{v_2, v_4, v_6, v_8\}$.

Case (i) Suppose we cannot move a pebble to the vertex v_9.

Clearly, the vertices of S_1 contain at most one pebble each and the vertices of S_2 contain at most three pebbles each. If any vertex of S_1 contains a pebble, then the neighbors of that vertex contain at most one pebble each. Otherwise, a pebble can be moved to v_9. Also, note that, if a vertex of S_2 contains two or three pebbles then the neighbors of that vertex contain zero pebbles and also the proceeding and succeeding vertex of that vertex in S_2 contain at most one pebble each. Otherwise, we can move a pebble to v_9. So, in any such distribution, the vertices of $J_{2,4}$ contain at most eight pebbles so that a pebble could not be moved to v_9-a contradiction to the total number of pebbles placed over the vertices of $J_{2,4}$. Thus, we can always move a pebble to v_9. So, our assumption (case(i)) is wrong.

Case (ii) Suppose we cannot move a pebble to a vertex of S_1. Without loss of generality, let v_1 be the target vertex.

Clearly, the neighbors v_2, v_8, and v_9 of v_1 contain at most one pebble each. The vertices of $S_1 - \{v_1\}$ contain at most three pebbles and the vertices of $S_2 - \{v_2, v_8\}$ contain at most seven pebbles. Also, note that, no two vertices of $S_1 - \{v_1\}$ contain two or three pebbles each and if a vertex of S_1-$\{v_1\}$ contain two or three pebbles then the neighbor of that vertex contain at most three pebbles. Also, both v_4 and v_6 cannot contain four or more (at most seven) pebbles each and if either v_4 or v_6 contains four or more pebbles then the neighbors of that vertex contain at most one pebble.

Suppose v_9 has a pebble on it. Clearly, the neighbors v_3, v_5, and v_7 of v_9 contain at most one pebble each. So the remaining ten pebbles are at v_4 and v_6. Thus either v_4 or v_6 receives at least five pebbles. This implies that, we can move a pebble to v_1 through v_9-a contradiction to case(ii). So assume that v_9 has zero pebbles on it.

Suppose v_2 has a pebble on it. Clearly, the neighbor v_3 of v_2 contains at most one pebble. If v_3 contains a pebble, then the vertex v_4 contains at most one pebble. Otherwise, v_4 contains at most three pebbles. So the path $v_4v_3v_2$ $v_1v_8v_7$ contains at most seven pebbles. Now, the path v_5v_6 contain at least nine pebbles. Clearly, v_6 contains at least six pebbles. If either v_5 or v_7 or v_8 contains one pebble then we can move a pebble to v_1. Otherwise, the vertex v_6 contains at least eight pebbles and so we get a contradiction to case(ii). So assume v_2 has zero pebbles on it. In a similar way, we may assume that v_8 has zero pebbles on it.

Suppose v_5 has two or three pebbles on it. Then the path $v_4v_5v_6$ contains at most six pebbles, otherwise we get a contradiction to case (ii). Thus either v_3 or v_7 contains at least five pebbles-a contradiction to case(ii). Thus, assume that v_5 has at most one pebble. The vertices v_3 and v_7 totally contain at most four pebbles. This implies that, the path $v_4v_5v_6$ contains at least twelve pebbles. Clearly we can move a pebble to v_1-a contradiction to case(ii). Thus, assume that v_5 has zero pebbles on it. So, either the path v_3v_4 or v_6v_7 contains at least eight pebbles. Clearly, we can move a pebble to v_1-a contradiction to case(ii). Thus, we can always move a pebble to v_9. So, our assumption (Case(ii)) is wrong.

Case (iii) Suppose we cannot move a pebble to a vertex of S_2. Without loss of generality, let v_2 be the target vertex.

As we discussed in case (i) and case (ii), we can give a simple argument which will conclude that our assumption (case(iii)) is wrong.

