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# The Pebbling Number of 4-star Graph 

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#### Abstract

A pebbling move on a graph G consists of taking two pebbles off one vertex and placing one pebble to an adjacent vertex. The pebbling number of a connected graph $\mathrm{G}, \mathrm{f}(\mathrm{G})$, is the least n such that any distribution of n pebbles on G allows one pebble to be moved to any specified but arbitrary vertex by a sequence of pebbling moves. In this paper we will determine the pebbling number of 4-star graph.


## 1.Introduction

One recent development in graph theory, suggested by Lagarias and Saks, called pebbling has been the subject of much research and substantive generalizations. It was first introduced into the literature by Chung [1], and has been developed by many others including Hulbert, who published a survey of pebbling results in [5]. Given a connected graph G , distribute k pebbles on its vertices in some configuration, C. Specifically, a configuration on a graph G is a function from $\mathrm{V}(\mathrm{G})$ to $\mathrm{N} \mathrm{U}\{0\}$ representing an arrangement of pebbles on G . We call the total
number of pebbles, k , the size of the configuration. Apebbling move is defined as the simultaneous removal of two pebbles from some vertex and addition of one pebble on an adjacent vertex. Chung [1] defined thepebbling number of a connected graph, which we denote $f(G)$, as follows : $f(G)$ is the minimum number of pebbles such that from any configuration of $f(G)$ pebbles on the vertices of $G$, any designated vertex can receive one pebble after a finite number of pebbling moves.

There are many known results in [5] regarding $f(G)$. If one pebble is placed at each other vertex than the target vertex, $v$, then no pebble can be moved to $v$. Also, if $w$ is at distance of d from v and $2^{\mathrm{d}}-1$ pebbles are placed at w , then no pebble can be moved to v . Thus, we have $\mathrm{f}(\mathrm{G}) \geq \max \left\{\mathrm{n}(\mathrm{G}), 2^{\operatorname{diam}(\mathrm{G})}\right\}$, where $\mathrm{n}(\mathrm{G})$ denotes the number of vertices in G . and diam(G) denotes the diameter of $G$. Graphs $G$ that satisfy $f(G)=n(G)$ are calledClass 0 graphs and graphs $G$ that satisfy $f(G)=n(G)+1$ are calledClass 1 graphs[2]. Class 0 graphs include the complete graph $K_{n}, n$-cube $Q_{n}[1,9]$, complete bipartite graphs $K_{m, n}$ [10], the product graph $\mathrm{C}_{5} \mathrm{X}_{5}$ [4] and many others. We find an elegant characterization about Class 1 graphs in [5]. The path $P_{n}[10]$, $n$-cube $Q_{n}[1,9]$, even cycle [9,10] are examples of graphs $G$ that satisfy $f(G)=2^{\text {diam(G) }}$, whereas the odd cycle $[9,10]$ is an example of a graph not satisfying either lower bounds. Another interesting result is the pebbling number of a tree, which is beautifully worked out in [8]. Hulbert [5] has written an excellent survey article on graph pebbling. Note that pebbling number does not exist for a disconnected graph. Throughout this paper, G will denote a simple connected graph. We now proceed to find the pebbling number of the 4 -star graph.

## 2. n-star graph

A formal group theoretic model called the Cayley Graph has been introduced in the literature for designing and analyzing symmetric interconnection networks. The two important members of this class are the star graph and the hypercube. An n-dimensional hypercube or n-cube, consists of $2^{\mathrm{n}}$ vertices labeled by ( 0,1 )-tuples of length n . Two vertices are adjacent if their labels are different in exactly one entry. Chung [1] proved that the n-cube satisfies $f\left(Q_{n}\right)=2^{n}$. This paper explores the pebbling number of 4-star graph. We were particularly intrigued by n -star graph since it has fewer interconnecting edges.

Definition : 2.1 [6] An n-star graph, denoted by $\mathrm{S}_{\mathrm{n}}$, is an undirected graph consisting of n ! vertices labeled with the $n$ ! permutations on $n$-symbols (we use symbols $1,2,, 2 n$ ) and such
that there is an edge between any two vertices if and only if, their labels differ only in the first (left most) and in any (one) other position.

Recursive Construction 2.2 [6] $\mathrm{S}_{\mathrm{n}}$ can be recursively constructed from n copies of $\mathrm{S}_{\mathrm{n}-1}$ as follows:

We first construct $n$ copies, $G_{1}, G_{2}, 2, G \quad{ }_{n}$, of $S_{n-1}$ and label each $G_{i}$ using all symbols 1 through $n$ except symbol $i$; then for each label in $G_{i}$ we add symbol $i$ as the last symbol (rightmost) in that label (or in any other fixed position); finally, we connect by an edge every pair of vertices $u$ and $v$ such that label of $v$ is obtained from that of $u$ by exchanging the first and last symbols of $u$.

Partitioning 2.3 [6] The n -star can be partitioned in $\mathrm{n}-1$ different ways into n copies of ( $\mathrm{n}-1$ ) stars. The different ways correspond to different symbol positions in the labels. For each symbol position $i$ other than the first position (left most position) we can partition $\operatorname{Sin}_{\mathrm{n}} \mathrm{inton}$ copies of ( $\mathrm{n}-1$ )-star denoted $1_{\mathrm{i}}, 2_{\mathrm{i}}, ? \mathrm{n} \quad \mathrm{i}$. Each $\mathrm{k}_{\mathrm{i}}$ contains all the vertices of $\mathrm{S}_{\mathrm{n}}$ with symbol k in the i -th position of their labels. If however we try to partition along the first position, we obtain $n$ collections of ( $\mathrm{n}-1$ ) ! isolated vertices, Figure 2.1.1 illustrates the partitioning of a 4star into four 3-stars (each 3-star is an hexagon) along the fourth position (rightmost). The four 3 -stars are denoted $1_{4}, 2_{4}, 3_{4}$ and $4_{4}$


Figure 2.1.1.: The 4 -star viewed as four interconnected 3 -stars

Theorem 2.4. The pebbling number of 3-star is $f\left(\mathrm{~S}_{3}\right)=8$.
Proof. Clearly 3-star is the cycle with six vertices i.e. C.
Therefore $\mathrm{f}\left(\mathrm{S}_{3}\right)=8$ [7].
Theorem 2.5 [7]. The t-pebbling number of the cycle $\mathrm{C}_{2 \mathrm{k}}$ is $\mathrm{f}_{\mathrm{t}}\left(\mathrm{C}_{2 \mathrm{k}}\right)=2^{\mathrm{k}} \mathrm{t}$.
Definition 2.6. We say that two vertices of $S_{3}$ are opposite to each other if they are at a distance of three from each other.

Clearly there are three pairs of opposite vertices in S.
We include some facts here, most of which are quite straightforward and can be easily verified.

Let $n$ be the number of pebbles distributed on the vertices of $S_{3}$ and let $(u, v)$ be a pair of opposite vertices in $\mathrm{S}_{3}$.

## FACTS

1. If $\mathrm{n}=7$ then either u or v can be 2-pebbled
2. If $\mathrm{n}=10$ and if u cannot be 2-pebbled then v be can be 4 -pebbled.
3. If $\mathrm{n}=11$ and if u cannot be 2 -pebbled then v can be 8 -pebbled.
4. If $\mathrm{n}=12$ and if u cannot be 2 -pebbled then v can be 10 -pebbled.
5. If $\mathrm{n}=13$ then either u or v can be 4 -pebbled.
6. If $\mathrm{n}=14$ and if u cannot be 2 -pebbled then v can be 12 -pebbled.
7. If $\mathrm{n}=15$ and if u cannot be 4 -pebbled then v can be 6 -pebbled.
8. If $\mathrm{n}=18$ and if u cannot be 4 -peebled then v can be 8 -pebbled.
9. If $\mathrm{n}=8$ and if u cannot be 2-pebbled and v can be 2-pebbled but cannot be 4-pebbled, then either $u_{1}$ can be 2-pebbled or $v_{1}$ can be 4-pebbled where $\left(u_{1}, v_{1}\right)$ is a pair of oppo site vertices such that $u_{1}$ is adjacent to $u$ and $v_{1}$ is adjacent to $v$.

Next, we find the pebbling number of $S_{4}$.

## 3.The pebbling number of $\mathbf{S}_{4}$

For our convenience, we represent $\mathrm{S}_{4}$ as in Figure 3.1.1.


Figure 3.1.1: $\mathrm{S}_{4}$
In Figure 3.1.1 the four copies of $\mathrm{S}_{3}$ are represented by $\mathrm{G}_{\mathrm{i}}, \mathrm{i}=1,2,3,4$ and the vertices of $\mathrm{S}_{4}$ are represented by $\mathrm{x}_{\mathrm{ij}}, \mathrm{i}=1,2,3,4 ; \mathrm{j}=1,2,2,6$.

Definition 3.1 [3]. Given a pebbling of G , a transmitting subgraph of G is a path $\mathrm{x}_{0}, \mathrm{x}_{1},,{ }^{2} \mathrm{x}_{\mathrm{k}}$ such that there are at least two pebbles on $x_{0}$ and at least one pebble on each of the other vertices in the path except possibly $x_{k}$. In this case, we can transmit a pebble from $\mathrm{x}_{\mathrm{t}}$ to $\mathrm{x}_{\mathrm{k}}$.

Theorem 3.2. The pebbling number of $S_{4}$ is $f\left(S_{4}\right)=4!+2$.
Proof : Let the target vertex be $\mathrm{x}_{31}$ of $\mathrm{G}_{3}$.
First, we prove $f\left(S_{4}\right) \geq 26$.
We consider the distribution of twenty five pebbles on the vertices of $\mathrm{S}_{4}$ as follows:
We place fifteen pebbles on $x_{44}$ and one pebble each on every vertex of $G_{2}$ except $x_{16}$ and one pebble each on every vertex of $\mathrm{G}_{2}$ other than $\mathrm{X}_{23}$ and we place zero pebbles on the rest of the vertices of $S_{4}$. In this distribution we cannot move a pebble to $x_{31}$.

Next, we prove $f\left(S_{4}\right) \leq 26$.
Suppose we distribute twenty six pebbies on the vettices of $S_{\text {d }}$. Lef $P$, denote the number of pebbles distrinitiod on the verticer at $G$, und lef $F_{i}$ denote the number of pebblen on $x_{i 1}$ initiaily. We preve:fixji, $S_{4} \mid \leq 26$. Then by symmetry, fi $\mathrm{x}_{( }\left(5_{4}\right) \leq 26$. That, it Sollows that $\left(S_{3}\right) \leq 26$. If $\beta_{1} \geq 8$ then we pebtre the target by Theorem 2.1.4. So, we take $\mathrm{F}_{5}<8$. Withour less of generality, we
 several steps. We brtakidown the possible configitation of twenty six petbles on Ss acconding to the distribumon of pehbles on $G_{3}$. There are eight sleps and we take $\mathrm{F}_{n}=k+1$ in the $k^{*}$ step. Hach ntep invalves sevetal eares and in coch chase we fix $\mathrm{P}_{4}$ und then as $\mathrm{P}_{1}+\mathrm{P}_{2}=26-\mathrm{P}_{3}=\mathrm{P}_{2}$ we consider the cases for each pait $\left(\beta_{1}, \beta_{2}\right)$, auch that $P_{1} \geqslant P_{2}$, fince similar procedure follows if we choose ( $p_{i}$, Fi) sech thut $p_{2} \geq p_{1}$
Noration 2.2.3. Let an deniste fhe number of pebble we piace on $x_{/ A / k}$ after
 $\mathrm{G}_{g_{g}}$ initially, then we write $\left.\left(x_{h / 1}, x_{h_{2} / 2}, \ldots, x_{i_{k}}, \ldots, x_{p}\right)_{r}\right)=\left(a_{1}, a_{2}, \ldots\right.$ a...., a), where $1 \leq 6 \leq 4,1 \leq k \leq 6$.

