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The Pebbling Number of 4-star Graph

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Abstract : A pebbling move on a graph G consists of taking two pebbles off one vertex and placing one pebble to an adjacent vertex. The pebbling number of a connected graph G, f(G), is the least n such that any distribution of n pebbles on G allows one pebble to be moved to any specified but arbitrary vertex by a sequence of pebbling moves. In this paper we will determine the pebbling number of 4-star graph.

1.Introduction

One recent development in graph theory, suggested by Lagarias and Saks, called pebbling has been the subject of much research and substantive generalizations. It was first introduced into the literature by Chung [1], and has been developed by many others including Hulbert, who published a survey of pebbling results in [5]. Given a connected graph G, distribute k pebbles on its vertices in some configuration, C. Specifically, a configuration on a graph G is a function from V (G) to N U {0} representing an arrangement of pebbles on G. We call the total number of pebbles, k, the size of the configuration. A pebbling move is defined as the simultaneous removal of two pebbles from some vertex and addition of one pebble on an adjacent vertex. Chung [1] defined the pebbling number of a connected graph, which we denote f(G), as follows : f(G) is the minimum number of pebbles such that from any configuration of f(G) pebbles on the vertices of G, any designated vertex can receive one pebble after a finite number of pebbling moves.

There are many known results in [5] regarding f(G). If one pebble is placed at each other vertex than the target vertex, v, then no pebble can be moved to v. Also, if w is at distance of d from v and 2^d-1 pebbles are placed at w, then no pebble can be moved to v. Thus, we have $f(G) \geq \max\{n(G), 2^{diam(G)}\}$, where n(G) denotes the number of vertices in G. and diam(G) denotes the diameter of G. Graphs G that satisfy f(G) = n(G) are called Class 0 graphs and graphs G that satisfy f(G) = n(G)+1 are called Class 1 graphs[2]. Class 0 graphs include the complete graph K_n, n-cube Q_n [1,9], complete bipartite graphs $K_{m,n}$ [10], the product graph C_5 ?C $_5$ [4] and many others. We find an elegant characterization about Class 1 graphs in [5]. The path P_n [10], n-cube Q_n [1,9], even cycle [9,10] are examples of graphs G that satisfy $f(G) = 2^{diam(G)}$, whereas the odd cycle [9,10] is an example of a graph not satisfying either lower bounds. Another interesting result is the pebbling number of a tree, which is beautifully worked out in [8]. Hulbert [5] has written an excellent survey article on graph pebbling. Note that pebbling number does not exist for a disconnected graph. Throughout this paper, G will denote a simple connected graph. We now proceed to find the pebbling number of the 4-star graph.

2. n-star graph

A formal group theoretic model called the Cayley Graph has been introduced in the literature for designing and analyzing symmetric interconnection networks. The two important members of this class are the star graph and the hypercube. An n-dimensional hypercube or n-cube, consists of 2^n vertices labeled by (0,1)-tuples of length n. Two vertices are adjacent if their labels are different in exactly one entry. Chung [1] proved that the n-cube satisfies $f(Q_n) = 2^n$. This paper explores the pebbling number of 4-star graph. We were particularly intrigued by n-star graph since it has fewer interconnecting edges.

Definition : 2.1 [6] An n-star graph, denoted by S_n, is an undirected graph consisting of n! vertices labeled with the n! permutations on n-symbols (we use symbols 1,2,?n) and such that there is an edge between any two vertices if and only if, their labels differ only in the first (left most) and in any (one) other position.

Recursive Construction 2.2 [6]. S_n can be recursively constructed from n copies of S_{n-1} as follows :

We first construct n copies, G_1,G_2, ?, G_n, of S_{n-1} and label each G_i using all symbols 1 through n except symbol i ; then for each label in G_i we add symbol i as the last symbol (rightmost) in that label (or in any other fixed position); finally, we connect by an edge every pair of vertices u and v such that label of v is obtained from that of u by exchanging the first and last symbols of u.

Partitioning 2.3 [6] The n-star can be partitioned in n-1 different ways into n copies of (n-1) stars. The different ways correspond to different symbol positions in the labels. For each symbol position i other than the first position (left most position) we can partition S_n into n copies of (n-1)-star denoted 1_i, 2_i, ? n_i. Each k_i contains all the vertices of S_n with symbol k in the i-th position of their labels. If however we try to partition along the first position, we obtain n collections of (n-1) ! isolated vertices, Figure 2.1.1 illustrates the partitioning of a 4-star into four 3-stars (each 3-star is an hexagon) along the fourth position (rightmost). The four 3-stars are denoted 1_4, 2_4, 3_4 and 4_4

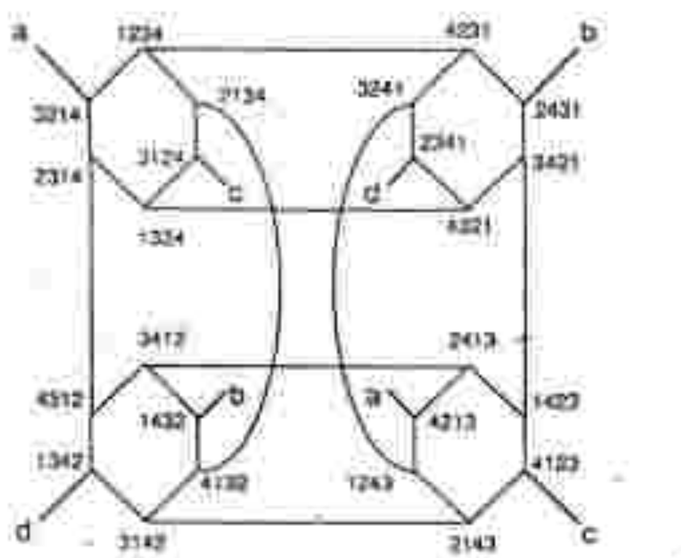

Figure 2.1.1. : The 4-star viewed as four interconnected 3 -stars

Theorem 2.4 . The pebbling number of 3-star is $f(S_3) = 8$.

Proof. Clearly 3-star is the cycle with six vertices i.e. C_6.

Therefore $f(S_3) = 8$ [7].

Theorem 2.5 [7]. The t-pebbling number of the cycle C_{2k} is $f_t(C_{2k}) = 2^k t$.

Definition 2.6. We say that two vertices of S_3 are opposite to each other if they are at a distance of three from each other.

Clearly there are three pairs of opposite vertices in S_3.

We include some facts here, most of which are quite straightforward and can be easily verified.

Let n be the number of pebbles distributed on the vertices of S_3 and let (u, v) be a pair of opposite vertices in S_3.

FACTS

1. If n = 7 then either u or v can be 2-pebbled
2. If n = 10 and if u cannot be 2-pebbled then v be can be 4-pebbled.
3. If n = 11 and if u cannot be 2-pebbled then v can be 8-pebbled.
4. If n = 12 and if u cannot be 2-pebbled then v can be 10-pebbled.
5. If n = 13 then either u or v can be 4-pebbled.
6. If n = 14 and if u cannot be 2-pebbled then v can be 12-pebbled.
7. If n = 15 and if u cannot be 4-pebbled then v can be 6-pebbled.
8. If n = 18 and if u cannot be 4-peebled then v can be 8-pebbled.
9. If n = 8 and if u cannot be 2-pebbled and v can be 2-pebbled but cannot be 4-pebbled, then either u_1 can be 2-pebbled or v_1 can be 4-pebbled where (u_1, v_1) is a pair of oppo site vertices such that u_1 is adjacent to u and v_1 is adjacent to v.

Next, we find the pebbling number of S_4.

3.The pebbling number of S_4

For our convenience, we represent S_4 as in Figure 3.1.1.

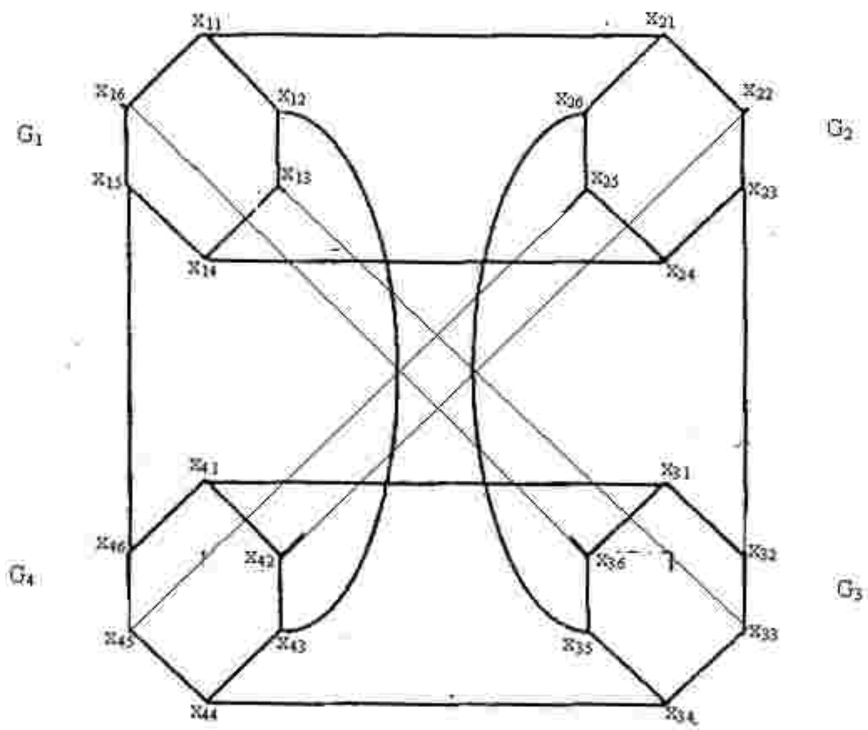

Figure 3.1.1 : S_4

In Figure 3.1.1 the four copies of S_3 are represented by G_i, $i = 1,2,3,4$ and the vertices of S_4 are represented by x_{ij}, $i = 1,2,3,4$; $j = 1,2,?6$.