Thus, $f(J_{2,4}) \leq 16$.

Therefore, $f(J_{2,4}) = 16$.

Theorem 2.5 For the Jahangir graph $J_{2,5}$, $f(J_{2,5}) = 18$.

Proof Put fifteen pebbles at v_6 and one pebble each at v_8 and v_{10}, and then we cannot move a pebble to v_2. Thus $f(J_{2,5}) \geq 18$.

Now, consider a distribution of eighteen pebbles on the vertices of $J_{2,5}$. Consider the sets $S_1 = \{v_1, v_3, v_5, v_7, v_9\}$ and $S_2 = \{v_2, v_4, v_6, v_8, v_{10}\}$.

Case (i) Suppose we cannot move a pebble to the vertex v_{11}.

The vertices of S_1 contain at most one pebble each and the vertices of S_2 contain at most three pebbles each. Also, note that, no two consecutive vertices of S_2 contain two or three pebbles each and if a vertex of S_2 contains two or three pebbles then the neighbors of that vertex contain zero pebbles. Also, if a vertex of S_1 contains one pebble then the neighbors of that vertex contain at most one pebble each. Thus, in any distribution, $J_{2,5}$ contains at most ten pebbles so that a pebble could not be moved to v_{11} –a contradiction to the total number of pebbles placed over the vertices of $J_{2,5}$. So, our assumption is wrong.

Case (ii) Suppose we cannot move a pebble to a vertex of S_1. Without loss of generality, let v_1 be the target vertex.

Clearly, the neighbors v_2, v_{10}, and v_{11} of v_1 contain at most one pebble each. The vertices of $S_1 - \{v_1\}$ contain at most three pebbles each and the vertices of $S_2 - \{v_2, v_{10}\}$ contain at most seven pebbles each. Also, note that, no two vertices of $S_1 - \{v_1\}$ contain two or three pebbles each and if a vertex of $S_1-\{v_1\}$ contains two or three pebbles then the neighbors of that vertex contain at most three pebbles each. Also, no two vertices of $S_2 - \{v_2, v_{10}\}$ contain four or more pebbles each and if a vertex of S_2-

$\{v_2, v_{10}\}$ contains four or more pebbles then the neighbors of that vertex contain at most one pebble each. Suppose v_{11} contains one pebble on it. Clearly the vertices of S_1-$\{v_1\}$ contain at most one pebble each. This implies that, the paths $v_3v_4v_5$ and $v_7v_8v_9$ contain at most three pebbles each. Thus, the vertex v_6 contains at least nine pebbles and so we can move a pebble to v_1 easily-a contradiction to case(ii). So assume that v_{11} has zero pebbles on it.

Suppose v_2 has one pebble on it. The vertex v_3 contains at most one pebble and the vertex v_4 contains at most three pebbles. If the vertex v_5 contains two or three pebbles then either v_6 or v_8 contains at least four pebbles. So we get a contradiction to case(ii). Thus, assume that v_5 contains at most one pebble. In a similar way, we may assume that both v_7 and v_9 contain at most one pebble each. Thus, the vertices v_6 and v_8 contain at least ten pebbles totally. So, in any distribution of these ten pebbles on v_6 and v_8, we can always move a pebble to v_1 through v_7 and v_{11}. So assume that v_2 has zero pebbles on it. In a similar way, we may assume that v_{10} has zero pebbles on it.