Step I: Let $F_{i}=0$
Case I (i). Let $P_{3} \geq 16$
We move at least two pebbles to $x_{41}$ by Theoreim 2,15 and so we move a pebble to $\mathrm{X}_{3}$.

Case 1 (ii) Lef $\mathrm{F}_{4}$ te etither fourtein or fittech.
Clanty $\beta_{1} \geq 6$ and so $P_{2} \leq 6$
 in the raulting destributinin of at least fiftoen pubbles on $\mathrm{G}_{2}$, we move fwo pebbles to Nur and so we are dome. If not, then $P_{1}-6$ and $P_{2}$ is cither five or
six. If a pehble is moved to $G_{1}$ from $G_{2}$ then we. ged seven pebbles in $G_{1}$ Int so we are done. Or, if a pebble is moved to $G_{4}$ either from $G_{1}$ ir from $G_{2}$ we are done, Onherwise, note that either $P_{44} \geq 12$ or we get twelve pebbles on $x_{44}$ by Face 6. Therefore, if a pebble is moved to G , either from G we move six pethiles to $x_{4}$ from $x_{4}$ und then we move a pebble to $x_{4}$. If not, then $P_{\pi}=1$ for every $j, 1 \leq j \leq 6$ if $k_{2}=6$ and $P_{31}=1$ except for anc $j_{1} 1 \leq j \leq$ 6 if $R_{2}=5$ if $P_{2}=1$ for every $j, 1 \leq j \leq 6$, then wo move a pebble to $x_{32}$ from $\mathrm{G}_{4}$ using at most four pebbles and so we place two pobbles on $x_{23}$ and then we move a pebble to $x_{22}$ - Now, as $x_{41}$ has at least eight pebbles affer the above thinve we add one more pebble fo $x_{2}$ and wo pebble the target: Now, let $P_{5}=$ I except for one $J_{1} \mid \leq j \leq 6$. Suppose we move a pebble to $G_{3}$, from $G_{s}$ then we get six pebbles vo $G_{2}$. We proceed as before to get two pebilles on $x$ xo Suppose not. Then $P_{t i}=1$ for every $1,1 \leq 1 \leq 6$. We putit two pebbles on $x_{2 s}$ as befors and wa petible the tragec.
Case 1 (iii), Let $P_{4}$, be cither twetve or ithirtem.
Clearly $P_{1}>7$ and $P_{2}<7$. If $\left(x_{4}\right)=(2)$, we are donie, Whiot, then $\left(x_{46}\right)=(10)$ by Fuet 4, Suppose either $F_{24}=12$ or $\left(x_{42}\right)=(12)$, after a move, we move six pebbles to $x_{74}$. As $\alpha_{1} \geq 7$, we mong a pebble to either $x_{n 1}$ or $x_{31}$ from $G_{1}$ and so we mive a pebbile to $x$, .
Suppose not Fitst, we take $F_{1}-7$. We move a pebbie to $G_{4}$ from $\bar{G}_{2}$. As either $P_{2}=6$ or $P_{3}=7$ we move a pebbic to $G_{4}$ from $G_{2}$ if $P_{:}=7$. Now in the tesulting distribution of fourtien pebbles on Gi, slearly we mave two pebbics to $x_{\text {gi }}$
Next, let P1: 28 . Suppose $\left(x_{15}\right)=(2)$ or $\left(x_{2}\right)=(4)$, we move either (me pebfile to $x_{10}$ of two pebbles to $x_{0}$. In the tesuhting distribution we muve two pebbles to . yt. Suppose not. Then $\left(x_{i} e\right)=(2)$ or $\left(X_{1}\right)=(4)$ by Fact 4. Then we thove
cithen ome pebble to $x_{w}$ or tiwo pebbles to $\pi \mu$. Now, since $\left(x_{44}\right)=(10) x_{17}$ Wb move four pebblas to $x_{36}$ frimi $x_{66}$ and en we potble she farget.
Case 1 (iv). Let $8 \leq P_{4} \leq 11$.
Cicarly $P_{1} \geq 8$.
Suppnse $\left(x_{4}\right)=8$.
If $p_{1} \geq 10$, then cither $\left(x_{+N}\right)=\{2)$ or $\left(x_{12}\right)=\{4\}$ und so we move cither ans pebble to $x_{\text {pit }}$ on two pebbies to $X_{i j}$. Now, we mave four pebblen to $x_{34}$ from $X_{4+}$ and so we are done If not, then $p_{1}=8$ of $p_{1}=9$.

Let $F_{T}=$ \& . Then one of the followise holds:
(i) $P_{i}=11$ and $P_{2}=7$.
(ii) $\mathrm{P}_{4}=10$ and $\mathrm{F}_{2}=8$ :

Now, in $\mathrm{G}_{1}$, if $\left(x_{12}\right)=(2)$ or $\left(x_{21}\right)=(4)$, we are done as before Or, in $\mathrm{G}_{2}$, if Either $\left(K_{23}\right)=(2)$ or $\left(x_{2}\right)=(4)$ then we are done. If mot, then, either $\left(x_{2} y\right)=(2)$ or
 either $\left(x_{23}\right)=\left(21\right.$ or $\left(x_{2}\right)=(4)$ if (i) holds (Fact 9). Now, we move the above pebbles to 6 and in the resulting distributisu we place iwo polshles on $x_{41}$ :
Now let $H_{1}=9$. Then $\sigma \leq F_{2} \leq 9$. If $P_{2} \geq 7$ then we move a pebble to $C_{n}$ and mow $G_{1}$ lus ten pebbles and so we we dote as befote: If not, then $\beta_{2}=6$. Also $\mathrm{B}_{4}=11$. If enther $\left(x_{2}\right)=(2)$ or $\left(x_{3}\right)$ - (4), we moye the above pebbles to Gi and then we move four pebbles to $x_{3}$ from $x_{31}$ and we pebible the target. Os. if we move a pebble to $G+$ frum $G+$ thif we are done. Suppose we move a pebble to $G_{4}$ trom $G_{2}$. If cither $\left(x_{1 n}\right)=(2)$ or $\left(x_{10}\right)=$ (4), we move the above pehbles to $G_{2}$. Now, we move four pebblen to $x_{14}$ from $x_{44}$ and we pebhic the target. If $n 96$, is $p_{1}=9$, by Fact 9 cither $\left(x_{11}\right)=(2)$ or $\left(x_{t 2}\right)=(4)$, we move the above pebbles to Ci4. In the tesulting destribution we place twos pebbles on $x_{41}$ Suppose not. We moye one pebbje in $G_{1}$ from $G_{1}$ and clearly we place two
pebbles on $\mathrm{x}_{3 s}$ and so we move a pebble to $\mathrm{x}_{12}$ - Another pebtble is moved to $\mathrm{x}_{32}$ using the ejpht pebbliss on $x_{4}$ and sio we pebble the turget.

## Suppose cight prebbles cannot be placed on $x_{a}$ -

Clearly $F_{6} \leqslant 10$ by Fact 3. If $\left.\left(x_{1}\right), x_{2}\right)=(4,2)$ we wat the trunsmittimg mubgraph $\left\{x_{21}, x_{22} x_{31} \mid\right.$ Or if $\left(x_{2 n} x_{46}\right)-(4,2)$ we get the tranminting suhgiuph $\left\{\mathrm{x}_{35}, \mathrm{x}_{36}, \mathrm{x}_{31}\right\}$.
If not, fing we take $\beta_{4}=10$. Therefore $P_{1} \geq 8$. If $\beta_{1} \geq 10$ then $\left(x_{12}\right)=\{2)$ or $\left(x_{22}\right)=(4)$ and so we mose efthicr ane pebble ta $x_{4}$ or two petables to $x_{6=}$. In the resultung distribution we place two pebbles on Xaw if: $F_{1}=8$ then $F_{2}=8$ and if $F_{1}=9$ then $F_{2}=8$. So, wh move two pabbier io $G_{+}$one fovm $G_{1}$, umther from G2. In the resalting distribition of twelve pebbles in $\mathcal{C}_{4}$ we miove tw) pebbles to $x_{6}$.
Now. lat $B_{i}=9$, Fint, we take $B_{Y}=9$. Thenefine $H_{2}=8$. (If cither $\left.\left(x_{1}\right)=-4\right)$ or $\left(x_{e-1}\right)=(4)$ then we miove at pebble to xu : Another pebble is added to $\mathrm{x} u \mathrm{e}$ Using the dine pebbles on $G_{4}$. Or. if either $\left(x_{1 k}\right)=(2)$ or $\left(x_{12}\right)=(4)$, we move these jetbiles to $G_{3}$ and we mave one more pebble to $G_{4}$ from $G_{2}$. In the resultug distribution, chearly we plate two pebbles on Xai, Stumfarly, if eithor $\left(x_{22}\right)=(2)$ 아 $\left(x_{5}\right)=(4)$, we cmi place two pehbles ont $x_{41}$ - Otherwise, ome cyil canilly check that by moving us miny pebbles as possible either $G_{1}$ frutu $G_{2}$ and G4, or to $G_{2}$ from $G_{1}$ and $\mathrm{G}_{3}$, we pehble the taract, Next, we ratic: $F_{1}=10$. Therefare $F_{2}=7$. Now, we mbve und pebble to $G_{4}$ froin $G_{2}$ and so we get ten pehbled on $G_{4}$ Now, both $G_{1}$ and $G_{2}$ ench bas ten pebbles and this fies been already discussed Next, we tale $\mid P_{i} \geq 11$. Or, if either $\left(x_{i s}\right)=(4)$ or $\left(x_{13}, x_{2}\right)$ (2, 2) or $\left(x_{12}\right)-(6)$ we move the above pohbles $10 \mathrm{G}_{3}$. It the resulting distribution wo piace two pebbles on xis. If not, cteatly by moving us matny pebbles as possible either to $G_{2}$ from $G_{1}$ and $G_{4}$ or to $G_{1}$ from $G_{3}$ and $G_{4+}$ we pebble the tirgast,