Definition 3.1 [3]. Given a pebbling of G, a transmitting subgraph of G is a path $x_0, x_1, ?x_k$ such that there are at least two pebbles on x_0 and at least one pebble on each of the other vertices in the path except possibly x_k. In this case, we can transmit a pebble from x_0 to x_k.

Theorem 3.2. The pebbling number of S_4 is $f(S_4) = 4! + 2$.

Proof : Let the target vertex be x_{31} of G_3.

First, we prove $f(S_4) \geq 26$.

We consider the distribution of twenty five pebbles on the vertices of S_4 as follows:

We place fifteen pebbles on x_{44} and one pebble each on every vertex of G_2 except x_{16} and one pebble each on every vertex of G_2 other than x_{23} and we place zero pebbles on the rest of the vertices of S_4. In this distribution we cannot move a pebble to x_{31}.

Next, we prove $f(S_4) \leq 26$.

Suppose we distribute twenty six pebbles on the vertices of S_4. Let Þ_i denote the number of pebbles distributed on the vertices of G_i and let Þ_{ij} denote the number of pebbles on x_{ij} initially. We prove $f(x_{31}, S_4) \leq 26$. Then by symmetry, $f(x_{ij}, S_4) \leq 26$. Thus, it follows that $f(S_4) \leq 26$. If $\text{Þ}_3 \geq 8$ then we pebble the target by Theorem 2.1.4. So, we take $\text{Þ}_3 < 8$. Without loss of generality, we assume $\text{Þ}_{16} \neq 4$, $\text{Þ}_{13} \neq 8$, $\text{Þ}_{23} \neq 4$, $\text{Þ}_{26} \neq 8$ and $\text{Þ}_{41} \neq 2$. The proof involves several steps. We breakdown the possible configuration of twenty six pebbles on S_4 according to the distribution of pebbles on G_3. There are eight steps and we take $\text{Þ}_3 = k-1$ in the k^{th} step. Each step involves several cases and in each case we fix Þ_4 and then as $\text{Þ}_1 + \text{Þ}_2 = 26 - \text{Þ}_3 - \text{Þ}_4$, we consider the cases for each pair (Þ_1, Þ_2) such that $\text{Þ}_1 \geq \text{Þ}_2$, since similar procedure follows if we choose (Þ_1, Þ_2) such that $\text{Þ}_2 \geq \text{Þ}_1$.

Notation 2.2.3. Let a_k denote the number of pebbles we place on $x_{i_k j_k}$ after some pebbling moves using only the Þ_{i_k} pebbles distributed on the vertices of G_{i_k} initially, then we write $(x_{i_1 j_1}, x_{i_2 j_2}, \ldots, x_{i_k j_k}, \ldots, x_{i_r j_r}) = (a_1, a_2, \ldots, a_k, \ldots, a_r)$, where $1 \leq i_k \leq 4$, $1 \leq j_k \leq 6$.

Step I : Let $\text{Þ}_3 = 0$

Case 1 (i). Let $\text{Þ}_4 \geq 16$

We move at least two pebbles to x_{41} by Theorem 2.1.5 and so we move a pebble to x_{31}.

Case 1 (ii) Let Þ_4 be either fourteen or fifteen.

Clearly $\text{Þ}_1 \geq 6$ and so $\text{Þ}_2 \leq 6$

If $\text{Þ}_1 \geq 7$ then we move a pebble to either x_{46} or x_{43} from G_1 by Fact 1. Clearly in the resulting distribution of at least fifteen pebbles on G_4 we move two pebbles to x_{41} and so we are done. If not, then $\text{Þ}_1 = 6$ and Þ_2 is either five or six. If a pebble is moved to G_1 from G_2 then we get seven pebbles in G_1 and so we are done. Or, if a pebble is moved to G_4 either from G_1 or from G_2 we are done. Otherwise, note that either $p_{44} \geq 12$ or we get twelve pebbles on x_{44} by Fact 6. Therefore, if a pebble is moved to G_3 either from G_1 or from G_2, then we move six pebbles to x_{34} from x_{44} and then we move a pebble to x_{31}. If not, then $p_{3j} = 1$ for every j, $1 \leq j \leq 6$ if $p_3 = 6$ and $p_{3j} = 1$ except for one j, $1 \leq j \leq 6$ if $p_3 = 5$. If $p_{3j} = 1$ for every j, $1 \leq j \leq 6$, then we move a pebble to x_{23} from G_4 using at most four pebbles and so we place two pebbles on x_{23} and then we move a pebble to x_{22}. Now, as x_{44} has at least eight pebbles after the above move we add one more pebble to x_{22} and we pebble the target. Now, let $p_{3j} = 1$ except for one j, $1 \leq j \leq 6$. Suppose we move a pebble to G_2 from G_1 then we get six pebbles on G_2. We proceed as before to get two pebbles on x_{12}. Suppose not. Then $p_{1j} = 1$ for every j, $1 \leq j \leq 6$. We put two pebbles on x_{34} as before and we pebble the target.

Case 1 (iii). Let p_4 be either twelve or thirteen.

Clearly $p_1 \geq 7$ and $p_2 \leq 7$. If $(x_{41}) = (2)$, we are done. If not, then $(x_{44}) = (10)$ by Fact 4. Suppose either $p_{44} = 12$ or $(x_{44}) = (12)$, after a move, we move six pebbles to x_{34}. As $p_1 \geq 7$, we move a pebble to either x_{36} or x_{31} from G_1 and so we move a pebble to x_{31}.

Suppose not. First, we take $p_1 = 7$. We move a pebble to G_4 from G_1. As either $p_2 = 6$ or $p_2 = 7$ we move a pebble to G_4 from G_2 if $p_2 = 7$. Now in the resulting distribution of fourteen pebbles on G_4, clearly we move two pebbles to x_{41}.

Next, let $p_1 \geq 8$. Suppose $(x_{15}) = (2)$ or $(x_{12}) = (4)$, we move either one pebble to x_{46} or two pebbles to x_{41}. In the resulting distribution we move two pebbles to x_{41}. Suppose not. Then $(x_{16}) = (2)$ or $(x_{11}) = (4)$ by Fact 9. Then we move either one pebble to x_{34} or two pebbles to x_{33}. Now, since $(x_{44}) = (10)$ x_{43} we move four pebbles to x_{34} from x_{44} and so we pebble the target.

Case 1 (iv). Let $8 \le \text{Þ}_4 \le 11$.

Clearly $\text{Þ}_1 \ge 8$.

Suppose $(x_{44}) = 8$.

If $\text{Þ}_1 \ge 10$, then either $(x_{14}) = (2)$ or $(x_{13}) = (4)$ and so we move either one pebble to x_{34} or two pebbles to x_{33}. Now, we move four pebbles to x_{34} from x_{44} and so we are done. If not, then $\text{Þ}_1 = 8$ or $\text{Þ}_1 = 9$.

Let $\text{Þ}_1 = 8$. Then one of the following holds:

(i) $\text{Þ}_4 = 11$ and $\text{Þ}_2 = 7$.

(ii) $\text{Þ}_4 = 10$ and $\text{Þ}_2 = 8$.

Now, in G_1, if $(x_{14}) = (2)$ or $(x_{13}) = (4)$, we are done as before. Or, in G_2, if either $(x_{23}) = (2)$ or $(x_{24}) = (4)$ then we are done. If not, then, either $(x_{14}) = (2)$ or $(x_{13}) = (4)$ by Fact 9 and either $(x_{23}) = (2)$ or $(x_{24}) = (2)$ if (i) holds (Fact 1), either $(x_{23}) = (2)$ or $(x_{24}) = (4)$ if (ii) holds (Fact 9). Now, we move the above pebbles to G_4 and in the resulting distribution we place two pebbles on x_{43}.

Now let $\text{Þ}_1 = 9$. Then $6 \le \text{Þ}_2 \le 9$. If $\text{Þ}_2 \ge 7$ then we move a pebble to G_1 and now G_1 has ten pebbles and so we are done as before. If not, then $\text{Þ}_2 = 6$. Also $\text{Þ}_4 = 11$. If either $(x_{23}) = (2)$ or $(x_{24}) = (4)$, we move the above pebbles to G_3 and then we move four pebbles to x_{34} from x_{44} and we pebble the target. Or, if we move a pebble to G_1 from G_2 then we are done. Suppose we move a pebble to G_4 from G_2. If either $(x_{14}) = (2)$ or $(x_{13}) = (4)$, we move the above pebbles to G_3. Now, we move four pebbles to x_{34} from x_{44} and we pebble the target. If not, as $\text{Þ}_1 = 9$, by Fact 9 either $(x_{13}) = (2)$ or $(x_{12}) = (4)$, we move the above pebbles to G4. In the resulting distribution we place two pebbles on x_{41}.