Suppose v_3 has two or three pebbles on it. Then clearly the total number of pebbles on the vertices v_6 and v_8 is at least nine. Thus, we can move a pebble to v_1 through v_7 and v_{11}. So, assume that v_3 contains at most one pebble. In a similar way, we may assume that v_9 contains at most one pebble. Suppose v_5 contains two or three pebbles on it then the vertices v_6 and v_8 totally contain at least nine pebbles, so we can move a pebble to v_1 through v_{11} –a contradiction to case(ii). So assume that v_5 contains at most one pebble. In a similar way, we may assume that v_7 contains at most one pebble. Now, any one of the vertices of S_2-$\{v_2, v_{10}\}$ contains at least four pebbles, say v_6. If both neighbors of that vertex v_6 contain one pebble each then we can move a pebble to v_1 or if v_4 contains two or three pebbles and if a neighbor of v_4 contains one pebble then also we can move a pebble to v_1. Otherwise, v_6 contains more than eight pebbles-a contradiction to case(ii). Thus, in any distribution of eighteen pebbles on the vertices of $J_{2,5}$, we can always move a pebble to v_1. So, our assumption (case(ii)) is wrong.

Case (iii) Suppose we cannot move a pebble to a vertex of S_2. Without loss of generality, let v_2 be the target vertex.

Clearly, the neighbors v_1, and v_3 of v_2 contain at most one pebble each and the vertices v_4, v_{10}, and v_{11} contain at most three pebbles each. The vertices of $S_1 - \{v_1, v_3\}$ contain at most seven pebbles each and the vertices of $S_2 - \{v_2, v_4, v_{10}\}$ contain at most fifteen pebbles each. Also, note that, no two vertices of $S_1 - \{v_1, v_3\}$ contain four or more pebbles each and if a vertex of $S_1 - \{v_1, v_3\}$ contains four or more pebbles then the neighbors of that vertex contain at most three pebbles each. Also, no two vertices of $S_2 - \{v_2, v_4, v_{10}\}$ contain eight or more pebbles each and if a vertex of $S_2 - \{v_2, v_4, v_{10}\}$ contains eight or more pebbles then the neighbors of that vertex contain at most three pebbles each.

Suppose v_3 has one pebble on it. Clearly, both v_4 and v_{11} contain at most one pebble each. If v_{11} has one pebble on it then the vertices of $S_1 - \{v_1, v_3\}$ contain at most one pebble each. This implies that, any one of the vertices v_6, v_8 or v_{11} contains at least four pebbles. Thus we can move a pebble to v_2. So, assume that v_{11} has zero pebbles on it. Then either one of the vertices of $S_2 - \{v_2, v_4\}$ receives at least four pebbles and a vertex of $S_1 - \{v_1, v_3\}$ receives at least two pebbles or any two vertices of $S_2 - \{v_2, v_4\}$ receive at least four pebbles each. So, we can always move a pebble to v_2-a contradiction to case(iii). So, assume that v_3 has zero pebbles on it. In a similar way, we may assume that v_1 has zero pebbles on it.

Suppose v_{11} has two or three pebbles on it. Then both v_4 and v_{10} contain at most one pebble each. This implies that, the total number of pebbles on the vertices v_6 and v_8 is at least nine. So, we can move a pebble to v_2-a contradiction to case(iii). Thus, assume v_{11} has at most one pebble.

Suppose v_4 contains two or three pebbles on it. Then one of the vertices of $\{v_6, v_8\}$ contains at least four pebbles and if a vertex of $S_1 - \{v_1, v_3\}$ contains two or three pebbles then we can move a pebble to v_2. Otherwise, the total number of pebbles on the vertices v_6 and v_8 is at least eleven -a contradiction to case(iii). So, assume v_4 contains at most one pebble. In a similar way, we may assume that v_{10} contains at most one pebble.

Suppose v_7 has four or more pebbles (at most seven pebbles) on it. Clearly, the total number of pebbles on the vertices v_5, v_7, and v_9 contain at most nine. Thus, one of the vertices of $\{v_6, v_8\}$ contains at least four pebbles. If one of the vertices of $\{v_5, v_9\}$ contains two or three pebbles then we can move a pebble to v_2 easily. So, assume both v_5 and v_9 contain at most one pebble each. If v_{11} contains a pebble and if v_7 contains at least six pebbles then we can move a pebble to v_2. So assume that v_7 contains at most five pebbles. In this case also, we can move a pebble to v_2, since the total number of pebbles on the vertices v_6 and v_8 is at least nine. Thus, we assume that v_7 contains at most three pebbles. In a similar way, we may assume that both v_5 and v_9 contain at most three pebbles each.