Lea Fi $\geq 8$. Chearly F $_{3} \geq 9$. Firat, We assume four pohbles cannot be moved to Ka4. If $\mathcal{F}_{2} \geq 7$ then we move wo pebhles to $G_{4}$ anc from $G_{8}$ and amother from Cy, and then we move fwo pehbles to $x_{2}$. If nit, then $F_{1} \geq 12$. Suppose We get ut least seven pebbles on the path $\left\{x_{21}, x_{0}\right\}$. As $\left(x_{12}\right)=(2)$ (itherwise $\left(x_{12}\right)=(10)$ and 50 we are donc) we move one pebbie to $x_{4}$ ) and sowe place two pebbles on $x_{\text {ar }}$. Suppose foat. Then as $\left(x_{\mathrm{rit}}\right)=(2)$ (otherwise, $\left(x_{12}\right)=(10)$ and we move five piebblen to $x_{r e}$ and then we fnove two pebbles to $x_{4 i}$, and we meve one pebble to: $x_{4}$ and clearly we place two petbles $00 x_{41}$ Next, we assume $\left(x_{4}\right)=(4)$. If either $\left(x_{T}\right)=(6)$ or if $\left(x_{\Pi,}, x_{21}\right)=(2,2)$ we urt one of the
 whe $P_{1}=9, P_{2}=9$. If either $\left(x_{12}, x_{12}\right)=(2,2)$ or $\left(x_{12}\right)=(6)$ then we move the above pehtites to $K_{4}$ and then we move at lemst vie pebble to $\mathrm{G}_{3}$ from $\mathrm{G}_{2}$ and clearify wo place two pebbles on $x_{4}$. Similarily, if cither $\left(x_{24}, x_{22}\right)=(2,2)$ or $\left(x_{21}\right)$ $\equiv 6$.we proceed as above. Otberwise it can bo easily verified thar by moving as many probles an pussible aither to $G_{1}$ frum $G_{1}$ and $G_{4}$ or to $G_{3}$ from $G_{1}$ and $G_{4}$ the turget can be polbblad. Noxt, we take either $P_{f}=10$ or $p_{1}=11$. If eithet $\left.f \mathrm{x}_{21}\right)=2$ or if $\left(\mathrm{x}_{\mu y}\right)=4$ we move the above pebbies to $\mathrm{G}_{4}$. Thert since either $\left(x_{15}\right)=(2)$ of $\left(x_{12}\right)$ - (4) by Fant 2 , we move there petbles to: $G_{4}$. In the raulting distritution we place two petbies in $x_{1}$. Or if either $\left(\mathrm{K}_{12}\right)=(6)$ of (8.4) - ( 6 ) then we mowe three pebbles to $G_{4}$ and we place two pebbles on $X_{i 6}$. If not, then cleatly the target can be pebbled by moving as many pebbles as: possible sither to $G_{2}$ from $G_{2}$ and $G_{3}$ or to $G_{2}$ from $G_{1}$ and $G_{3}$. Nest, Ist $P_{y} \geq 12$. If either $\left(x_{10}\right)=4$ or $\left(x_{i 1}\right)=6$ we move the above pebbles to $G_{4}$ and We place कौा pebbles on Xo, Otherwise we pebble the target by moving pebbles either to $G_{1}$ or to $G_{2}$ as before.
Case 1 (v). Lel $4 \leq P_{1} \leq 7$.

$$
\text { Clearly } P_{1} \geq 10 \text { and } \beta_{3} \leq 11 \text {. }
$$

## Suppose $\left(x_{i}\right)=[4)$ after nome movesi.

First, we take $F_{1}=10$. Thenefore either $P_{2}=9$ or $P_{2}=10, \quad\left(X_{0}\right)=2$ (otherwise, ( $x_{i},=4$ by Fact 2 and so we iere done) Also, if $\left(x_{21}\right)=(2)$ then we

 So, we asome $x_{y}$ is not 2-pebbled. Thien $\left(\dot{x}_{11}\right)=4$ by Fact 2, Now, if vither $\left(x_{11}\right)=(6)$ or $\left(x_{2 i}\right)=6$, then we are done. Otherwise, clearly $\left(x_{12}\right)=6$ or $\left(x_{i 2}=\right.$ $\left.\delta_{19}\right)=(2,2)$ in $G_{1}$ and in $G_{2}$ cither $\left(\mathrm{x}_{22}\right)=2$ or $\left(\mathrm{x}_{21}\right)=4$. We move the above pebbles to $G_{4}$ and in the resulting distribution we place two pebbles on $\boldsymbol{x}_{4}$.
Next, Let $F_{1} \geq 11$. Therefore $F_{2} \leq 1.1$. Lee $F_{2} \geq 7$. If $\left(x_{2}\right)=2$ then we get the transmittime suthraph $\left\{x_{1+}, x_{32}, x_{34}, x_{11}\right\}_{\text {, since }}\left(x_{13}\right)=2$ by Fat 3, If mot, fhen $\left(X_{3}\right)=2, ~ A 60\left(x_{10}\right)=2$ by Fact 3 and so we get the turaixitting subjataph $\left\{x_{3 i}, x_{3}, x_{y e} x_{4 i}\right\}$. Now, let $P_{2} \leq 6$. If $P_{2}=6$, hen enther $P_{1} \geq 13$ if either $\left(x_{2}\right)=(4)$ or $\left(x_{20}\right)-2$ then we are dote. Or if we move iwo pebbles to $G_{1}$ from C) then we gec at least fiffeen pelables on $G$, and sin we plave eitiver four pobbles on $x_{\text {is }}$ or six pebbles on $x_{u s}$ by Fact 7 and sor we ate done, If not, we move a pebble to $G_{2}$ from $G_{4}$ (since $\left(X_{\mu}\right)=(4)$ ) and clearly we place two pebbles on $x_{2 /}$. Ax $\left(x_{i},\right)=4$ by Fact 5 we ect the मangmitting subgeraph $\left\{x_{31}, x_{\text {ph }} x_{i 1}\right\}$. Now, if $P_{2}=5$ then $P_{12} \geq 14$ If $P_{y}=14$ and if $x_{13}$ is not $6-p$ bbled then either $\left(x_{11}\right)=6$ on $\left(x_{12}\right) E(10)$ or $\left(x_{12}, x_{12}\right)=(2,8)$, we mave the nbove pebbles Io $\mathrm{O}_{4}$ and then clearty we pluce two peltiles on $x_{51}$. If not. Then we move as muny pebbles as possible to $G_{2}$ from $G_{2}$ and $G_{4}$ then we pebble the torget. In all the other distrihutionf, as $P_{1} \geq 15$, cither $\left(x_{10}\right)=4$ ar $\left(x_{2}\right)=6$ by Fact 7 and so we are done

## Suppose four pebbles cannot be placed on $x_{4 u}$

If either $\left(\mathrm{X}_{0}, \mathrm{~S}_{2}\right)=(4,2)$ or $\left(\mathrm{X}_{0 /} \mathrm{x}_{2}\right)=(2,4)$, we get one of the transmitting sthgraph $\left\{x_{21}, x_{1}, x_{11}\right\}$ • $\left\{x_{28}, x_{20}, x_{21}\right\}$.

If not, first we talke $F_{4}=7^{\prime}$. Therefore $K_{1} \geq 10$ and $\beta_{2} \leq 9$. If $F_{2} \geq 7$ we get $F_{i} \leq 12$. Therofore, either $\left(x_{i s}\right)=(2)$ or $\left(x_{2}\right)=4$. We move the ubove pebbles to G. Now, we move ono pebbie to (is from fiss in the resulting dimponton We place two pehbles on $x_{41}$. If $\beta_{2} \leq 6$ theth $\beta_{1} \geq 13$. Supenc $\left(x_{6}\right)=6$. Then vithor $(x ; y)-2$ or $(3,2)-(8)$ by Fact 3 and we move the whove pebbles to Gi4 Now, in the resulng distribumon we place two pebhles on ait. $^{\text {. Suppose not. }}$ Thim, if sithar $\left(x_{16}\right)=(4)$ or $\left(x_{13}\right)=$ ( 8$)$ we are donc If not, cleatly stither $\left(x_{11}\right)=$ (4) or $\left(x_{12}\right)=(6)$ and so we mave the abiove pobbles to $\mathrm{G}_{4}$ and then we place rwo pebbles on Nil-

Next. Ler $F_{A}=6$, Therefore $F_{t} \geq 10$ and $P_{2} \leq 10$. Finn, we uke $P^{\prime} \geq 7$. We
 pebbles to $x_{4}$ Also, we move ono pebble to $\mathrm{G}_{4}$ from $\mathrm{O}_{2}$ and then clearly we
 thove the atvve probilis to $G_{4}$ Now, we mive one pehble to $G_{4}$ from $G_{2}$ and in the resultine distribution we move two pebbles to $x_{4}$. Or if both $\left(x_{32}, x_{2}\right)=$ (2J) wo move these pobbles to $\mathrm{G}_{4}$. Also, sither $(\mathrm{x} i \mathrm{i})=2$ on $\left(\mathrm{x}_{12}\right)=4$ by Fapt 2 and we muve the above pebblas to $G_{4}$ Now, we plate two putbles on $s_{6}$, Otherwise the target can be pebbled by moving as many pisbbles as posaible enther to $G_{1}$ from $G_{2}$ or to $G_{2}$ from $G_{2}$ Nexi, wo take $P_{z} \leq 6 ; \mathrm{So}_{3} P_{i} \geq 14$ : If $x_{3}$ is not I-pebbled thea $\left(x_{21}\right)-(10)$ by Foct 4 and 80 we move five pebbies to $X_{a t}$ und in the resulting distribution of eleven pebbles on $\mathrm{G}_{\text {si }}$ clearly we place two pepbles on $x_{41}$ : So, we ansume $\left(x_{10}\right)=2$. If cuther $\left(x_{11}\right)=6$ or $\left.\left(x_{11}\right)_{-} x_{12}\right\}=$ $(4,2)$ of $\left(x_{13}, x_{12}\right)-(2,6)$. we move these pebbles io $G_{6}$ and in the ressiting distrinition we place two pebbles oni xal. If not, supposes either $\left(x_{21}\right)=(2)$ or $\left(\mathrm{X}_{31}\right)=2$ then we move one pehbie to either $\mathrm{X}_{1}$ or $\mathrm{X}_{14}$. Clearly, in the resultin" ifistribution of fifteat pebbles on $G_{h}$ one of the above bolds Soppoce twot.