Suppose not. We move one pebble to G_2 from G_1 and clearly we place two pebbles on x_{55} and so we move a pebble to x_{12}. Another pebble is moved to x_{32} using the eight pebbles on x_{44} and so we pebble the target.

Suppose eight pebbles cannot be placed on x_{44}.

Clearly $Þ_4 \leq 10$ by Fact 3. If $(x_{13}, x_{23}) = (4,2)$ we get the transmitting subgraph $\{x_{21}, x_{22}, x_{17}\}$. Or if $(x_{26}, x_{16}) = (4,2)$ we get the transmitting subgraph $\{x_{35}, x_{36}, x_{31}\}$.

If not, first we take $Þ_4 = 10$. Therefore $Þ_1 \geq 8$. If $Þ_1 \geq 10$ then $(x_{11}) = (2)$ or $(x_{12}) = (4)$ and so we move either one pebble to x_{44} or two pebbles to x_{47}. In the resulting distribution we place two pebbles on x_{41}. If $Þ_1 = 8$ then $Þ_2 = 8$ and if $Þ_1 = 9$ then $Þ_2 = 8$. So, we move two pebbles to G_4, one from G_1, another from G_2. In the resulting distribution of twelve pebbles on G_4 we move two pebbles to x_{41}.

Now, let $Þ_4 = 9$. First, we take $Þ_1 = 9$. Therefore $Þ_2 = 8$. If either $(x_{13}) = (4)$ or $(x_{12}) = (4)$ then we move a pebble to x_{41}. Another pebble is added to x_{41} using the nine pebbles on G_4. Or, if either $(x_{13}) = (2)$ or $(x_{12}) = (4)$, we move these pebbles to G_4 and we move one more pebble to G_4 from G_2. In the resulting distribution, clearly we place two pebbles on x_{41}. Similarly, if either $(x_{22}) = (2)$ or $(x_{25}) = (4)$, we can place two pebbles on x_{41}. Otherwise, one can easily check that by moving as many pebbles as possible either G_1 from G_2 and G_4 or to G_2 from G_1 and G_4, we pebble the target. Next, we take $Þ_1 = 10$. Therefore $Þ_2 = 7$. Now, we move one pebble to G_4 from G_2 and so we get ten pebbles on G_4. Now, both G_1 and G_2 each has ten pebbles and this has been already discussed. Next, we take $Þ_1 \geq 11$. Or, if either $(x_{13}) = (4)$ or $(x_{13}, x_{12}) = (2, 2)$ or $(x_{12}) = (6)$ we move the above pebbles to G_4. In the resulting distribution we place two pebbles on x_{41}. If not, clearly by moving as many pebbles as possible either to G_2 from G_1 and G_4 or to G_1 from G_2 and G_4, we pebble the target.

Let $\text{Þ}_4 = 8$. Clearly $\text{Þ}_1 \geq 9$. First, We assume four pebbles cannot be moved to x_{44}. If $\text{Þ}_2 \geq 7$ then we move two pebbles to G_4, one from G_1 and another from G_2 and then we move two pebbles to x_{41}. If not, then $\text{Þ}_1 \geq 12$. Suppose we get at least seven pebbles on the path $\{x_{33}, x_{42}\}$. As $(x_{12}) = (2)$ (otherwise $(x_{12}) = (10)$ and so we are done) we move one pebble to x_{43} and so we place two pebbles on x_{41}. Suppose not. Then as $(x_{13}) = (2)$ (otherwise, $(x_{12}) = (10)$ and we move five pebbles to x_{42} and then we move two pebbles to x_{41}), and we move one pebble to x_{46} and clearly we place two pebbles on x_{41}. Next, we assume $(x_{44}) = (4)$. If either $(x_{11}) = (6)$ or if $(x_{11}, x_{21}) = (2,2)$ we get one of the transmitting subgraphs $\{x_{14}, x_{15}, x_{34}, x_{11}\}$, $\{x_{14}, x_{25}, x_{22}, x_{21}\}$. If not, first we take $\text{Þ}_1 = 9$, $\text{Þ}_2 = 9$. If either $(x_{11}, x_{12}) = (2,2)$ or $(x_{12}) = (6)$ then we move the above pebbles to G_4 and then we move at least one pebble to G_4 from G_2 and clearly we place two pebbles on x_{41}. Similarly, if either $(x_{24}, x_{22}) = (2,2)$ or $(x_{21}) = 6$ we proceed as above. Otherwise, it can be easily verified that by moving as many pebbles as possible either to G_1 from G_2 and G_4 or to G_2 from G_1 and G_4 the target can be pebbled. Next, we take either $\text{Þ}_1 = 10$ or $\text{Þ}_1 = 11$. If either $(x_{21}) = 2$ or if $(x_{25}) = 4$ we move the above pebbles to G_4. Then, since either $(x_{15}) = (2)$ or $(x_{12}) = (4)$ by Fact 2, we move these pebbles to G_4. In the resulting distribution we place two pebbles on x_{41}. Or if either $(x_{12}) = (6)$ or $(x_{24}) = (6)$ then we move three pebbles to G_4 and we place two pebbles on x_{41}. If not, then clearly the target can be pebbled by moving as many pebbles as possible either to G_1 from G_2 and G_4 or to G_2 from G_1 and G_4. Next, let $\text{Þ}_1 \geq 12$. If either $(x_{10}) = 4$ or $(x_{12}) = 6$ we move the above pebbles to G_4 and we place two pebbles on x_{41}. Otherwise we pebble the target by moving pebbles either to G_1 or to G_2 as before.

Case I (v). Let $4 \leq \text{Þ}_4 \leq 7$.

Clearly $\text{Þ}_1 \geq 10$ and $\text{Þ}_2 \leq 11$.

Suppose $(x_{44}) = (4)$ after some moves.

First, we take $P_1 = 10$. Therefore, either $P_2 = 9$ or $P_2 = 10$. $(x_{13}) = 2$ (otherwise, $(x_{14}) = 4$ by Fact 2 and so we are done). Also, if $(x_{23}) = (2)$ then we get the transmitting subgraph $\{x_{34}, x_{33}, x_{32}, x_{31}\}$. If not, then $(x_{26}) = 2$. Therefore, if $(x_{16}) = 2$ then we get the transmitting subgraph $\{x_{34}, x_{35}, x_{36}, x_{31}\}$. So, we assume x_{34} is not 2-pebbled. Then $(x_{13}) = 4$ by Fact 2. Now, if either $(x_{13}) = (6)$ or $(x_{26}) = 6$, then we are done. Otherwise, clearly $(x_{12}) = 6$ or $(x_{15}, x_{13}) = (2,2)$ in G_1 and in G_2 either $(x_{22}) = 2$ or $(x_{23}) = 4$. We move the above pebbles to G_4 and in the resulting distribution we place two pebbles on x_{41}.

Next, Let $P_1 \ge 11$. Therefore $P_2 \le 11$. Let $P_2 \ge 7$. If $(x_{23}) = 2$ then we get the transmitting subgraph $\{x_{34}, x_{33}, x_{32}, x_{31}\}$, since $(x_{13}) = 2$ by Fact 3. If not, then $(x_{26}) = 2$. Also $(x_{16}) = 2$ by Fact 3 and so we get the transmitting subgraph $\{x_{34}, x_{35}, x_{36}, x_{31}\}$. Now, let $P_2 \le 6$. If $P_2 = 6$, then either $P_1 \ge 13$. If either $(x_{23}) = (4)$ or $(x_{26}) = 2$ then we are done. Or if we move two pebbles to G_1 from G_3 then we get at least fifteen pebbles on G_1 and so we place either four pebbles on x_{14} or six pebbles on x_{13} by Fact 7 and so we are done. If not, we move a pebble to G_2 from G_4 (since $(x_{44}) = (4)$) and clearly we place two pebbles on x_{23}. As $(x_{13}) = 4$ by Fact 5 we get the transmitting subgraph $\{x_{13}, x_{12}, x_{11}\}$. Now, if $P_2 = 5$ then $P_1 \ge 14$. If $P_1 = 14$ and if x_{13} is not 6-pebbled then either $(x_{15}) = 6$ or $(x_{12}) = (10)$ or $(x_{15}, x_{12}) = (2,8)$, we move the above pebbles to G_4 and then clearly we place two pebbles on x_{41}. If not, then we move as many pebbles as possible to G_2 from G_1 and G_4 then we pebble the target. In all the other distributions, as $P_1 \ge 15$, either $(x_{16}) = 4$ or $(x_{13}) = 6$ by Fact 7 and so we are done.

Suppose four pebbles cannot be placed on x_{44}.