Suppose all the three vertices v_5, v_7, and v_9 contain two or three pebbles each. Then, the vertices v_6 and v_8 totally contain at least six pebbles. So, we can move at least one pebble to v_{11} using the pebbles at v_6 and v_8. So, we can move a pebble to v_2. Thus, assume that two of the vertices of $\{v_5, v_7, v_9\}$, say v_5 and v_9 contain at least two or three pebbles each. Then, the total number of pebbles on the vertices v_6 and v_8 is at least eight. If v_7 contains one pebble then we can move a pebble to v_2- a contradiction to case(iii). Now, exactly one vertex from $\{v_5, v_7, v_9\}$, say v_5 contains two or three pebbles. Then the total number of pebbles on the vertices v_6 and v_8 is at least ten. If either v_{11} or v_4 contains a pebble then we can move a pebble to v_2 easily. So assume both v_4 and v_{11} contain zero pebbles on it. Then, the total number of pebbles on the vertices v_6 and v_8 is at least twelve. If v_7 contains one pebble on it then also we can move a pebble to v_2. Otherwise, we can move a pebble to v_2 easily, since the total number of pebbles on the vertices v_6 and v_8 is at least thirteen. Thus assume that all the three vertices v_5,v_7,v_9 contain at most one pebble each. Now, the total number of pebbles on the vertices v_6 and v_8 is at least twelve. If both v_7 and v_{11} contain one pebble each then we can move a pebble to v_2. So, assume v_{11} has zero pebbles on it.

Suppose all the three vertices v_5, v_7, v_9 contain one pebble each. One of the vertices $\{v_6 , v_8\}$ receives at least six pebbles and hence we can move a pebble to v_2. So, we may assume v_7 has zero pebbles on it. If both v_4 and v_{10} have one pebble each then we can move a pebble to v_2, since the total number of pebbles on the vertices v_6 and v_8 is fourteen. For the other distributions, it is easy to see that, a pebble can be moved to v_2- a contradiction to case(iii). This implies that $f(J_{2,5}) \leq 18$.

Therefore, $f(J_{2,5}) = 18$.

Theorem 2.6 For the Jahangir graph $J_{2,6}$, $f(J_{2,6}) = 21$.

Proof Put fifteen pebbles at v_6, three pebbles at v_{10} and one pebble each at v_8 and v_{12}.

Then, we cannot move a pebble v_2. Thus, $f(J_{2,6}) \geq 21$.

Now, consider a distribution of twenty one pebbles on the vertices of $J_{2,6}$. Consider the sets $S_1=\{v_1, v_3, v_5, v_7, v_9, v_{11}\}$ and $S_2 = \{v_2, v_4, v_6, v_8, v_{10}, v_{12}\}$.

Case (i) Suppose we cannot move a pebble to the vertex v_{13}.

The vertices of S_1 contain at most one pebble each and the vertices of S_2 contain at most three pebbles each. Also, note that, no two consecutive vertices of S_2 contain two or three pebbles each and if a vertex of S_2 contains two or three pebbles then the neighbors of that vertex contain zero pebbles. Also, if a vertex of S_1 contains one pebble then the neighbors of that vertex contain at most one pebble each. Thus, in any distribution, $J_{2,6}$ contains at most twelve pebbles so that a pebble could not be moved to v_{13} –a contradiction to the total number of pebbles placed over the vertices of $J_{2,6}$. So, our assumption (case(i)) is wrong.

Case (ii) Suppose we cannot move a pebble to a vertex of S_1. Without loss of generality, let v_1 be the target vertex.