Then we miove at least six pebbles to $\mathrm{G}_{2}$ from $\mathrm{G}_{1}$ and we ploce four pebble's on $x=$ and so we are done.
Now, let $\mathrm{K}_{4}=5$. Therefore $\mathrm{F}_{1} \geq 11$ and $\mathrm{K}_{2} \leq 10$. Finst, We assame neithor
 and so we move four pebbles to $\mathrm{X}_{4}$ - Also, we tnove a pebble to $\mathrm{O}_{4}$ from $\mathrm{G}_{2}$ as $F_{2} \geq 7$ whan $F_{1} \leq 14$. Now alearly we place two pebbler on $x_{4 i}$ If $\left(x_{11}\right)=12$ ) then $\left(x_{n}\right)=(2)$ sinnaltumeously, we mowe these petbies to $G_{i}$ and then we move one mase petble to $\mathrm{G}_{6}$ from $\mathrm{G}_{7}$, Now, we place two pebbles on $\mathrm{X}_{4}$. Next, we talke $P_{j} \geq 15$. If $x_{14}$ is not 2 -pebbled then $\left(x_{12}\right)=(12 y$ by Fact 6 and
 then $\left(x_{\square}\right)=(2)$ smmitaneously, we nove two pebbles to $G_{4}$ und clearly we place two pebbles on $\mathrm{Kin}_{1}$. Now, we asoume etither $\left(\mathrm{x}_{81}\right)-(2)$ or $\left(\mathrm{X}_{42}\right)-(2)$, If $\mathrm{F}_{2}$ 27 and if ether $\left(x_{2}, 2\right)=(4) \cdot \operatorname{ar}\left(x_{4}\right)=(6)$, then we-move the above pebbles to $G_{4}$. Also cither $\left(x_{13}\right)=(2)$ or $\left(x_{\square 1}\right)=(4)$ as $P_{1} \geq 11$. We move these pebbies to $\mathrm{G}_{4}$ and in the resuling distribution clatly we place two pebbles on $x_{\text {tr }}$. Or if $\left(x_{\text {ps }}\right.$ $\left.x_{15}\right)=(6,2)$, then we move tbose pebbles to Gui Also, we move al pebble to $G_{4}$ form $G_{2}$ and now we place two pebbles on $x_{i n}$ if $\psi_{2} \leq 6$ then $\beta_{1} \geq 15$, if either $\left(x_{11}\right)=(6)$ on $\left(x_{13}, x_{15}\right)=(8,2)$, then we move these pebbles to $\mathrm{C}_{4}$ and we plase two pebbles on $x_{a 1}$. If all the abovs fait, then we move a pehble cither to G) if $\left(x_{0}\right)=(2)$ or to $\mathrm{G}_{2}$ if $\left(\mathrm{x}_{0}\right)=(2)$. Now, we pebble fie target by moving as mumy pebbles ar possible cither to $G_{1}$ form $G_{3}$ or $\sigma_{0} G_{2}$ from $G_{1}$.
Next, let $\beta_{4}=4$. Clearly $\beta_{1} \geq 11$ and $\beta_{2} \leq 11, \mid f$ cithor $\left(x_{10}\right)=(4)$ or $\left(x_{11}\right)=$ (8) ar $\left(x_{21}\right)=(4)$ ar $\left(x_{3}\right)=(8)$, we are done, Or if either $\left(x_{1+2}, x_{30}\right)=(2,4)$ or $\left(x_{1}, K_{2}\right)=(4,2)$, we get one of the transmitting subgraphs $\left\{x_{34}+x_{3}, x_{11}\right),\left\{x_{31}\right.$ $\mathrm{x}_{2} ; \mathrm{X} \| 1, \mathrm{Or}$, if either $\left(\mathrm{x}_{3}+\mathrm{x}_{124} \mathrm{x}_{5} \mathrm{x}, 7\right)=(2,4,2,4)$ or $\left(\mathrm{x}_{1} 5 ; \mathrm{x}_{2}\right)=(4,4)$, we mare the alove pebbles to $\mathrm{G}_{4}$ and we place two pehbhes on sn: Otherwise, clearly
the tanget can be pebbled only by moving as many pebbles as possible either to Gif $_{6}$ from $\mathrm{G}_{2}$ or to $\mathrm{G}_{2}$ from $\mathrm{G}_{3}$
Case 1 (vi). Let $\mathrm{b}_{1} \leq 3$
Clearly $\beta_{1} \geq 12$
Fiss, we take $p_{1}=12$. Therefore, either $p_{2}=11$ or $p_{2}=12$. $\operatorname{Suppose}\left(x_{r}\right)=$
 Suppose not. Then $x_{i s}$ is 2 -pehbled but not 4 -pebbled. Similarly, we get $x_{\text {an }}$ is 2peoblied but not 4-pebbled. Now, elearly we move at heast two pubbles to $G_{1}$ from $G_{2}$ and we place four pebbles on $x_{i n}$ and so we ate done. Next, we take $P_{1} \geq 13$. Therefore $P_{2} \leq 13$. Let $10 \leq p_{2} \leq 13$. $\ln G_{j+}$ if $x_{10}$ is not 4 -pebbled then $\left(x_{i 2}\right)=(4)$ (Fact 5$)$. Therefore, if $\left(x_{y}\right)=(2)$, we get the transminting subgroph $\left\{x_{20}, x_{03}, x_{3}\right\}$. If not, then $\left(x_{x}\right)=(4)$. As $p_{1} \geq 13_{i}\left(x_{i s}\right)=$ (2) (otherwise $\left(x_{11}\right)=(8)$ and so we ate dono). So, we get the transmitting sübigraph $\left\{x_{i 1}, x_{x+} x_{n 1}\right\}$. Next, let $7 \leq \beta_{2} \leq 9$. Then $F_{1} \geq 14, \mathrm{So}_{+}\left(x_{i j}\right)=(4)$ (otherwise. $\left(\mathrm{x}_{\mathrm{i}}\right)=(4)$ and so we are done). If $\left(\mathrm{x}_{2}\right)=(2)$, then we ger the transmitting subgraph $\quad\left(x_{33}, x_{3}, x_{3}\right\}$. If not, then either we move at least one pebfite to $\mathrm{G}_{i}$ front $\mathrm{G}_{2}$, md we put eithur four pebbiles on $\mathrm{x}_{66}$ or eight pebbles an $X_{1 s}$ or we move at leait five pebbles to $G_{2}$ from $G_{1}$ and we put four pebbles on $x$ 分 and so we are done: Next, fot $P_{2} \leq 6$. Then $\beta_{1} \geq 17$, if $\beta_{1} \geq$ 18, then either $\left(x_{20}\right)=(4)$ or $\left(x_{12}\right)=(8)$ (Fact 8$)$ and so we pebble the targat. If $k_{1}=17$ und if we mave u pubble to $G_{1}$ from $G_{7}$ that we are done. If not, clearly we move seven pebbles to $G_{2}$ from $G_{1}$ and we put four pebbles on $x_{23}$ and hence the case is complete.

Step 2. Let $b_{3}=1$.
Care 2 (i). Let $\beta_{4} \geq 15$
If $\mathrm{P}_{4} \geq 15$, then we move seven probles to x 4 and ag we get enght pubthles on $X_{4}$ and so we are done. If not, elcarly $\left(x_{3}\right)=(2)$ and as we pebble the therest.

Case 2 (ii). Let $\mathrm{B}_{4}=14$.
If $\mathrm{P}_{44}=14$ then we mibve weven pebbles to $\mathrm{x}_{\mathrm{in}}$. If $\mathrm{P}_{44} \leq 12$ then clearly $\left(\mathrm{x}_{41}\right)=$
 $P_{1} \geq 6$ and $P_{2} \leq 5$. If $P_{1} \geq 7_{2}$ we move n pebble to $G_{4}$ from $G_{1}$ and in the resulting distribution we move two pebbles to Kaj . If not, then $\$_{i}-6$ and $\mathrm{F}_{2}=$ 5. If we move a pehible to sither $G_{4}$ in $G_{1}$ 而 $G_{1}$ from $G_{2}$ then we ane done. If not, then $\mathrm{F}_{21}=1 \mathrm{except}$ for onk $j, I \leq j \leq 6$. Now, if a pebble is moved $\mathrm{C}_{2}$ from $G_{(, ~ t h e n ~ w e ~ m o v e ~ o n e ~ m o r e ~ p o b b l e ~ t o ~} G_{i}$ from $G_{i}$ aging at most four pebbles. Now, in the reanting fistribution, we plice two pehbles on $x_{r}$ atd so we unde a pebbir to xas As there ar fopht nime pehbits on $\mathrm{X}_{4}$ after the above move, we
 I for every j. I $\leq j \leq 6$. Now, we use at most four pebbles on cith to move a pebble to $x_{11}$ and then we move a pebble to $x_{31}$ fram $G_{3}$. Now, as thare ane nine pobbles on $x_{44}$; we move four pebbles to $x_{4}$ and so we pebble the taryet

Case 2 (iii), Let $\beta$, be either twelve or thirteen.
Chearly $F_{f} \geq 6$ und $\varphi_{1} \leq 6$.
Let $P_{1} \geq 7_{\text {. If }}\left(x_{3}\right)=(12)$, then we move six pebbles to $x_{34}$. From $x_{u 4}$ and we move one mov pobtile to $G_{7}$ from $G$ and so we are done If not, first we take $P_{1} \geq 8$. If eithur $\left(x_{51}\right)=(2)$ or $\left(x_{13}\right)=(4)$, we mave the above pobbles to $G_{2}$. Now, as sither $\left(x_{4}\right)=(2)$ or $\left(x_{4}\right)=(10)$ by Fact 4 , we move fonr petbles to $x_{4}$
 9. We move these pebbles to $G_{4}$ and we place two pehbies on $x_{41}$. Noxt, Iet R1 $=7$. Therofors cittier $R_{2}=5$ ot $\beta_{2}=6$ Let $p_{2}=$ 6. If a pebble is moyed to other $G_{1}$ or $G_{1}$ or $G_{1}$ from $G_{3}$ we are dome. If now, then $P_{2}=1$ for every j. I $\leq 1$
$\leq 6$. We move a pebble to $G_{2}$ from $G_{1}$ and we place two pebbles on $x=1$ und so we move $a$ pebble to $x_{2}=$ Now, we move four pebbles to $x_{4}$ from $x_{4}$ and we are done Let $P_{2}=5$, We mave a pebble to $G_{3}$ Irmin $G_{5}$ and we place two pebbles ofl $x_{41}$. Next, let $k_{1}=6$. Then $k_{2}=6$ and $k_{4}=13$. If we move a pebble either to $\mathrm{G}_{1}$ from $\mathrm{G}_{2}$ or $\mathrm{G}_{2}$ from $\mathrm{G}_{1}$, we are dooe as before, Or, if either $\left(\mathrm{x}_{81}\right)=(2)$ or $\left(x_{10}\right)=(2)$ or $\left(x_{21} x_{11}\right)=(2,2)$, we move the above pebbles in $G_{7}$ and then we Hove flve pebbles to $x_{24}$ from $X_{44}$ and so we are done. Or, if cither $\left(x_{i 1}\right)=(2)$ ot ( $\mathrm{K}_{2}=1$ - (2) or $\left(\mathrm{x}_{22} ; \mathrm{x}_{23}\right)$ - (2,2), we mpve these pebbles to $\mathrm{G}_{4}$ and then clearly we move two pebbles to $x_{10}$ Or. if eitier $\left(x_{24}, x_{11}\right)=(2,2)$ or $\left(x_{20}, x_{12}\right)=(2,2)$, we move one pebble to (iy and one pebble to (i). Now, in $\mathrm{G}_{6}$ either we move two pebbles to $x$, or wo move twelve pebblea to $x$. by Fact 6 amd soo we ure done. If all the above fati, then $P_{\|}=1$ for either for every $\mid-1$ or $i=2$ or $i-1,2,1 \leq \mid \leq$ 6. If either $b_{i s}=1$ or $Q_{3}=1$, we move a pebble to $G_{3}$ using at midest four pebbles from $\mathrm{G}_{3}$ and then we move a pebble to $\mathrm{x}_{\mathrm{y}}$. Now, an there are at loast six pehbles on $x_{24}$, we move three pehbles to $x_{3}$, from $x_{42}$ and so we are done. If not, then we mave a pebble to $\mathrm{G}_{2}$ from $\mathrm{G}_{4}$ and we move a pebble to $\mathrm{X}_{12}$. Nuw, We move three pebbles to $x_{5}$ frome $x_{4+}$ and sat we are done.
Case 2 (iv). Let $k$, be either ten of eleven.