If either $(x_{13}, x_{23}) = (4,2)$ or $(x_{16}, x_{26}) = (2,4)$, we get one of the transmitting subgraphs $\{x_{23}, x_{22}, x_{21}\}$, $\{x_{25}, x_{26}, x_{21}\}$.

If not, first we take $P_4 = 7$. Therefore $P_1 \geq 10$ and $P_2 \leq 9$. If $P_2 \geq 7$ we get $P_1 \leq 12$. Therefore, either $(x_{15}) = (2)$ or $(x_{12}) = 4$. We move the above pebbles to G_4. Now, we move one pebble to G_4 from G_2. In the resulting distribution we place two pebbles on x_{41}. If $P_2 \leq 6$ then $P_1 \geq 13$. Suppose $(x_{45}) = 6$. Then either $(x_{15}) = 2$ or $(x_{12}) = (8)$ by Fact 3 and we move the above pebbles to G_4. Now, in the resulting distribution we place two pebbles on x_{41}. Suppose not. Then, if either $(x_{16}) = (4)$ or $(x_{13}) = (8)$ we are done. If not, clearly either $(x_{15}) = (4)$ or $(x_{12}) = (6)$ and so we move the above pebbles to G_4 and then we place two pebbles on x_{41}.

Next, Let $P_4 = 6$, Therefore $P_1 \geq 10$ and $P_2 \leq 10$. First, we take $P \geq 7$. We assume x_{17} is not 2-pebbled. Therefore $(x_{11}) = 4$ by Fact 2 and so we move two pebbles to x_{46}. Also, we move one pebble to G_4 from G_2 and then clearly we place two pebbles on x_{41}. So, we assume $(x_{15}) = 2$. If $(x_{12}, x_{13}) = (2,2)$, then we move the above pebbles to G_4. Now, we move one pebble to G_4 from G_2 and in the resulting distribution we move two pebbles to x_{41}. Or if both $(x_{22}, x_{24}) = (2,2)$ we move these pebbles to G_4. Also, either $(x_{15}) = 2$ or $(x_{12}) = 4$ by Fact 2 and we move the above pebbles to G_4. Now, we place two pebbles on x_{41}. Otherwise the target can be pebbled by moving as many pebbles as possible either to G_1 from G_2 or to G_2 from G_1. Next, we take $P_2 \leq 6$. So, $P_1 \geq 14$. If x_{15} is not 2-pebbled then $(x_{12}) = (10)$ by Fact 4 and so we move five pebbles to x_{41} and in the resulting distribution of eleven pebbles on G_4, clearly we place two pebbles on x_{41}. So, we assume $(x_{16}) = 2$. If either $(x_{11}) = 6$ or $(x_{13}, x_{12}) = (4,2)$ or $(x_{15}, x_{12}) = (2,6)$, we move these pebbles to G_4 and in the resulting distribution we place two pebbles on x_{41}. If not, suppose either $(x_{21}) = (2)$ or $(x_{24}) = 2$ then we move one pebble to either x_{11} or x_{14}. Clearly, in the resulting distribution of fifteen pebbles on G_1 one of the above holds. Suppose not.

Then we move at least six pebbles to G_2 from G_1 and we place four pebbles on x_{23} and so we are done.

Now, let $\bar{p}_4 = 5$. Therefore $\bar{p}_1 \geq 11$ and $\bar{p}_2 \leq 10$. First, we assume neither x_{45} nor x_{43} is 2-pebbled. If $\bar{p}_1 \leq 14$ and if x_{13} is not 2-pebbled then $(x_{12}) = (8)$ and so we move four pebbles to x_{43}. Also, we move a pebble to G_4 from G_2 as $\bar{p}_2 \geq 7$ when $\bar{p}_1 \leq 14$. Now, clearly we place two pebbles on x_{41}. If $(x_{11}) = (2)$ then $(x_{12}) = (2)$ simultaneously, we move these pebbles to G_4 and then we move one more pebble to G_4 from G_2. Now, we place two pebbles on x_{41}. Next, we take $\bar{p}_1 \geq 15$. If x_{11} is not 2-pebbled then $(x_{12}) = (12)$ by Fact 6 and so we move six pebbles to x_{43}. Now, we move two pebbles to x_{41}. If $(x_{15}) = (2)$ then $(x_{12}) = (2)$ simultaneously, we move two pebbles to G_4 and clearly we place two pebbles on x_{41}. Now, we assume either $(x_{45}) = (2)$ or $(x_{43}) = (2)$. If $\bar{p}_2 \geq 7$ and if either $(x_{25}) = (4)$ or $(x_{24}) = (6)$, then we move the above pebbles to G_4. Also either $(x_{13}) = (2)$ or $(x_{12}) = (4)$ as $\bar{p}_1 \geq 11$. We move these pebbles to G_4 and in the resulting distribution clearly we place two pebbles on x_{41}. Or if $(x_{12}, x_{13}) = (6,2)$, then we move those pebbles to G_4. Also, we move a pebble to G_4 form G_2 and now we place two pebbles on x_{41}. If $\bar{p}_2 \leq 6$ then $\bar{p}_1 \geq 15$. If either $(x_{11}) = (6)$ or $(x_{12}, x_{15}) = (8,2)$, then we move these pebbles to G_4 and we place two pebbles on x_{41}. If all the above fail, then we move a pebble either to G_1 if $(x_{43}) = (2)$ or to G_2 if $(x_{45}) = (2)$. Now, we pebble the target by moving as many pebbles as possible either to G_1 form G_2 or to G_2 from G_1.

Next, let $\bar{p}_4 = 4$. Clearly $\bar{p}_1 \geq 11$ and $\bar{p}_2 \leq 11$. If either $(x_{16}) = (4)$ or $(x_{12}) = (8)$ or $(x_{22}) = (4)$ or $(x_{26}) = (8)$, we are done. Or if either $(x_{16}, x_{26}) = (2,4)$ or $(x_{13}, x_{23}) = (4,2)$, we get one of the transmitting subgraphs $\{x_{35}, x_{36}, x_{11}\}$, $\{x_{33}, x_{32}, x_{11}\}$. Or, if either $(x_{14}, x_{13}, x_{25}, x_{23}) = (2,4,2,4)$ or $(x_{15}, x_{22}) = (4,4)$, we move the above pebbles to G_4 and we place two pebbles on x_{41}. Otherwise, clearly the target can be pebbled only by moving as many pebbles as possible either to G_1 from G_2 or to G_2 from G_1.

Case I (vi). Let $p_4 \leq 3$

Clearly $p_1 \geq 12$

First, we take $p_1 = 12$. Therefore, either $p_2 = 11$ or $p_2 = 12$. Suppose $(x_{12}) = (4)$. As $(x_{25}) = (2)$ (Fact 3) we get the transmitting subgraph $\{x_{13}, x_{23}, x_{31}\}$. Suppose not. Then x_{15} is 2-pebbled but not 4-pebbled. Similarly, we get x_{26} is 2-pebbled but not 4-pebbled. Now, clearly we move at least two pebbles to G_1 from G_2 and we place four pebbles on x_{16} and so we are done. Next, we take $p_1 \geq 13$. Therefore $p_2 \leq 13$. Let $10 \leq p_2 \leq 13$. In G_1, if x_{16} is not 4-pebbled then $(x_{13}) = (4)$ (Fact 5). Therefore, if $(x_{23}) = (2)$, we get the transmitting subgraph $\{x_{15}, x_{32}, x_{31}\}$. If not, then $(x_{26}) = (4)$. As $p_1 \geq 13$, $(x_{16}) = (2)$ (otherwise $(x_{13}) = (8)$ and so we are done). So, we get the transmitting subgraph $\{x_{14}, x_{36}, x_{31}\}$. Next, let $7 \leq p_2 \leq 9$. Then $p_1 \geq 14$. So, $(x_{13}) = (4)$ (otherwise, $(x_{16}) = (4)$ and so we are done). If $(x_{23}) = (2)$, then we get the transmitting subgraph $\{x_{33}, x_{32}, x_{31}\}$. If not, then either we move at least one pebble to G_1 from G_2 and we put either four pebbles on x_{16} or eight pebbles on x_{13} or we move at least five pebbles to G_2 from G_1 and we put four pebbles on x_{23} and so we are done. Next, let $p_2 \leq 6$. Then $p_1 \geq 17$. If $p_1 \geq 18$, then either $(x_{16}) = (4)$ or $(x_{13}) = (8)$ (Fact 8) and so we pebble the target. If $p_1 = 17$ and if we move a pebble to G_1 from G_2 then we are done. If not, clearly we move seven pebbles to G_2 from G_1 and we put four pebbles on x_{25} and hence the case is complete.

Step 2 . Let $\text{Þ}_3 = 1$.

Case 2 (i). Let $\text{Þ}_4 \geq 15$

If $\text{Þ}_{44} \geq 15$, then we move seven pebbles to x_{34} and so we get eight pebbles on x_{34} and so we are done. If not, clearly $(x_{41}) = (2)$ and so we pebble the target.

Case 2 (ii). Let $\text{Þ}_4 = 14$.