Clearly, the neighbors v_2, v_{12}, and v_{13} of v_1 contain at most one pebble each. The vertices of $S_1 - \{v_1\}$ contain at most three pebbles each and the vertices of $S_2 - \{v_2, v_{12}\}$ contain at most seven pebbles each. Also, note that, no two vertices of $S_1 - \{v_1\}$ contain two or three pebbles each and if a vertex of S_1-$\{v_1\}$ contains two or three pebbles then the neighbors of that vertex contain at most three pebbles each. Also, no two vertices of $S_2 - \{v_2, v_{12}\}$ contain four or more pebbles each and if a vertex of S_2-$\{v_2, v_{12}\}$ contains four or more pebbles then the neighbors of that vertex contain at most one pebble each.

Suppose v_{13} has a pebble on it. Then clearly the vertices of S_1-$\{v_1\}$ contain at most one pebble each and the vertices of S_2-$\{v_2, v_{12}\}$ contain at most three pebbles each. Thus in any distribution, $J_{2,6}$ contains at most eleven pebbles so that a pebble could not be moved to v_1-a contradiction to the total number of pebbles placed over the vertices of $J_{2,6}$. So, assume that v_{13} has zero pebbles on it.

Suppose v_2 has a pebble on it. Clearly, the neighbor v_3 contains at most one pebble and the vertex v_4 contains at most three pebbles. If v_5 has two or three pebbles, then the vertices of S_1-$\{v_1, v_5\}$ contain at most one pebble each. Thus, one of the vertices of $\{v_8, v_{10}\}$ contains at least four pebbles and hence, we can move a pebble to v_1 through v_{13}-a contradiction to case(ii). So, assume that v_5 has at most one pebble. In a similar way, we may assume that the vertices v_7, v_9, and v_{11} contain at most one pebble each. Thus, one of the vertices of $\{v_6, v_8, v_{10}\}$ contains at least six pebbles, say v_8. If one of the neighbors of v_8 contains one pebble then we can move a pebble to v_1 through v_{13}-a contradiction to case(ii). Otherwise, v_8 contains at least eight pebbles. So, we can move a pebble to v_1. So, assume that v_2 has zero pebbles on it. In a similar way, we may assume that the vertex v_{12} has zero pebbles.

Suppose v_3 has two or three pebbles on it. Clearly, the vertices of S_1-$\{v_1, v_3\}$ contain at most one pebble each and the vertices of S_2-$\{v_2, v_{12}\}$ contain at most three pebbles each. Also, if a vertex of S_2-$\{v_2, v_{12}\}$ contains two or three pebbles then the proceeding and succeeding vertices in S_2 contain at most one pebble. Thus in any distribution, $J_{2, 6}$ contains at most eleven pebbles so that a pebble could not be moved to v_1-a contradiction. So, assume v_3 has at most one pebble. In a similar way, we may assume that, the vertices v_5, v_7, v_9, and v_{11} contain at most one pebble each. Clearly, the total number of pebbles on the vertices v_4, v_6, v_8, and v_{10} is at least sixteen. Thus, two of the vertices of $\{v_4, v_6, v_8, v_{10}\}$ contain four or more pebbles each or one of the vertices $\{v_4, v_6, v_8, v_{10}\}$ receives at least eight pebbles. Thus, we can always move a pebble to v_1-a contradiction to case(ii). So, our assumption (case(ii)) is wrong.

Case (iii) Suppose we cannot move a pebble to a vertex of S_2. Without loss of generality, let v_2 be the target vertex.