$$
\text { Clearly } p_{1} \geq 7 \text { anil } b_{2} \leq 7
$$

## Supprose $\left(\mathrm{x}_{\mathrm{i}}\right)=(\mathbf{1 0})$.

Finst, we take $P_{1} \geq 7$ and $P_{2} \geq 7$. We move two pebbites io $\mathcal{G}_{2}$, one from $G_{1}$ and another from $G_{2}$ and then we move five pebibles to $x_{3}$ from $x_{4+}$
Next, fet $\beta_{j} \geq 10$. We move sither one pehble to $x_{4}$ or two pebbles to $x_{3 i}$ using Fats 2. Now, we move five petibles to $x_{4}$ from $x_{41}$ and we pebble the target.
Now, we take $k_{1}=8$ and $k_{2}=6.15$ cither $\left(x_{19}\right)=(2)$ or $\left(x_{10}\right)=(4)$,we are done as above. If not, then eithef $\left(x_{11}\right)=[2)$ or $\left(x_{1}\right)=(4)$ thy Fact 8$)$ We move these pebbles to $G_{i}$. only if a pebble is moved to $G_{4}$ from $G_{7}$ and se we move two
pebbles to $x_{4}$. Or, if a pebble to moved to $G_{3}$ fram $G_{1}$ we move one more pebhle to $G_{3}$ from $G_{1}$, and then we nowe five pebbles to $x_{j 4}$ from $x_{i t}$ Or. if two pebhliss are moved to $G_{1}$ from $G_{2}$, we are done If not, we move a pebble to $G_{2}$ frum Gi and clearly we place two pebble on $x_{21}$, mid we move a pebble to $x_{1}$ Nosw, we move five pebbles to $x_{34}$ from $\times 4$ and so we aro done:
Neat, let $\beta_{1}=9$. So, cither $\beta_{2} \equiv 6 \pi \beta_{2}=5$. If a pebbie is moved to cither $G_{1}$ or $G_{3}$ or $G_{\text {, from }} G_{2}$, we are slone. If not, then $P_{2}=1$ at least fin five vaties of j. $1 \leq j \leq 6$. Now, wo move two pebbles to $G_{2}$, one from $G_{5}$ and another from G, As there are at least six pebbles on Xu after the above move, we move three pebbles to $\mathrm{x}_{74}$. Now, in $\mathrm{C}_{2}$. we pit two pebbles on $x_{22}$ and we move a pebble to
 pebbles on $x$ ss and we move a pebble to $x a y$ and so we are done,

## Suppose ten pebbles cannot be placed on $x_{4-}$.

First, we assume $\left(x_{4}\right)=(6)$.
If $P_{1} \geq 10$ then $\left(x_{14}\right)=(2)$ oo $\left(x_{2}\right)=(4)$ and we move the above pethbles to Cis $^{2}$ Now, we move thire petbles to $x_{\mu s}$ from $x_{a s}$ if $\phi_{\mu}=1$ and then we move a pebblit to $x_{31}$. If $\mathrm{B}_{4} \neq 1$ and if $\left(x_{i 4}\right)=(8)$, we inove four pebbles to $x_{4}$ and so
 from $G_{1}$ using Fact 2 and we plone two pebbles on $x_{\pi}-$
Suppose $b_{3}=9$. Let $b_{\mu}=1$ If eithar $\left(x_{14}\right)=(2)$ or $\left(x_{1} \mid\right)-(4)$ flien we ars dorie as above. If not, cither $\left(x_{i 1}\right)=(2)$ or $\left(x_{j 2}\right)=(4)$ by Fact 9 und wo move the above pebbles to GFe $_{6}$ when $\varphi_{4}=11$ and in the resulting distribution we plate two pebbles on $x_{4}$. Wheti $B_{4}=10, \mathrm{k}_{2}=6$. Il a pebble can be moved either to $G_{i}$ or to $G_{i}$ from $G_{2}$ then we get etther ten pebbten on $G_{i}$ on efoven pebbles on $G_{4}$ and in we proveed ati before. $O$, if a pethble is mowed to $\lambda_{2}$ from $G_{2}$ we are done If rian, we move at least one pebble to $G_{1}$ from $G$ and clearty we plate rwo pebble's on Xe and so we move a pebble to xis - Now we move three pebblts
10. $\mathrm{X}_{4}$ from $\mathrm{x}_{4}$ and sa we pebble to target. Now, (e1 $\mathrm{P}_{34} \neq 1$. Suppose $\left(x_{a}\right)=$ (8). If either $\left(s_{1}\right)=(2)$ of $\left(x_{21}\right)=(4)$, we move shexe pehbles to GF and then we move four pebbles to $x_{34}$ frimi $x_{44}$ and so we aro done. If not, firse we take $k_{2}=$ 5. As: (Xis) $=(2)$ or $\left(\mathrm{X}_{1}\right)=$ (4) by Fact 9 , we mpve these pebbles to $\mathrm{G}_{1}$ and we place two pebbles on $x_{11}$. Now, let $\beta_{2}=6$. If a pebble is moved to either $G_{2}$ of Gy or $G_{2}$ from $\mathrm{O}_{2}$ we are done as above. Hnot, $\mathrm{B}_{2}=1$ for every $\mathrm{j}_{2} 1 \leq j \leq 6: \mathrm{So}_{+}$ wo move a pebble to $G_{2}$ from $G_{1}$ and we place two pehblef on $x_{2}$ and wo move a pebble to $x_{y}$ : Niw, we move fow pebties to $x_{H}$ from $\Sigma_{4}$ and $s y$ are are dome. Suppose eight pebbles cannot be thoved to $x_{4+}$. Thati $\dot{P}_{4}=10$ by Fact 3, We move a getbble to Ga from $G_{i}$ and clearly we plane rwo pebbles on $X_{4 i}$.
 above pebbles to $\sigma_{y}$. Now, we move three pebbles to $x_{4}$ from $x_{11}$ and so wo pebble the targen. If not, by Fact 9 e etifer $\left(x_{8}\right)=(2)$ or $\left.\left(x_{2}\right)=14\right)$ and we nove these pebbles to $\mathrm{G}_{4}$ Abo, if $\boldsymbol{F}_{4}=10$ we move one more pebble to On $_{4}$ from $\mathrm{G}_{2}$ Is $P_{4}=7$-in such caic. Now, in the resulting ditritution we place two pebbles
 are dome. If not, then we move two pehhles to $G_{2}$, noe from $G_{1}$, amother from $G_{2}$ and we pethile the turect if $P_{2}=3$, If $\xi_{1}=6$, then, since either $\left.\left(\xi_{10}\right)=12\right)$ or $\left(\mathrm{X}_{3}\right)=(4)$ by Fact 9, we move these pebbles to $\mathrm{G}_{4}$ and wis place two pebbles on $x_{n}$. Suppose not. Then $P_{1}=10$ by Fact 3 . We move two pobbles to $G_{h}$ one from $G_{1}$ ansher from $G_{2}$ and we place two pebbles on $K$ (r
Finally, if $\mathrm{P}_{1}=7$ then $\beta_{2}=7$ and $\mathrm{F}_{4}=11$. So, we move two pebbles to $G_{A 4}$ one from $\mathrm{Gil}_{1}$ and another from $\mathrm{G}_{2}$ Now, clearly we move two pebbles to $x_{61}$

Now, we ussume we cannot place tix pebbles on $x_{3}$ -
Then, clearly $F_{4}=10$ by Fact 3. As $\mathrm{F}_{1} \geq \mathrm{x}$, we move at least one pebble io $\mathrm{G}_{4}$ from $G_{1}$ and in tha resuling distritution clearly we place two pebbles on $x_{4}$.

Case 2(v) L-at $6 \leq)_{4} \leq 9$.
Liet $\beta_{54}=1$.

## Suppose $(\mathrm{X}+4)=(6)$.

Firsit, we convider the cale $H_{4}=9$ and $P_{2}=R_{\text {. }}$ if enther $\left(x_{14}\right)=(2)$ or $\left(x_{13}\right)=(4)$ of $\left\langle x_{n}\right)-\left(2\right.$ ) or $(\kappa \geqslant)=(4)$ we anove these pebbles to $G_{j}$ and then we move three pehbles $10 x_{34}$ from $x_{i 4}$ tind so we are done. If not, by Fact 9 , cither $\left(x_{0}\right)=(2)$ or $(x \mid y)=(4)$ and eifher $\left(\mathrm{X}_{3}\right)=(2)$ or $\left(x_{2}\right)=(4)$. Now, we move the above pebbles is $G_{4}$ and in the resthtu distribution we move two petbies to Kn- In all the other cases, etither $P_{i}=10$ or the number of pebbles in $\mathrm{G}_{1}$ can be made ten tifter moving a pebble from $\mathrm{G}_{2}$ to $\mathrm{G}_{1} . \mathrm{So}_{4}$ elther $(\mathrm{x}(\mathrm{a})=(2)$ or $(\mathrm{N}, 9)=(\hat{4})$ and wo move these pebblis to $\sigma_{5}$. Now. we move three pebilics to $x_{4}$ from . Far ath then we pebble the target.

## Suppose six pebbles cumnot be placed oo xif

First. We astume $\beta_{4}=9$. Then $p_{1} \geq$ 各 and $\beta_{2} \leq 8$. If $\beta_{2} \geq 7$ shen we move two pebbles to $G_{4}$, one from $G_{2}$ and another from $G_{2}$. In the resulting distribution of eleven pebbies on $G_{12}$ we place two pebbles on $x_{41}$. If $P_{2} \leq 6$ then $P_{1} \geq 10$. So. either $\left(\mathrm{K}_{1}\right)=(2)$ or $\left(\mathrm{X}_{12}\right)=(4)$ and we move these pebbles to $\mathrm{Ci}_{4}$. Now, we pilade two pebbles on X Xj .
Next, let $F_{4}=8$. Therefore, $P_{1} \geq 9$. und $F_{2}, 58$ in follows from Fuet 1 that if $\delta_{a n}$ is mot 2 -petbical sher $\left(x_{4}\right)=(2)$. Suppose $k_{1}=9$ and $k_{2}=8$. If either $\left(x_{1}\right.$ ) $=(2)$ or $\left(x_{i 2}\right)=(4)$, we move these petbles to (ij. Now, we move ont more pebble to $G_{4}$ frum $G_{2}$ - and clearly we place twa pebbles on $x_{4}$. Similurly, if eithor $\left(x_{a}\right)=(2)$ or $\left(x_{21}\right)=(4)$, we proced in tofore Or. if either $\left(x_{i 1}\right)=(6)$ ar
 Su and so we are done, Or, if $\left(k_{1}=x_{r 3}\right)=(2,2)$ or if $\left(x_{1} \in x_{2}\right)=(2,2)$ we get
 then we move at least one pebble to $G_{1}$ from $G_{2}$ and one pebble to $G_{1}$ from $G_{2}$
and we place four pebbies an $x_{10}$ if $F_{1}=10$ then either $\left(x_{n 3}\right)=(2)$ or $\left(x_{12}\right)=(4)$ and wo move these pebbies to $G_{L}$. Aiso; we move a pebble to $G_{4}$ from $G_{1}$ Now, in the resalting distribution we place two pebbles on $\mathrm{X}_{31}$. Next, - let $\mathrm{P}_{1} \geq 11$. Claurfy $\left(x_{i t}\right)=(2)$ (otherwine, $\left(x_{12}\right)=(B)$ and $s 0$ we move four pebbles to $x_{61}$ and ften we place (wo pebbles on $\left.x_{i 1}\right)$. If $\left(x_{i j}\right)$ ) (4) we mowe swo peblies fo $x_{\text {ss }}$ and in the regulting distribution we place two pebbles on $x_{4}$ - If ont, then ( $x_{1 s}$.
 x
Let $P_{4}=7$ Then $B_{1} \geq 9$ and $k_{2} \leq 9$. If $x_{41}$ is not 2 -pehbled then $\left(x_{41}\right)=\{2\}$. Suppose $P_{1}=9$ and $P_{3}=9$. Let $\left(x_{14}\right)=(2)$ if $\left(x_{3}\right)=(2)$ then wo pet the