If $\text{Þ}_{44} = 14$ then we move seven pebbles to x_{34}. If $\text{Þ}_{44} \leq 12$ then clearly $(x_{41}) = (2)$. If $\text{Þ}_{44} = 13$, then either $\text{Þ}_{42} = 1$ or $\text{Þ}_{43} = 1$ (otherwise, $(x_{41}) = (2)$). Also $\text{Þ}_1 \geq 6$ and $\text{Þ}_2 \leq 5$. If $\text{Þ}_1 \geq 7$, we move a pebble to G_4 from G_1 and in the resulting distribution we move two pebbles to x_{41}. If not, then $\text{Þ}_1 = 6$ and $\text{Þ}_2 = 5$. If we move a pebble to either G_4 or G_3 or G_1 from G_2 then we are done. If not, then $\text{Þ}_{2j} = 1$ except for one j, $1 \leq j \leq 6$. Now, if a pebble is moved G_2 from G_1, then we move one more pebble to G_2 from G_4 using at most four pebbles. Now, in the resulting distribution, we place two pebbles on x_{21} and so we move a pebble to x_{32}. As there at least nine pebbles on x_{44} after the above move, we move four pebbles to x_{34} from x_{44} and so we pebble the target. Otherwise, $\text{Þ}_{2j} = 1$ for every j, $1 \leq j \leq 6$. Now, we use at most four pebbles on x_{44} to move a pebble to x_{13} and then we move a pebble to x_{30} from G_1. Now, as there are nine pebbles on x_{44}, we move four pebbles to x_{34} and so we pebble the target.

Case 2 (iii). Let Þ_4 be either twelve or thirteen.

Clearly $\text{Þ}_1 \geq 6$ and $\text{Þ}_2 \leq 6$.

Let $\text{Þ}_1 \geq 7$. If $(x_{44}) = (12)$, then we move six pebbles to x_{34} from x_{44} and we move one more pebble to G_3 from G_1 and so we are done. If not, first we take $\text{Þ}_1 \geq 8$. If either $(x_{14}) = (2)$ or $(x_{13}) = (4)$, we move the above pebbles to G_3. Now, as either $(x_{41}) = (2)$ or $(x_{44}) = (10)$ by Fact 4, we move four pebbles to x_{34} and so we pebble the target. If not, then either $(x_{13}) = (2)$ or $(x_{12}) = (4)$ by Fact 9. We move these pebbles to G_4 and we place two pebbles on x_{41}. Next, let $\text{Þ}_1 = 7$. Therefore, either $\text{Þ}_2 = 5$ or $\text{Þ}_2 = 6$. Let $\text{Þ}_2 = 6$. If a pebble is moved to either G_1 or G_4 or G_3 from G_2, we are done. If not, then $\text{Þ}_{2j} = 1$ for every j, $1 \leq j$

≤ 6. We move a pebble to G_3 from G_1 and we place two pebbles on x_{31} and so we move a pebble to x_{32}. Now, we move four pebbles to x_{34} from x_{44} and we are done. Let $\text{Þ}_2 = 5$. We move a pebble to G_4 from G_1 and we place two pebbles on x_{41}. Next, let $\text{Þ}_1 = 6$. Then $\text{Þ}_2 = 6$ and $\text{Þ}_4 = 13$. If we move a pebble either to G_1 from G_2 or G_2 from G_1, we are done as before. Or, if either $(x_{21}) = (2)$ or $(x_{16}) = (2)$ or $(x_{26}, x_{13}) = (2,2)$, we move the above pebbles to G_3 and then we move five pebbles to x_{34} from x_{44} and so we are done. Or, if either $(x_{11}) = (2)$ or $(x_{22}) = (2)$ or $(x_{12}, x_{22}) = (2,2)$, we move these pebbles to G_4 and then clearly we move two pebbles to x_{41}. Or, if either $(x_{25}, x_{13}) = (2,2)$ or $(x_{26}, x_{12}) = (2,2)$, we move one pebble to G_4 and one pebble to G_3. Now, in G_4, either we move two pebbles to x_{41} or we move twelve pebbles to x_{44} by Fact 6 and so we are done. If all the above fail, then $\text{Þ}_{ij} = 1$ for either for every $i = 1$ or $i = 2$ or $i = 1,2$, $1 \leq j \leq 6$. If either $\text{Þ}_{35} = 1$ or $\text{Þ}_{36} = 1$, we move a pebble to G_3 using at most four pebbles from G_4 and then we move a pebble to x_{34}. Now, as there are at least six pebbles on x_{44}, we move three pebbles to x_{34} from x_{44} and so we are done. If not, then we move a pebble to G_2 from G_4 and we move a pebble to x_{32}. Now, we move three pebbles to x_{34} from x_{44} and so we are done.

Case 2 (iv). Let Þ_4 be either ten or eleven.

Clearly $\text{Þ}_1 \geq 7$ and $\text{Þ}_2 \leq 7$.

Suppose $(x_{44}) = (10)$.

First, we take $\text{Þ}_1 \geq 7$ and $\text{Þ}_2 \geq 7$. We move two pebbles to G_3, one from G_1 and another from G_2 and then we move five pebbles to x_{34} from x_{44}.

Next, let $\text{Þ}_1 \geq 10$. We move either one pebble to x_{36} or two pebbles to x_{33} using Fact 2. Now, we move five pebbles to x_{34} from x_{44} and we pebble the target.

Now, we take $\text{Þ}_1 = 8$ and $\text{Þ}_2 = 6$. If either $(x_{16}) = (2)$ or $(x_{13}) = (4)$, we are done as above. If not, then either $(x_{13}) = (2)$ or $(x_{12}) = (4)$ by Fact 9. We move these pebbles to G_4, only if a pebble is moved to G_4 from G_2 and so we move two pebbles to x_{41}. Or, if a pebble is moved to G_3 from G_2, we move one more pebble to G_3 from G_1 and then we move five pebbles to x_{34} from x_{44}. Or, if two pebbles are moved to G_1 from G_2, we are done. If not, we move a pebble to G_2 from G_1 and clearly we place two pebbles on x_{21} and we move a pebble to x_{12}. Now, we move five pebbles to x_{34} from x_{44} and so we are done.

Next, let $p_1 = 9$. So, either $p_2 = 6$ or $p_2 = 5$. If a pebble is moved to either G_1 or G_4 or G_3 from G_2, we are done. If not, then $p_{2j} = 1$ at least for five values of j, $1 \leq j \leq 6$. Now, we move two pebbles to G_3, one from G_1 and another from G_4. As there are at least six pebbles on x_{44} after the above move, we move three pebbles to x_{34}. Now, in G_2, we put two pebbles on x_{23} and we move a pebble to x_{32} if either $p_{34} = 1$ or $p_{33} = 1$ or $p_{32} = 1$. Otherwise, in G_3, we place two pebbles on x_{36} and we move a pebble to x_{33} and so we are done.

Suppose ten pebbles cannot be placed on x_{44}.

First, we assume $(x_{44}) = (6)$.

If $p_1 \geq 10$ then $(x_{14}) = (2)$ or $(x_{13}) = (4)$ and we move the above pebbles to G_3. Now, we move three pebbles to x_{34} from x_{44} if $p_{34} = 1$ and then we move a pebble to x_{31}. If $p_{34} \neq 1$ and if $(x_{44}) = (8)$, we move four pebbles to x_{34} and so we are done. Otherwise, we move either one pebble to x_{46} or two pebbles to x_{42} from G_1 using Fact 2 and we place two pebbles on x_{41}.

Suppose $p_1 = 9$. Let $p_{34} = 1$. If either $(x_{14}) = (2)$ or $(x_{11}) = (4)$ then we are done as above. If not, either $(x_{16}) = (2)$ or $(x_{12}) = (4)$ by Fact 9 and we move the above pebbles to G_4 when $p_4 = 11$ and in the resulting distribution we place two pebbles on x_{41}. When $p_4 = 10$, $p_2 = 6$. If a pebble can be moved either to G_1 or to G_4 from G_2 then we get either ten pebbles on G_1 or eleven pebbles on G_4 and so we proceed as before. Or, if a pebble is moved to x_{32} from G_2 we are done. If not, we move at least one pebble to G_2 from G_1 and clearly we place two pebbles on x_{23} and so we move a pebble to x_{32}. Now we move three pebbles to x_{34} from x_{44} and so we pebble to target. Now, let Þ$_{34} \neq 1$. Suppose $(x_{44}) = (8)$. If either $(x_{14}) = (2)$ or $(x_{13}) = (4)$, we move these pebbles to G_3 and then we move four pebbles to x_{34} from x_{44} and so we are done. If not, first we take Þ$_2 = 5$. As $(x_{15}) = (2)$ or $(x_{12}) = (4)$ by Fact 9, we move these pebbles to G_4 and we place two pebbles on x_{41}. Now, let Þ$_2 = 6$. If a pebble is moved to either G_1 or G_3 or G_4 from G_2 we are done as above. If not, Þ$_{2j} = 1$ for every j, $1 \leq j \leq 6$. So, we move a pebble to G_2 from G_1 and we place two pebbles on x_{23} and we move a pebble to x_{32}. Now, we move four pebbles to x_{34} from x_{44} and so are are done. Suppose eight pebbles cannot be moved to x_{44}. Then Þ$_4 = 10$ by Fact 3. We move a pebble to G_4 from G_1 and clearly we place two pebbles on x_{41}.