Clearly, the neighbors v_1, and v_3 of v_2 contain at most one pebble each and the vertices v_4, v_{12}, and v_{13} contain at most three pebbles each. The vertices of $S_1 - \{v_1, v_3\}$ contain at most seven pebbles each and the vertices of $S_2 - \{v_2, v_4, v_{12}\}$ contain at most fifteen pebbles each. Also, note that, no two vertices of $S_1 - \{v_1, v_3\}$ contain four or more pebbles each and if a vertex of $S_1 - \{v_1, v_3\}$ contains four or more pebbles then the neighbors of that vertex contain at most three pebbles each. Also, no two vertices of $S_2 - \{v_2, v_4, v_{12}\}$ contain eight or more pebbles each and if a vertex of $S_2 - \{v_2, v_4, v_{12}\}$ contains eight or more pebbles then the neighbors of that vertex contain at most three pebbles each.

Suppose v_3 has a pebble on it. Clearly, the vertices v_4 and v_{13} contain at most one pebble each and at most one of the vertices of $S_1 - \{v_1, v_3\}$ contains two or three pebbles. Thus, the total number of pebbles on the vertices v_6, v_8, v_{10}, and v_{12} is at least twelve. So, in any distribution of these twelve pebbles on the vertices v_6, v_8, v_{10}, and v_{12}, we must have the following cases: either one of the vertices receives eight or more pebbles or two of the vertices of $\{v_6, v_8, v_{10}, v_{12}\}$ receive four or more pebbles each or all the vertices receive two or three pebbles each. So, we can move a pebble to v_2. So, assume that v_3 has zero pebbles on it. In a similar way we may assume that v_1 has zero pebbles on it.

Suppose v_{13} has two or three pebbles on it. Clearly the vertices of S_1-$\{v_1, v_3\}$ contain at most three pebbles each and no two vertices of S_1-$\{v_1, v_3\}$ contain two or three pebbles each. Thus, we can move at least two pebbles to v_{13}, since the total number of pebbles on the vertices of $\{v_4, v_6, v_8, v_{10}, v_{12}\}$ is at least thirteen (note that the vertices v_4 and v_{12} contain at most one pebble each). So, assume that v_{13} contains at most one pebble.

Suppose v_4 has two or three pebbles on it. Then, the vertex v_5 contains at most three pebbles and the vertex v_{13} contains at most one pebble. Thus the total number of pebbles on the vertices of $\{v_6, v_8, v_{10}, v_{12}\}$ is twelve. So, we can move a pebble to v_3 and then move a pebble to v_2 using the pebbles at v_4-a contradiction to case(iii). So, assume that v_4 has at most one pebble on it. In a similar way, we may assume that v_{12} has at most one pebble.

Suppose v_7 has four or more pebbles (at most seven pebbles) on it. Then the total number of pebbles on the vertices $\{v_5, v_7, v_9, v_{11}\}$ is at most ten. Thus the total number of pebbles on the vertices of $\{v_6, v_8, v_{10}\}$ is at least nine. If v_{13} has one pebble on it then we can move a pebble to v_2 through v_{13} using the pebbles at v_7. So, assume that v_{13} has zero pebbles on it. Also, if v_7 contains six or seven pebbles then we can move a pebble to v_2. So, assume that the vertex v_7 contains at most five pebbles. Thus the total number of pebbles on the vertices of $\{v_6, v_8, v_{10}\}$ is at least nine. If one of the vertices of $\{v_6, v_8, v_{10}\}$ contains two or three pebbles then we can move a pebble to v_2. Otherwise, it is easy to see that, we can move a pebble to v_2. So, assume v_7 contains at most three pebbles. In a similar way, we assume that the vertices v_5, v_9, and v_{11} contain at most three pebbles each.

If the four vertices v_5, v_7, v_9 and v_{11} contain two or three pebbles each then clearly we can move a pebble to v_2. So, assume that at most three of them contain two or three pebbles each. Suppose the vertices v_5, v_7, and v_9 contain two or three pebbles each then the total number of pebbles on the vertices of $\{v_6, v_8, v_{10}\}$ is at least nine. Thus we can move a pebble to v_2 easily. Next assume that any two of the vertices of $\{v_5, v_7, v_9, v_{11}\}$ contain two or three pebbles each. Then the total number of pebbles on the vertices of $\{v_6, v_8, v_{10}\}$ is at least ten. If v_{13} contains one pebble then clearly we can move a pebble to v_2. Otherwise, we can move a pebble to v_2 easily according to the distribution of eleven pebbles at v_6, v_8, and v_{10}.