 detribution is siach that if $\left(x_{1}\right.$ a $=$ (2) then (4) then $x_{t i}$ canmot be 2-pebbled. Now, if $\left(x_{i 1}\right)=$ ( $⿻$ ( $)$ then wes move three pebblet to $G_{4}$ (two pebbles from $x_{1 s}$ and one pebble from $G_{2}$ ) and we place two pebbles on $x_{42}$. Sünilarly, it $\left(x_{22}\right)=(4)$, we are dane. If not, then clearly wo
 from $\mathrm{G}_{4}$ and then we place four pehbles int $\mathrm{N}, \mathrm{s}$ or we move at least three pebbles to $\mathrm{G}_{2}$ frum $\mathrm{C}_{1}$, wid owe more pebble $10 \mathrm{C}_{2}$ frum $\mathrm{G}_{3}$ und we place four pebties on $x_{21} . N e x t$, let $P_{i}=10$. Than $P_{2}=8$. Clearly $\left(x_{12}\right)=(2)$ (otherwise $\left(x_{16}\right)=[4)$ by Fact 4 and so we are dome). If $\left(x_{2}\right)=$ (2) then we get the tninmmithime subgraph $\left\{x_{14}, x_{31}, x_{12}, x_{11}\right\}$. If
 $\left(x_{20}, x_{44}\right)-(4,4)$ them we move thexe pebbles to $G$, and we pebble the target, Or, if ether $\left(x_{t}\right)=(6)$ or $\left(x_{s}\right)=(6)$ then we move three pebbles to oither $x_{i d}$ of
 (6) then we maye three pehbles to $x_{4}$ and we move ond more pebble to $G_{4}$ from
$\mathrm{G}_{2}$ and thith wo place two pebibles on $\mathrm{X}_{4}$, Similarly ич proceed if $\left(\mathrm{X}_{\underline{z}}\right)=(6)$ II all the above fail, then clearly $\left(\mathrm{x}_{\mathrm{in}}, \mathrm{X}_{12}\right)=(2,2)$ \#nd so we move two pebbles to $\mathrm{G}_{2}$ We move one mbre pebble $10 G_{1}$ from $G_{2}$ and then we move two pebbles to


 $x_{i 4} \downarrow$. Next, let $b_{1} \geq 12$ If either $\left(x_{10}\right)=(4)$ or $\left(x_{i n}\right)=\left(0_{0}\right)$ we pebble the tafget. Or, if ecther $\left(x_{0}\right)-(6)$ or $\left[x_{13}, x_{12}\right)-(4,2)$ or $\left(x_{12}\right)-(10)$ we move the above
 move thite pebbles to xys . Now, we move a pebble to $x_{y}$ from $x_{4}$ and we pebble the target. Or, if a pebble in moved to $\mathrm{G}_{\mathrm{j}}$ from $\mathrm{G}_{2}$ then we pet a trammitting subgraph as in the case if $\beta_{1}=11$ and $\beta_{1}=7_{2} O_{\text {r }}$, if a pebble is moved to $G_{2}$ from $G_{2}$ then we get cight pebbles on $G_{2}$ and this situation has been afreaty disoussed where we use conly the pebbles on $G_{1}, G_{3}$ and $G_{2}$ to pebtile the targat. If all the alove fail, we pehole the tatyet by moving as many pebbles as possible cither $10 \mathrm{G}_{1}$ from $\mathrm{G}_{2}$ and $\mathrm{G}_{4}$ or to $\mathrm{G}_{2}$ from $\mathrm{G}_{1}$ and $\mathrm{G}_{2}$ -
Next, let $B_{i}=6$. Suppose rwo pehbles cannot be placed on $\mathrm{X}_{24}$ First, we assume $\phi_{2} \geq 7$, As $\phi_{1} \geq 10$, ether $\left(x_{n 1}\right)=(2)$ er $\left(x_{12}\right)=(4)$. We move the above pebbles to $G_{2}$ and then we move one more pebble to. $G_{2}$ from $G_{4}$. In the resultime dastribuion, wa place rwo petbles oo $\mathrm{Ma}_{1}$. Next, we take $\psi_{2} \leq 6$. Then $\beta_{1} \geq 13$. If ether $\left(x_{10}\right)=(4) \operatorname{or}\left(x_{2}\right)=(8)$ ws are dots. If not, then tither $\left(x_{e 3}\right)=$ $(4)$ or $\left(x_{4 \%}, x_{14}\right)=(2,2)$. We move the above pebbles to $G_{4}$ and now we place two pebbles on $x_{4 i}$ Suppose $\left.\left(x_{4}\right)=\beta_{2}\right)$. Ws assume $\beta_{1}=10$ and $p_{2}=q_{2}$ Withosit loss of generality, we assume if $\left.\left(x_{14}\right)-f_{2}\right)$ then $x_{\text {air }}$ campor be 2 pebblect and if $\left(\mathrm{x}_{51}\right)=(2)$ then $\mathrm{x}_{2}$ cannot be 2 -pebbled. (learly $\left(\mathrm{x}_{11}\right)=(2)$ (otherwise $\left(x_{1}\right)=(4)$ and 59 we are done). Thepsfore $\left(\mathrm{X}_{41}\right)=(2)$ and $x_{21}$ cannos be 2-pebbled. $\mathrm{So}_{0}$, if $\left(\mathrm{S}_{46}\right)=(2)$, we are done if not, then $\left(\mathrm{K}_{0}\right)=(4)$ by Fact 2 . If
$(x \leq 0)=(6)$ we move three pebbles to $x$ ys and we move a pebble to $x_{y 4}$ from $x_{44}$ and so we are done, Or. if $\left(x_{2}, x_{3}\right)=(6,6)$ then we move these pebbles to $G_{4}$ and we place two pebbles on $x_{41}$. Or, if $\left(x_{12}, x_{12}\right)=(2,2)$ and $\left(x_{21}, x_{22}\right)=(2,2)$ thet we move the above pebties to $G_{4}$ and we place two pebbles on $x_{1}$. If all the above fail, ether we move at leant two pobbles to $G_{1}$ from $G_{2}$ and che more pebble to $G_{1}$ from $G_{i}$ and we place four pebbles on Xis ow we move at leant two pebbles to $G_{2}$ from $G_{2}$ and one more pobble to $G_{2}$ from $G_{4}$ and we place fout pebbles on $\mathrm{X}_{23}$. Next, tet $\mathrm{F}_{1}$ be either ciever or twelve. As $\mathrm{F}_{2} \geqq 7_{7}$, if $\left(\mathrm{x}_{23}\right)$ - (2)
 $x_{11}$. If not, then $x_{s}$ is 2-pebbled. Ajain, by Fact $3_{s}\left(x_{20}\right)=(2)$ and so wet get the tramsmition subzraph $\left\{\mathrm{x}_{3}, \mathrm{X}_{31} \mathrm{x}_{20} \mathrm{x}_{11} 1\right.$. Next, ict $\mathrm{F}_{1} \geq 13$, If pither $\left(\mathrm{x}_{2}\right)=(2)$ $m\left(x_{31}\right)=(2)$ then we ate done. $O r$, if $\left(x_{n}\right)=(6)$, we nove three pebbles to $\mathrm{Xm}_{\mathrm{m}}$ and then we move a pebble to $x_{\mu}$ from $x_{4}$ and sio we are done. Or, if either ( $x_{1}$ ) $=(f)$ or $\left(x_{12}\right)=(10)$ or if $\left(x_{1}, x_{i 1}\right)=(4,4)$, we move the above pehbies to $G_{2}$ and we pline two pebties on $x_{4}$. Otherwise, we move either one pebble to $G_{1}$ from $G_{4}$ or at least five pebbles to $\mathrm{G}_{2}$ from $\mathrm{G}_{1}$ and one pebble to $\mathrm{G}_{2}$ from $\mathrm{G}_{4}$ and then wo pebble the target.

Different cases can be unilly verified if $p_{34} \neq 1$,
Case 2(vi), Let $\mathrm{F}_{4} \leq 5$.
Let $P_{3}=1$.
Suppose $\left(X_{4}\right)=(2)$.
Cluaty $F_{1} \geq 10$ and $P_{2} \leq 10$. Fims, we take $F_{1}=P_{2}=10$. Therefiore $P_{4}=5$. Without loss of benerality, we assume $x_{y}$ s is nat $2-p e b b l e d$ in $Q_{1}$ and $x_{22}$ is not 2-pebhied in G.- (Ohherwise, we get a transmitting subgraph by Fact 2), Thersfore $\left(x_{2,}, x_{2}\right)=(4,4)$ by Fist 2 . If either $\left(x_{2,}\right)=(6)$ or $\left(x_{2 x}\right)=(6)$. we move three pebbles to $G_{3}$ and we move one pebble to $x_{4}$ from $x_{k}$ and so we pebble the target, Or, if $\left(\lambda_{12}, x_{23}\right)=(6,6)$ then we move fix pebbles to $G_{1}$ and
 $G_{4}$ and onie moire pebble to $G_{2}$ from $G_{1}$ ind theu we place four pebbles on Xen Next. fer $\phi_{1} \geq 11$. Fint, we take $\beta_{2} \geq 7$. Therefore we mow a pebble to cether $\lambda_{4}$ or $x_{3}$ from $G_{2}, S_{0}$, applyim: Fatt 3 to $G_{1}$, we get a tranmmititis supaph. Now, let $p_{2} \leq 6$. Then $p_{3} \geq 14$. We abiume $p_{1}=14$. Then $p_{2}=6$ and $p_{2}=5$;
 (2) we ave done Or, if a pohtile is moved to $G_{1}$ from $G_{2}$ then clearly either $X_{1}$ is 4 -pebbled of $X_{31}$ is 6 -piebbled and so we are donc. Or, if either $\left(x_{11}\right)$ - (4) or $\left(x_{2}\right)=(8)$ and $\left(x_{22}\right)-(2)$, we mpve the above pebblen to Gind we place two pebわles on $x_{2}$, Or, if $\left(x_{12}, x_{11}\right)=(2,2)$ and erither $\left(x_{22}\right)=(2)$ of $\left(x_{32}\right)=(2)$, we Bove the above pebbles to $\mathrm{G}_{4}$. Now, ith the fescultait distribution we place two pebbles on $x_{3}$. If all the above fail then we move at leat five pebbles to $\sigma_{2}$ from $\mathrm{G}_{5}$ ant one pebber in $\mathrm{G}_{2}$ from $\mathrm{G}_{5}$ und then we place four potbles on $\mathrm{X}_{21}$ in all the othet distributions we have $F_{;} \geq 15$ mad no either $\left(X_{12}\right)=(4)$ or $(X E)=(6)$ and ro we are dome