Next, let Þ$_1 = 8$. Let Þ$_{34} = 1$. If either $(x_{16}) = (2)$ or $(x_{13}) = (4)$, we move the above pebbles to G_3. Now, we move three pebbles to x_{34} from x_{44} and so we pebble the target. If not, by Fact 9, either $(x_{15}) = (2)$ or $(x_{12}) = (4)$ and we move these pebbles to G_4. Also, if Þ$_4 = 10$ we move one more pebble to G_4 from G_2 as Þ$_2 = 7$ in such case. Now, in the resulting distribution we place two pebbles on x_{41}. Let Þ$_{34} \neq 1$. Suppose $(x_{44}) = (8)$. If either $(x_{16}) = (2)$ or $(x_{13}) = (4)$, we are done. If not, then we move two pebbles to G_3, one from G_1, another from G_2 and we pebble the target if Þ$_2 = 7$. If Þ$_2 = 6$, then, since either $(x_{15}) = (2)$ or $(x_{12}) = (4)$ by Fact 9, we move these pebbles to G_4 and we place two pebbles on x_{41}. Suppose not. Then Þ$_4 = 10$ by Fact 3. We move two pebbles to G_4, one from G_1 another from G_2 and we place two pebbles on x_{41}.

Finally, if Þ$_1 = 7$ then Þ$_2 = 7$ and Þ$_4 = 11$. So, we move two pebbles to G_4, one from G_1 and another from G_2. Now, clearly we move two pebbles to x_{41}.

Now, we assume we cannot place six pebbles on x_{44}.

Then, clearly Þ$_4 = 10$ by Fact 3. As Þ$_1 \geq 8$, we move at least one pebble to G_4 from G_1 and in the resulting distribution clearly we place two pebbles on x_{41}.

Case 2(v). Let $6 \leq \text{Þ}_4 \leq 9$.

Let $\text{Þ}_{34} = 1$.

Suppose $(x_{44}) = (6)$.

First, we consider the case $\text{Þ}_4 = 9$ and $\text{Þ}_2 = 8$. If either $(x_{14}) = (2)$ or $(x_{13}) = (4)$ or $(x_{23}) = (2)$ or $(x_{24}) = (4)$ we move these pebbles to G_3 and then we move three pebbles to x_{34} from x_{44} and so we are done. If not, by Fact 9, either $(x_{13}) = (2)$ or $(x_{12}) = (4)$ and either $(x_{25}) = (2)$ or $(x_{22}) = (4)$. Now, we move the above pebbles to G_4 and in the resulting distribution we move two pebbles to x_{41}. In all the other cases, either $\text{Þ}_1 = 10$ or the number of pebbles in G_1 can be made ten after moving a pebble from G_2 to G_1. So, either $(x_{16}) = (2)$ or $(x_{15}) = (4)$ and we move these pebbles to G_3. Now, we move three pebbles to x_{34} from x_{44} and then we pebble the target.

Suppose six pebbles cannot be placed on x_{44}.

First, we assume $\text{Þ}_4 = 9$. Then $\text{Þ}_1 \geq 8$ and $\text{Þ}_2 \leq 8$. If $\text{Þ}_2 \geq 7$ then we move two pebbles to G_4, one from G_1 and another from G_2. In the resulting distribution of eleven pebbles on G_4, we place two pebbles on x_{41}. If $\text{Þ}_2 \leq 6$ then $\text{Þ}_1 \geq 10$. So, either $(x_{13}) = (2)$ or $(x_{12}) = (4)$ and we move these pebbles to G_4. Now, we place two pebbles on x_{41}.

Next, let $\text{Þ}_4 = 8$. Therefore, $\text{Þ}_1 \geq 9$ and $\text{Þ}_2 \leq 8$. It follows from Fact 1 that if x_{41} is not 2-pebbled then $(x_{44}) = (2)$. Suppose $\text{Þ}_1 = 9$ and $\text{Þ}_2 = 8$. If either $(x_{13}) = (2)$ or $(x_{12}) = (4)$, we move these pebbles to G_4. Now, we move one more pebble to G_4 from G_2 and clearly we place two pebbles on x_{41}. Similarly, if either $(x_{22}) = (2)$ or $(x_{21}) = (4)$, we proceed as before. Or, if either $(x_{13}) = (6)$ or $(x_{26}) = (6)$, we move three pebbles to G_3 and then we move a pebble to x_{34} from x_{44} and so we are done. Or, if $(x_{15}, x_{25}) = (2, 2)$ or if $(x_{16}, x_{26}) = (2, 2)$, we get one of the transmitting subgraphs $\{x_{34}, x_{35}, x_{36}, x_{31}\}$, $\{x_{34}, x_{35}, x_{32}, x_{31}\}$. If not, then we move at least one pebble to G_1 from G_2 and one pebble to G_1 from G_4

and we place four pebbles on x_{16}. If $P_1 = 10$ then either $(x_{13}) = (2)$ or $(x_{12}) = (4)$ and we move these pebbles to G_4. Also, we move a pebble to G_4 from G_2. Now, in the resulting distribution we place two pebbles on x_{41}. Next, let $P_1 \geq 11$. Clearly $(x_{12}) = (2)$ (otherwise, $(x_{12}) = (8)$ and so we move four pebbles to x_{43} and then we place two pebbles on x_{41}). If $(x_{13}) = (4)$ we move two pebbles to x_{46} and in the resulting distribution we place two pebbles on x_{41}. If not, then $(x_{12}, x_{14}) = (2, 2)$ and so we move two pebbles to G_4. Now, we place two pebbles on x_{41}.

Let $P_4 = 7$. Then $P_1 \geq 9$ and $P_2 \leq 9$. If x_{41} is not 2-pebbled then $(x_{44}) = (2)$. Suppose $P_1 = 9$ and $P_2 = 9$. Let $(x_{14}) = (2)$. If $(x_{26}) = (2)$ then we get the transmitting subgraph $\{x_{34}, x_{35}, x_{36}, x_{31}\}$. If not, then $(x_{23}) = (2)$. Therefore, if $(x_{13}) = (2)$ we get the transmitting subgraph $\{x_{34}, x_{33}, x_{32}, x_{31}\}$. Otherwise, the distribution is such that if $(x_{14}) = (2)$ then x_{34} cannot be 2-pebbled and if $(x_{13}) = (2)$ then x_{33} cannot be 2-pebbled. Now, if $(x_{14}) = (4)$ then we move three pebbles to G_4 (two pebbles from x_{15} and one pebble from G_2) and we place two pebbles on x_{41}. Similarly, if $(x_{22}) = (4)$, we are done. If not, then clearly we move either at least three pebbles from G_2 to G_1 and we move one pebble to G_1 from G_4 and then we place four pebbles on x_{14} or we move at least three pebbles to G_2 from G_1 and one more pebble to G_2 from G_4 and we place four pebbles on x_{21}. Next, let $P_1 = 10$. Then $P_2 = 8$. Clearly $(x_{13}) = (2)$ (otherwise $(x_{16}) = (4)$ by Fact 4 and so we are done). If $(x_{23}) = (2)$ then we get the transmitting subgraph $\{x_{34}, x_{33}, x_{32}, x_{31}\}$. If not, then $(x_{26}) = (2)$. So, if $(x_{16}) = (2)$, we get the transmitting subgraph $\{x_{34}, x_{35}, x_{36}, x_{31}\}$. Otherwise, $(x_{13}) = (4)$ by Fact 4. If $(x_{26}, x_{44}) = (4, 4)$ then we move these pebbles to G_3 and we pebble the target. Or, if either $(x_{13}) = (6)$ or $(x_{36}) = (6)$ then we move three pebbles to either x_{33} or x_{33}. We move a pebble to x_{34} from x_{44} and so we pebble the target. Or, if $(x_{13}) = (6)$ then we move three pebbles to x_{43} and we move one more pebble to G_4 from

G_3 and then we place two pebbles on x_{41}. Similarly we proceed if $(x_{14}) = (6)$. If all the above fail, then clearly $(x_{12}, x_{11}) = (2, 2)$ and so we move two pebbles to G_4. We move one more pebble to G_4 from G_2 and then we move two pebbles to x_{41}. Next, let $p_1 = 11$ and $p_2 = 7$. Clearly $(x_{13}) = (2)$ by Fact 4. So, if $(x_{23}) = (2)$, we get the transmitting subgraph $\{x_{34}, x_{33}, x_{32}, x_{31}\}$. If not, then $(x_{26}) = (2)$. Also $(x_{16}) = (2)$ by Fact 3 and so we get the transmitting subgraph $\{x_{34}, x_{35}, x_{36}, x_{31}\}$. Next, let $p_1 \geq 12$. If either $(x_{16}) = (4)$ or $(x_{13}) = (8)$, we pebble the target. Or, if either $(x_{13}) = (6)$ or $(x_{13}, x_{12}) = (4, 2)$ or $(x_{12}) = (10)$, we move the above pebbles to G_4 and clearly we place two pebbles on x_{41}. Or, if $(x_{11}) = (6)$ we move three pebbles to x_{35}. Now, we move a pebble to x_{34} from x_{44} and we pebble the target. Or, if a pebble is moved to G_3 from G_2 then we get a transmitting subgraph as in the case if $p_1 = 11$ and $p_2 = 7$. Or, if a pebble is moved to G_4 from G_2 then we get eight pebbles on G_4 and this situation has been already discussed where we use only the pebbles on G_1, G_3 and G_4 to pebble the target. If all the above fail, we pebble the target by moving as many pebbles as possible either to G_1 from G_2 and G_4 or to G_2 from G_1 and G_4.