Assume one of the vertices from the set $\{v_5, v_7, v_9, v_{11}\}$ contains two or three pebbles on it. Then the total number of pebbles on the vertices of $\{v_6, v_8, v_{10}\}$ is at least thirteen. So, we can move a pebble to v_2 if v_{13} contains one pebble on it. If not, then the total number of pebbles on the vertices of $\{v_6, v_8, v_{10}\}$ is at least fourteen. Thus, one of the vertices of $\{v_6, v_8, v_{10}\}$ contains at least four pebbles, say v_8.

Suppose v_8 has six or seven pebbles. If one of the neighbors of v_8 contains one pebble, then we can move a pebble to v_2. Otherwise, the total number of pebbles on the vertices of $\{v_6, v_8, v_{10}\}$ is at least sixteen. So, we can move a pebble to v_2. Thus assume v_8 contains at most five pebbles. Then clearly we can move a pebble to v_2, since the total number of pebbles on the vertices of $\{v_6, v_8, v_{10}\}$ is at least fourteen. So, assume that v_8 contains at most three pebbles. In a similar way, we may assume that the vertices v_6 and v_{10} contain at most three pebbles each. But, we get a contradiction to the total number of pebbles placed on $J_{2,6}$. So, assume that the vertices v_5, v_7, v_9 and v_{11} contain at most one pebble each. Thus, the total number of pebbles on the vertices of $\{v_6, v_8, v_{10}\}$ is at least fourteen. A similar argument shows that v_8 contain at least four pebbles and so we can move a pebble to v_2. So, our assumption is wrong.

Thus, $f(J_{2,6}) \leq 21$.

Therefore, $f(J_{2,6}) = 21$.

Theorem 2.7 For the Jahangir graph $J_{2,7}$, $f(J_{2,7}) = 23$.

Proof Put fifteen pebbles at v_6, three pebbles at v_{10} and one pebble each at v_8, v_{14}, v_{12}, and v_{13}. Then, we cannot move a pebble to v_2. Thus, $f(J_{2,7}) \geq 23$.

As we argued in previous theorems, we can show that $f(J_{2,7}) \leq 23$.

Hence $f(J_{2,7}) = 23$.

Conjeture 2.8 For the Jahangir graph $J_{2,m}$ $(m \geq 8)$, $f(J_{2,m}) = 2m + 10$.

Conclusion. In this paper, we have computed the pebbling number for Jahangir graph $J_{2,m}$ where $3 \leq m \leq 7$. We strongly believe that the pebbling number of the Jahangir graph $J_{2,m}$ where $m>7$ is $f(J_{2,m}) = 2m + 10$.

References :

[1] F.R.K.Chung, Pebbling in Hypercubes, SIAM J. Discrete Mathematics 2 (1989), 467-472.

[2] David Moews, Pebbling Graphs, Journal of Combinatorial Theory, Vol 55, 1992, No. 2, 244-252.

[3] G. Hurlbert, Recent progress in graph pebbling, graph Theory Notes of New York, XLIX (2005), 25 – 34.

[4] D. A. Mojdeh and A. N. Ghameshlou, Domination in Jahangir Graph $J_{2,m}$, Int. J. Contemp. Math. Sciences, Vol. 2, 2007, No. 24, 1193 – 1199.

[5] L. Pachter, H. S. Snevily and B. Voxman, On Pebbling Graphs, Congr. Numer.107 (1995),65-80.

[6] C. Xavier and A. Lourdusamy, Pebbling numbers in Graphs, Pure Appl.Math.Sci., 43 (1996), no. 1-2,73-79.