## Suppose two pebhles cantot he placed on $x_{\text {si }}$.

Lut $F_{4}=5$. If $F_{2}=P_{2}-10$ then cither $\left(x_{i 3}\right)-(2)$ or $\left(x_{1}\right)=(4)$ either $\left(x_{2}\right)=$ (2) of $\left(x_{21}\right)=$ (4) by Eact 2 We move the tibove pebbles to $G_{s}$ and dlearb wt

 imvice a petble tis $\mathrm{G}_{1}$ from $\mathrm{G}_{2}$ and clearly, we move two pebbles to $\mathrm{x}_{4}$. Suppoxe not, then $(x / s)-(2)$. So, we move a pebble to $x_{\text {as }}$ Alse, we move a pebbic to G. from $G_{2}$ Now, if $\left(x_{i 1}\right)=(2)$ we are done. If not, then as $\left(x_{(2)}\right)=(2)$ by Face 3 . instcad of moving a phetble to $\mathrm{x}_{\mathrm{ss}}$ from $G_{1}$, We mave a pebble to $\mathrm{x}_{7+}$. Alrearly,
 two pebbles on $x_{61}$. Nex1, let $P_{1}=14$. Therefore $B_{2}=6$. If ether $\left(x_{21}\right)=[2]$ or $(x \times 5)=(2)$ we are done $O$, if a petrile is moved to $G$, from $G_{2}$ then we get

Lifteein peibbles an $G_{1}$ and so either $x_{i s}$ is 6 -pebbled or $x_{i s}$ is 4 -pebbled and $x_{15}$ is 2-pebbled or $x_{15}$ is 2-pebiled and $x_{i r}$ is 4-pebbled. Wo move the atbove petribes to $G_{4}$ und clarly we place two prbbles on $x_{1}-O$, if a pebblet is moved to $G_{i}$, from $\mathrm{G}_{2}$ thon we pet six pebbles on $\mathrm{G}_{+}$In the resultite distribution if two pehbies tannot be moved to $x_{4}$ then we proceed as in Case 2(v). Suppose two pebbles am be moved to $x_{22}$. Then if $\left(x_{12}\right)=(6)$ we mave three pebifics to $x_{2}$ 'and we mivie a pebble to $x_{4}$ from $x_{4}$ and we pebble the tariket. Or, if eithier $\left(x_{01}\right)=(6)$ or $\left(\mathrm{x}_{12}\right)=(6)$ we move the above nebbles to $G_{4}$ and we place two pebbles on $\mathrm{X}_{\mathrm{n}}$. If not, we take the initial distribution as such and we move as miny pehbles as possible cifber to $G_{8}$ from $G_{2}$ ot to $G_{2}$ from $G_{1}$ und $G_{2}$ we pebble the targer Next, let $P=5$, Suppoge $P_{41}=1$ except for one j. $1 \leq 1 \leq 6$ As $P_{1} \geq 15$, if either $\left(x_{26}\right)=(4)$ or $\left(x_{13}\right)=(8)$ we are done Otherwise, either ( $\left.\mathrm{Xi}_{2}, \mathrm{x}_{-1}\right)=(2,2)$ of $\left(\mathrm{x}_{12}\right)=(6)$ or $\left(\mathrm{x}_{3}\right)=(6)$. We move the nbove pebblein to $\left(\mathrm{i}_{4}\right.$ and we place two pebhles on $x_{10}$ Suppose there existe a vertes in $G_{1}$ with at least two pebbles. Now, if either $\left(x_{i s}\right)=(6)$ or $\left(x_{2}, x_{i s}\right)=(8,2)$ itun we move the above pebbles to $G_{4}$ and ciearly we place two pebbles on $x$ at. If riot. then
 from C3. Thus we move as muny petibles us jpisible either to $\mathrm{G}_{7}$ froun $\mathrm{G}_{2}$ or to $\mathrm{G}_{2}$ from $\mathrm{G}_{3}$ - we pebbie target.
Next, fet $\beta_{4} \leq 4$. Therefore $R_{\mathrm{t}} \geq 11$ and $P_{2} \leq 10$. If the distribution is such that either $\left(x_{16}, x_{3 n}\right)=(2,4)$ or $\left\{x_{t i}, x_{3}\right\}=(4,2)$, we get one of the tramimitting
 $\left.x_{40}, x_{23} x_{21}\right)(3,4,2,4)$ 아 $\left(x_{15}, x_{22} Y(4,4)\right.$, we move the above pehbies to $G_{4}$ and we place two pebbles on $x 4$. Citherwise, we move as many pebbles as possible enther to $G_{2}$ from $G_{2}$ and $G_{i}$ or to $G_{2}$ frum $G_{i}$ and $G_{i}$ we pebble the Linget:
If ${ }^{+1}+1$, eases can be similarly veritied.

Step 3, $P_{1}=2$
Cake $3(i)$. Le: $\mathrm{F}_{4} \geq 14$.
If $x_{41}$ is not 2 -pebbled then by Fact $\left.\sigma_{1}\left(x_{4}\right)=112\right)$. So, we moye six pebsbles to $x_{4}$ fromill $x_{24}$
Cake 3(ii). Let F, bee either wwelve or thirteen:
Clearly $H_{1} \geq 6$ and $P_{2} \leq 6$.
If $\left(x_{14}\right)=(2)$ then we pebble the turget. If nut, by Fact $4,\left(x_{42}\right)=(10)$. If $P_{1} \geq 7$ then we mofe a pebble to $G_{4}$ from $G_{1}$ and then we move five pebbles to $x_{3}$ fram $x_{4}$. If $p_{1}=6$ then cither $R_{2}=6$ or $p_{2}=5$. Finst we take $F_{3}=\beta_{2}=6$. If is petable is moved to $G_{3}$ from $\mathcal{G}_{3}$, we are done. If a peblice is moved to $G_{2}$ from Qu. then we move a gecbite to $\mathrm{C}_{1}$ from $\mathrm{G}_{1}$ and then we move five pebbles to $\mathrm{x}_{34}$ fram $x_{4}$ Similarly, if a pehble is moved to either $G_{1}$ (o) $G_{1}$ torm $G_{j}$, we wre done. On, if two pebbles are moved to $\mathrm{G}_{2}$ either from $\sigma_{1}$ 估 from $\mathrm{G}_{2}$ ar ane from
 Case 3li, If not, then $v_{2}=1$. for cithen $i=1$ or $i=2$ or $\bar{j}=1,2$ and for every $\bar{j}$, $1 \leq j \leq 6$. Withou loss of yencralify, we oscume $F_{5}=1,1 \leq j \leq 6$ Now, we move i pebble to $\mathrm{G}_{\mathrm{L}}$ from $\mathrm{G}_{\mathrm{a}}$ using In mosit four pebbles and we put two pebhles on $x_{1 w}$ and $x 0$ we moys a pebble to $x_{90}$. Now, as there are at leats six pebtiles an
 $P_{2}=5$ : If a pebbic:is moved to erther $G_{j}$ on $G_{j}$ or $G_{4}$ from $G_{y}$ we afe done. If not, then $P_{i=1}-1$ encoptifor ime $1,1 \leq j \leq 6$ Now, if a pebble is mmyed to wither
 move one more petble to $G_{y}$ from $G_{1}$ uximg at most four pebbles and then we move a pebble to $x_{\rightarrow 2}$ frum $G_{3}$. Nonv,we move three pehbles to $x_{4}$ from $x_{44}$ and 30 we ute dane, If not, then $0_{1 f}=1$ for every $j_{1}, i \leq j \leq 6$ and so we proceed as betore:

As the other cajes are similar we leave the proof here.

Step 4. F, $=3$.
Case4(i) Let $P_{x} \geq 12$
If $x_{n}$ is toe 2 -petbled then by Fact $4,\left(x_{4}\right)=(10)$ ind so we movalive pebbles to $\mathrm{X}_{1+}$
Case 4 (ii) Lef $P_{4}-11$
Glearly $F_{n} \geq 6$ and $F_{2} \leq 6$. If $p_{1} \geq 7$, we move a pebble to $G_{n}$ frim $G_{1}$. Then. as $\left(x_{41}\right)-\{8)$ by Faci 3 ; we move Four pobbles to $x_{4}$ from $x_{3}$. If $\mathrm{S}_{1}-6$ then $\rho_{2}-$ 6. If we move a pebble to either $G_{2}$ or $G_{5}$ or Gis from $^{G_{1} \text {, we ure douc, if not }}$ than $\phi_{N}=1$ for every $1_{1} 1 \leq j \leq 6 . S_{6}$, as $\left(x_{a d}\right)=(8)$, we move a pebble to $\mathrm{C}_{i}$ using it most four pebbles of $\mathrm{G}_{4}$ and we pitit two pebbles on Xic Now, moye a pebble to $x_{50}$ from $x_{i s}$ and we move two gepbles to $x_{94}$ from $x_{i n}$ and we pebble the targe

The other caves dan be similarly verified
Srep $5, F_{3}=4$
Case 5(i). Let $\psi_{a} \geq 11_{1}$
If $x_{4}$ is not 2 -pebticd then $\left(x_{44}\right)=(8)$ by Fact 3. S0, we move lour pebbles to $X_{34}$ from $X_{44}$
Cass 5in). Lect $P_{4}=10$.
Clearly $\beta_{4} \geq$ frand $\vec{F}_{2} \leq 6$. If $\beta_{1} \geq 7$, we mowe a pebble $10 G_{4}$ fiom $G_{2}$ and we procect as in Cuse S(i) If $P_{1}=6$ मhen $\mathrm{F}_{2}=6$. If we move a pebble to cither $\mathrm{G}_{2}$ or $\mathrm{G}_{3}$ from $\mathrm{G}_{\mathrm{n}}$, we are some. Or, if $w e$ move cither ofre pebble to $x_{1}$ or two pobbles to $x_{i j}$ from $G_{i,}$, then as $\left(x_{44}\right)=(4)$ by Fact 2 , we movo two pobbles to Xor and we pehble the target Similarly, if we move al pehble to: enther $G_{1}$ or $\mathbf{G i}_{1}$ or $x_{y}$ or $t w r y c b b l e s ~ t o x_{j 1}$ from $G_{z}$ we are dore. Otherwifc, if $p_{4} \geq 8$, we move four pobbles to $x_{4}$. H'not clearly $\tilde{F}_{e 4} \geq 1$ or $P_{4} \geq 1$ und $F_{4} \geq 4$, $\mathrm{So}_{4}$ wo
 and we plate tswo pehbles on $x_{1 n}$ and then we move a pebtic 10 so Now, as

Thete are ot least two pebbles on $\mathrm{x}_{4-}$. We move a pebble to x a and we pebble the trirget.