Next, let $p_4 = 6$. Suppose two pebbles cannot be placed on x_{44}. First, we assume $p_2 \geq 7$. As $p_1 \geq 10$, either $(x_{13}) = (2)$ or $(x_{12}) = (4)$. We move the above pebbles to G_4 and then we move one more pebble to G_4 from G_3. In the resulting distribution, we place two pebbles on x_{41}. Next, we take $p_2 \leq 6$. Then $p_1 \geq 13$. If either $(x_{16}) = (4)$ or $(x_{13}) = (8)$ we are done. If not, then either $(x_{13}) = (4)$ or $(x_{13}, x_{14}) = (2, 2)$. We move the above pebbles to G_4 and now we place two pebbles on x_{41}. Suppose $(x_{44}) = (2)$. We assume $p_1 = 10$ and $p_2 = 9$. Without loss of generality, we assume if $(x_{16}) = (2)$ then x_{26} cannot be 2-pebbled and if $(x_{13}) = (2)$ then x_{23} cannot be 2-pebbled. Clearly $(x_{11}) = (2)$ (otherwise $(x_{18}) = (4)$ and so we are done). Therefore $(x_{28}) = (2)$ and x_{23} cannot be 2-pebbled. So, if $(x_{16}) = (2)$, we are done. If not, then $(x_{13}) = (4)$ by Fact 2. If

$(x_{13}) = (6)$ we move three pebbles to x_{13} and we move a pebble to x_{34} from x_{44} and so we are done. Or, if $(x_{12}, x_{23}) = (6, 6)$ then we move these pebbles to G_4 and we place two pebbles on x_{41}. Or, if $(x_{12}, x_{14}) = (2, 2)$ and $(x_{25}, x_{22}) = (2, 2)$ then we move the above pebbles to G_4 and we place two pebbles on x_{41}. If all the above fail, either we move at least two pebbles to G_1 from G_2 and one more pebble to G_1 from G_4 and we place four pebbles on x_{16} or we move at least two pebbles to G_2 from G_1 and one more pebble to G_2 from G_4 and we place four pebbles on x_{25}. Next, let Þ_1 be either eleven or twelve. As $\text{Þ}_2 \geq 7$, if $(x_{23}) = (2)$ then as $(x_{13}) = (2)$ by Fact 3, we get the transmitting subgraph $\{x_{34}, x_{31}, x_{32}, x_{11}\}$. If not, then x_{26} is 2-pebbled. Again, by Fact 3, $(x_{16}) = (2)$ and so we get the transmitting subgraph $\{x_{34}, x_{31}, x_{36}, x_{11}\}$. Next, let $\text{Þ}_1 \geq 13$. If either $(x_{23}) = (2)$ or $(x_{26}) = (2)$ then we are done. Or, if $(x_{13}) = (6)$, we move three pebbles to x_{13} and then we move a pebble to x_{34} from x_{44} and so we are done. Or, if either $(x_{13}) = (6)$ or $(x_{12}) = (10)$ or if $(x_{12}, x_{13}) = (4, 4)$, we move the above pebbles to G_4 and we place two pebbles on x_{41}. Otherwise, we move either one pebble to G_1 from G_4 or at least five pebbles to G_2 from G_1 and one pebble to G_2 from G_4 and then we pebble the target.

Different cases can be easily verified if $\text{Þ}_{34} \neq 1$.

Case 2(vi). Let $\text{Þ}_4 \leq 5$.

Let $\text{Þ}_{34} = 1$.

Suppose $(x_{44}) = (2)$.

Clearly $\text{Þ}_1 \geq 10$ and $\text{Þ}_2 \leq 10$. First, we take $\text{Þ}_1 = \text{Þ}_2 = 10$. Therefore $\text{Þ}_4 = 5$. Without loss of generality, we assume x_{16} is not 2-pebbled in G_1 and x_{25} is not 2-pebbled in G_2. (Otherwise, we get a transmitting subgraph by Fact 2). Therefore $(x_{13}, x_{26}) = (4, 4)$ by Fact 2. If either $(x_{13}) = (6)$ or $(x_{26}) = (6)$, we move three pebbles to G_1 and we move one pebble to x_{34} from x_{44} and so we pebble the target. Or, if $(x_{12}, x_{25}) = (6, 6)$ then we move six pebbles to G_4 and we place two pebbles on x_{42}. If not, we move at least three pebbles to G_2 from G_1 and one more pebble to G_2 from G_4 and then we place four pebbles on x_{21}. Next, let $\text{Þ}_1 \geq 11$. First, we take $\text{Þ}_2 \geq 7$. Therefore we move a pebble to either x_{12} or x_{35} from G_2. So, applying Fact 3 to G_1, we get a transmitting subgraph. Now, let $\text{Þ}_2 \leq 6$. Then $\text{Þ}_1 \geq 14$. We assume $\text{Þ}_1 = 14$. Then $\text{Þ}_2 = 6$ and $\text{Þ}_4 = 5$. If either $(x_{16}) = (4)$ or $(x_{13}) = (6)$ we are done. Or, if either $(x_{21}) = (2)$ or $(x_{26}) = (2)$ we are done. Or, if a pebble is moved to G_1 from G_2 then clearly either x_{16} is 4-pebbled or x_{13} is 6-pebbled and so we are done. Or, if either $(x_{11}) = (4)$ or $(x_{12}) = (8)$ and $(x_{22}) = (2)$, we move the above pebbles to G_4 and we place two pebbles on x_{41}. Or, if $(x_{12}, x_{11}) = (2, 2)$ and either $(x_{25}) = (2)$ or $(x_{22}) = (2)$, we move the above pebbles to G_4. Now, in the resulting distribution we place two pebbles on x_{41}. If all the above fail then we move at least five pebbles to G_2 from G_1 and one pebble to G_2 from G_4 and then we place four pebbles on x_{23}. In all the other distributions we have $\text{Þ}_1 \geq 15$ and so either $(x_{16}) = (4)$ or $(x_{13}) = (6)$ and so we are done.

Suppose two pebbles cannot be placed on x_{44}.

Let $\text{Þ}_4 = 5$. If $\text{Þ}_1 = \text{Þ}_2 = 10$ then either $(x_{13}) = (2)$ or $(x_{12}) = (4)$ either $(x_{21}) = (2)$ or $(x_{24}) = (4)$ by Fact 2. We move the above pebbles to G_4 and clearly we place two pebbles on x_{41}. Now, let $\text{Þ}_1 \geq 11$. First, we take $\text{Þ}_2 \geq 7$. Suppose x_{15} is not 2-pebbled. Then $(x_{12}) = (8)$ and so we move four pebbles to x_{41}. Now, we move a pebble to G_4 from G_2 and clearly we move two pebbles to x_{41}. Suppose not, then $(x_{15}) = (2)$. So, we move a pebble to x_{46}. Also, we move a pebble to G_4 from G_2. Now, if $(x_{41}) = (2)$ we are done. If not, then as $(x_{13}) = (2)$ by Fact 3, instead of moving a pebble to x_{46} from G_1, we move a pebble to x_{47}. Already, we moved a pebble to G_4 from G_2. Clearly, in the resulting distribution we place two pebbles on x_{41}. Next, let $\text{Þ}_1 = 14$. Therefore $\text{Þ}_2 = 6$. If either $(x_{21}) = (2)$ or $(x_{26}) = (2)$ we are done. Or, if a pebble is moved to G_1 from G_2 then we get fifteen pebbles on G_1 and so either x_{15} is 6-pebbled or x_{15} is 4-pebbled and x_{12} is 2-pebbled or x_{15} is 2-pebbled and x_{12} is 4-pebbled. We move the above pebbles to G_4 and clearly we place two pebbles on x_{41}. Or, if a pebble is moved to G_4 from G_2 then we get six pebbles on G_4. In the resulting distribution if two pebbles cannot be moved to x_{44} then we proceed as in Case 2(v). Suppose two pebbles can be moved to x_{44}. Then if $(x_{13}) = (6)$ we move three pebbles to x_{33} and we move a pebble to x_{44} from x_{34} and we pebble the target. Or, if either $(x_{15}) = (6)$ or $(x_{12}) = (10)$ we move the above pebbles to G_4 and we place two pebbles on x_{41}. If not, we take the initial distribution as such and we move as many pebbles as possible either to G_1 from G_2 or to G_2 from G_1 and G_3, we pebble the target. Next, let $\text{Þ}_2 \leq 5$. Suppose $\text{Þ}_{4j} = 1$ except for one j, $1 \leq j \leq 6$. As $\text{Þ}_1 \geq 15$, if either $(x_{16}) = (4)$ or $(x_{13}) = (8)$, we are done. Otherwise, either $(x_{16}, x_{23}) = (2, 2)$ or $(x_{12}) = (6)$ or $(x_{15}) = (6)$. We move the above pebbles to G_4 and we place two pebbles on x_{41}. Suppose there exists a vertex in G_4 with at least two pebbles. Now, if either $(x_{13}) = (6)$ or $(x_{12}, x_{15}) = (8, 2)$ then we move the above pebbles to G_4 and clearly we place two pebbles on x_{41}. If not, then clearly either $(x_{41}) = (2)$ or $(x_{43}) = (2)$. So, we move a pebble to either G_1 or G_2 from G_4. Then we move as many pebbles as possible either to G_1 from G_2 or to G_2 from G_1, we pebble target.