The other calues can be discusted in a simitar way,
Step 6. $\mathrm{P}_{5}>5$.
Clearly, we pebble the target if cother we move throe pebbles to $x_{s y}$ or we move a pebtle to sither $x_{22}$ ar $x_{16}$ or we move two pebbles to esther $x_{21}$ of $x_{3}$ using the pebbles from $G_{j,} i \neq 3$.
Now, if $\psi_{1} \equiv p_{2}=p_{1}=7$, then wv move throe pobbles to $G_{1}$ one from each sopy $G_{6} i \neq 1$
If not, first we take $\mathrm{P}_{4} \geq 9$. If $\left\{\mathrm{X}_{21}\right\}=(2)$ we are dote. If not, suppione $\left(\mathrm{X}_{2}\right)=$ (6). Then we move three pebblen to $x_{24}$ Suppose not $I f\left(x_{14}\right)=(2)^{2}$, wo mave a pehble to $x_{4 i}$ and in the resulting slatribution we place two pebbles an $x_{4+1}$ Othervise, as $\left(x_{2}\right)=(2)$, we moxe a pebble to $x_{4}$ and we pebble the target.
Let $B_{4}=8$. Then $\beta_{1} \geq 7,50$, we move a pebble to G G from $_{2} G_{2}$ As thete are nine pebbles oni $G_{4}$, we procesd as betote.
Let $F_{1}=7$. Then $\mathcal{F}_{1} \geq 2$ and $\mathrm{F}_{3} \leq 7$. If $\mathrm{F}_{3}=\mathrm{F}_{2}=7$, we are done. If $\mathrm{F}_{1}=8$ thim $P_{2}=6$. Suppore $F_{14} \geq 4$. If $\left(x_{44}\right)=(4)_{\text {, wo move two pobbles to } x+w}$ Wo Hove anc mone piobhle fo $G_{1}$ from $G$ and we polble the target If not, thes we nove two pebbles to $x_{4}$ from $x_{3}$ ind we movivene core pebble to $\mathrm{G}_{4}$ from $\mathrm{G}_{2}$. In the resulting distritution of ten pebbles on $\mathrm{G}_{4}$ we move two pebbies to Xet Suppose $P_{4+} \leq 3$. Wempove two pebbles to $G_{5}$ one from $G_{i}$ and unother frum G4. Now, tin the resultity distribution of seven pebble on Gry we pebble the targer Next, if $F_{1} \geq 9$, thon we move a pebble to $G_{1}$ from $G_{1}$. As there are at
 either one pebble to N n or two pebbles to Ne .
Ler $P_{4}=6$. Then $P_{2} \geq 8$, If $P_{1} \geq 10$ mo proced as before. It $P_{1} \equiv 8$ then $F_{2}=$ 7. If $\left(x_{4}\right)$ - (2), we move a pebble we $x_{34}$ fram $\mathrm{X}_{46}$. Now, we move tho pebbles
to $G_{3}$, whe from $G_{1}$ and another from $G_{7}$, If neg, suppose $\phi_{24} \geq 4$. Then we muve two pebbles to $\pi_{4}$. Also, we move two pebbles to $\mathrm{G}_{4}$, one from $\mathrm{C}_{j}$ and anather from $G_{3}$. Now, we place two pebbles on $x_{\text {aj }}$. Otherwise; $P_{4} \leq 3$. We muve two pebbles to $G_{i}$, one from $G_{1}$ and another from $G_{2}$ and we pebble the taryet. If $F_{1}=9$ then $P_{2}=6$, If $\left(x_{i+1}\right)=(4)$, we nove two pebbles to $x_{14}$ and we move one mote pebble to (i) fom $\mathrm{G}_{1}$. Or, if a pebble is muved to $\mathrm{G}_{1}$ fram $\mathrm{G}_{4}$ then we gat tim pebbles on Gi and so we are done. Or, if a pebble is moved to $G_{2}$ from $G_{4}$ then we move a pebble to $G_{1}$ from $G_{s}$. As there are ten pebbles on $\mathrm{G}_{1}$. We ant dome. Or, if either $\left(\mathrm{x}_{16}\right)=(2)$ or $\left(\mathrm{x}_{13}\right)$ - (4), we move either otte pebble to $X_{\text {in }}$ or'twa pebbles io $x_{n 2}$. Otherwise, if $\left(x_{24}\right)=(4)$ we nove twa pebbles to $x_{42}$ Also, elther $\left(x_{12}\right)=(2)$ of $\left(x_{12}\right)=(4)$ by Fact 9 und we mivie these puebbles to cia. Now, we piaco two pebbles on $x_{i-}$. If not, then $5_{i} \leq 3$ and if $\left(x_{+\infty}\right)=(2)$ we move one pebble to $x_{\text {搔 }}$ And, we move one more pobble to $G_{1}$ from $G_{7}$ and so we pebble the target. If all the above tall, then $P_{5}=I$ for cvery I. $1 \leq j \leq 6, S o$, we move de pebble to $G$, from $G_{1}$ and we place two pebbles on $\mathrm{X}_{\text {ij }}$.

Let $P_{4}=5$, Then $P_{1} \geq R_{8}$ and $P_{2} \leq 8$. Let $R_{1}=P_{2}=8$. If $R_{34} \leq 3$ wy move two pebbles to Gi, one from Gi and another from G. Now, clearly we pehble the target. If not, then $P_{32} \geq 4$. Supposic $\left(x_{44}\right)=(2)$, we mave a pebble to $x_{4}$. and we move two pebbles to $G_{3}$, one from $G_{L}$ and another from $G_{1}$. Suppose not. If cither $\left(\mathrm{x}_{20}\right)=(2)$ ov $\left(\mathrm{x}_{1}\right)=(4)$ or $\left(\mathrm{x}_{21}\right)=(2)$ or $\left(\mathrm{x}_{26}\right)=(4)$ we are done. Otherwisc, either $\left(x_{11}\right)=(2)$ of $\left(x_{12}\right)=(4)$ nud cither $\left(x_{0}\right)=(2)$ or $\left(x_{21}\right)=(4)$ by Fact 9. Now, we move the above pebbles to $G_{1,}$. Also we move two pebbles to
 $\Rightarrow 9$ then $F_{2}=7$, we move a pebble to $G_{1}$ from $G_{2}$ As there are in pebbles on G1. we proceed as before. If $p_{i} \geq 10$, we ate done.

Nexi, litt $P_{4} \leq 4$. Therctore $\beta_{1} \geq 9$ and $p_{2} \leq 9$. That $f$, cither $F_{1} \geq 10$ or the number of pebbles oo $G_{1}$ tam be made ten uflet a pebbling move troni $G_{2}$ und so wo are done.

Step 7. $\beta_{2}-66$
If we meve cither two pebbles to $G$ y ar one pebble to cither $3 y$ or $X$ xs, using thit pebbles frym Gin $i \neq 3$, we cum pobble the fareet.
Suppose there are twe coppies among $\mathrm{On}_{0} \mathrm{G}_{\mathrm{F}}$ and $\mathrm{G}_{3}$ each with at least seven
 that $\mu_{1} \geq 10$ then we are done by fuct 2 .
Suppose not. Let $\mp_{4}=9$. Then $R_{1} \geq 6 . I_{1} R_{1} \geq 7$, we move two nchion is $\boldsymbol{C}_{2}$ one from $G_{2}$ athe unother from $\alpha_{4}$ If $P_{1}=6$ then $P_{3}=5$. If a pebble is moved it $G_{4}$ from $G$, then we get fen pebhles on $G$, and so weare done, Or, if a pebble is movid to $G_{2}$ from $G_{1}$. we move une mure pehble is $G_{3}$ tism $G_{4}$. Or, it a pebble is moved to either $G_{1}$ or $G_{j}$ or $G_{4}$ from $G_{2}$ we ure done If not, then $P_{3}$ $\equiv 1$ except for one $j 1 \leq j \leq 6$, Suppose we move a pehble to $G_{2}$ from $\mathrm{O}_{\mathrm{N}}$, then we misve one mone pebble tis $G_{1}$ from $G_{4}$ mid we place twe pebbles on $x_{5}$, und
 We move a pebble to $G_{1}$ from $G_{4}$ and dearly we plate two pebbles on $X_{1 n}$ Sio, we move is pebblic to $x_{\text {in }}$
Let $P_{4}=8$. Then $P_{1} \geq 6$ and $\beta_{2} \leq 6$. If $B_{1} \geq 7$. We move two pebhles to Kh. one from $G_{1}$ and miother from $G_{+}$if $R_{1}=6$, then $P_{2}=6$. If we move a pebble to efther $\mathrm{G}_{y} \mathrm{or}_{2} \mathrm{G}_{2}$ fram $\mathrm{G}_{1}$ ot if wo move a pobble in either $\mathrm{G}_{3}$ or $\mathrm{G}_{1}$ frimin $\mathrm{G}_{2}$, wo are done. Oc, if we move wo pebbles to $G_{4}$ either from $G_{1}$ or from $G_{2}$ or one from $G_{1}$ another from $G_{2}$, we ate donc. If mot, then $\beta_{5}-1$, for $i=1$ or $i=2$ or $i$ $=1,2$ and for every $3,1 \leq 1 \leq 6$. Without lons of generaily, we assume $p_{1} \|=1$, $1 \leq j \leq 6$. Now, we move a pebble to $G_{2}$ from $G$ and we place two pobhler on $\mathrm{X}_{\text {IN }} . \mathrm{So}_{0}$, we wove a pebtle io $\mathrm{X}_{\mathrm{an}}$
f.et $P_{2}=7$, Then $P_{1} \geq 7$, So, we rove two pehbles to $G_{0}$ one from $G_{4}$ and ansther from $\mathrm{G}_{2}$ -
Let $p_{4}=6$. Then $p_{1} \geq 7$ and $p_{2} \leq 7$, If $p_{1}-p_{2}=7$, we are done If $p_{1}=8$ then $p_{2}=6$. We proceed an in the pase if $p_{1}=8, p_{i}=p_{1}=6$. If $p_{1}=9$ then $\beta_{2}=5$. We proeed at in the case if $\beta_{4}=9, \beta_{1}=6$ and $\beta_{2}=5$, If $p_{1} \geq 10$ then we afe dons.
Let $k_{4}=3$. Then $p_{1} \geq 8$ and $p_{2} \leq 7$. If $p_{1}=8$ and $p_{2}=7$ then we move two pebbliss is $G_{1}$, one fath $G_{1}$ and another from $G_{2}$. If $P_{1}=9$ than $\varphi_{2}=6$. We procesd as betore. If $\mathrm{P}_{\mathrm{y}} \geq 10$ then we aro done.
Ler $P_{y} \leq 4$. Then $R_{y} \geq 8$ and $p_{2} \leq 8$. If $P_{1} \geq 7$ and $R_{z} \geq 7$ dien we move two pebbles to $\mathrm{G}_{2}$ Otherwise, $p_{2} \geq 10$ and so we are dooc.
Step 8. $\quad F_{x}=7$
Clearly, thete exists at least one $i, i \neq 3$ such that $F_{i} \geq 7$. So, we move a pebble to Gs from G.

## 4.Conclusion and open problem

We have tound the pebbling number of $\mathrm{S}_{+}$. Computation of (1) pebbling mumber (ii) t-pebbling number and (iii) cover pebhing number of $S_{\mathrm{i}}$ will be anshar interotime area of neseareli.
Conjecture 4.1 The pebbling nimber of $\mathrm{S}_{\mathrm{n}}$ is $\mathrm{f}\left(\mathrm{S}_{\mathrm{n}}\right)=\mathrm{nI}+2$.

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