Next, let $\text{Þ}_4 \leq 4$. Therefore $\text{Þ}_1 \geq 11$ and $\text{Þ}_2 \leq 10$. If the distribution is such that either $(x_{16}, x_{26}) = (2, 4)$ or $(x_{13}, x_{23}) = (4, 2)$, we get one of the transmitting subgraphs $\{x_{35}, x_{36}, x_{31}\}$, $\{x_{33}, x_{32}, x_{31}\}$. Or, if the distribution is such that $(x_{15}, x_{12}, x_{22}, x_{23}) = (2, 4, 2, 4)$ or $(x_{15}, x_{22}) = (4, 4)$, we move the above pebbles to G_4 and we place two pebbles on x_{41}. Otherwise, we move as many pebbles as possible either to G_1 from G_2 and G_4 or to G_2 from G_1 and G_4 we pebble the target.

If $\text{Þ}_{34} \neq 1$, cases can be similarly verified.

Step 3. $Þ_3 = 2$.

Case 3(i). Let $Þ_4 \geq 14$.

If x_{41} is not 2-pebbled then by Fact 6, $(x_{44}) = (12)$. So, we move six pebbles to x_{34} from x_{44}.

Case 3(ii). Let $Þ_4$ be either twelve or thirteen.

Clearly $Þ_1 \geq 6$ and $Þ_2 \leq 6$.

If $(x_{41}) = (2)$ then we pebble the target. If not, by Fact 4, $(x_{44}) = (10)$. If $Þ_1 \geq 7$ then we move a pebble to G_3 from G_1 and then we move five pebbles to x_{34} from x_{44}. If $Þ_1 = 6$ then either $Þ_2 = 6$ or $Þ_2 = 5$. First, we take $Þ_1 = Þ_2 = 6$. If a pebble is moved to G_3 from G_1, we are done. If a pebble is moved to G_2 from G_1, then we move a pebble to G_3 from G_2 and then we move five pebbles to x_{34} from x_{44}. Similarly, if a pebble is moved to either G_1 or G_3 from G_2, we are done. Or, if two pebbles are moved to G_4 either from G_1 or from G_2 or one from G_1, another from G_2, we get fourteen pebbles on G_4 and so we proceed as in Case 3(i). If not, then $Þ_{ij} = 1$, for either $i = 1$ or $i = 2$ or $i = 1, 2$ and for every j, $1 \leq j \leq 6$. Without loss of generality, we assume $Þ_{1j} = 1$, $1 \leq j \leq 6$. Now, we move a pebble to G_1 from G_4 using at most four pebbles and we put two pebbles on x_{16} and so we move a pebble to x_{36}. Now, as there are at least six pebbles on x_{44}, we move three pebbles to x_{34} and we pebble the target. Next, let $Þ_1 = 6$ and $Þ_2 = 5$. If a pebble is moved to either G_1 or G_3 or G_4 from G_2 we are done. If not, then $Þ_{2j} = 1$ except for one j, $1 \leq j \leq 6$. Now, if a pebble is moved to either G_3 or G_4 from G_1, we are done. Or, if a pebble is moved to G_2 from G_1, then we move one more pebble to G_2 from G_4 using at most four pebbles and then we move a pebble to x_{32} from G_2. Now,we move three pebbles to x_{34} from x_{44} and so we are done. If not, then $Þ_{1j} = 1$ for every j, $1 \leq j \leq 6$ and so we proceed as before.

As the other cases are similar we leave the proof here.

Step 4. $p_3 = 3$.

Case 4(i). Let $p_4 \geq 12$.

If x_{43} is not 2-pebbled then by Fact 4, $(x_{44}) = (10)$ and so we move five pebbles to x_{34}.

Case 4(ii). Let $p_4 = 11$.

Clearly $p_1 \geq 6$ and $p_2 \leq 6$. If $p_1 \geq 7$, we move a pebble to G_3 from G_1. Then, as $(x_{44}) = (8)$ by Fact 3, we move four pebbles to x_{34} from x_{44}. If $p_1 = 6$ then $p_2 = 6$. If we move a pebble to either G_2 or G_3 or G_4 from G_1, we are done. If not then $p_{1j} = 1$ for every j, $1 \leq j \leq 6$. So, as $(x_{44}) = (8)$, we move a pebble to G_1 using at most four pebbles on G_4 and we put two pebbles on x_{14}. Now, move a pebble to x_{34} from x_{14} and we move two pebbles to x_{34} from x_{44} and we pebble the target.

The other cases can be similarly verified.

Step 5. $p_3 = 4$.

Case 5(i). Let $p_4 \geq 11$.

If x_{43} is not 2-pebbled then $(x_{44}) = (8)$ by Fact 3. So, we move four pebbles to x_{34} from x_{44}.

Case 5(ii). Let $p_4 = 10$.

Clearly $p_1 \geq 6$ and $p_2 \leq 6$. If $p_1 \geq 7$, we move a pebble to G_4 from G_1 and we proceed as in Case 5(i). If $p_1 = 6$ then $p_2 = 6$. If we move a pebble to either G_2 or G_4 from G_1, we are done. Or, if we move either one pebble to x_{34} or two pebbles to x_{13} from G_1, then, as $(x_{44}) = (4)$ by Fact 2, we move two pebbles to x_{34} and we pebble the target. Similarly, if we move a pebble to either G_1 or G_4 or x_{32} or two pebbles to x_{13} from G_2 we are done. Otherwise, if $p_{44} \geq 8$, we move four pebbles to x_{34}. If not clearly $p_{42} \geq 1$ or $p_{43} \geq 1$ and $p_{44} \geq 4$. So, we move a pebble to either G_1 or G_2, say G_1, using at most three pebbles from G_4 and we place two pebbles on x_{14} and then we move a pebble to x_{34}. Now, as there are at least two pebbles on x_{44}, we move a pebble to x_{34} and we pebble the target.

The other cases can be discussed in a similar way.

Step 6. $\text{Þ}_3 = 5$.

Clearly, we pebble the target if either we move three pebbles to x_{34} or we move a pebble to either x_{32} or x_{34} or we move two pebbles to either x_{31} or x_{33} using the pebbles from G_i, $i \neq 3$.

Now, if $\text{Þ}_1 = \text{Þ}_2 = \text{Þ}_4 = 7$, then we move three pebbles to G_{3}, one from each copy G_i, $i \neq 3$.

If not, first we take $\text{Þ}_4 \geq 9$. If $(x_{41}) = (2)$ we are done. If not, suppose $(x_{44}) = (6)$. Then we move three pebbles to x_{34}. Suppose not. If $(x_{34}) = (2)$, we move a pebble to x_{44} and in the resulting distribution we place two pebbles on x_{41}. Otherwise, as $(x_{44}) = (2)$, we move a pebble to x_{34} and we pebble the target.

Let $\text{Þ}_4 = 8$. Then $\text{Þ}_1 \geq 7$. So, we move a pebble to G_4 from G_1. As there are nine pebbles on G_4, we proceed as before.

Let $\text{Þ}_4 = 7$. Then $\text{Þ}_1 \geq 7$ and $\text{Þ}_2 \leq 7$. If $\text{Þ}_1 = \text{Þ}_2 = 7$, we are done. If $\text{Þ}_1 = 8$ then $\text{Þ}_2 = 6$. Suppose $\text{Þ}_{34} \geq 4$. If $(x_{44}) = (4)$, we move two pebbles to x_{34}. We move one more pebble to G_3 from G_1 and we pebble the target. If not, then we move two pebbles to x_{44} from x_{34} and we move one more pebble to G_4 from G_1. In the resulting distribution of ten pebbles on G_4 we move two pebbles to x_{41}. Suppose $\text{Þ}_{34} \leq 3$. We move two pebbles to G_3, one from G_1 and another from G_4. Now, in the resulting distribution of seven pebbles on G_3, we pebble the target. Next, if $\text{Þ}_1 \geq 9$, then we move a pebble to G_1 from G_4. As there are at least ten pebbles on G_1, either x_{14} is 2-pebbled or x_{13} is 4-pebbled. So, we move either one pebble to x_{34} or two pebbles to x_{33}.

Let $\text{Þ}_4 = 6$. Then $\text{Þ}_1 \geq 8$. If $\text{Þ}_1 \geq 10$ we proceed as before. If $\text{Þ}_1 = 8$ then $\text{Þ}_2 = 7$. If $(x_{44}) = (2)$, we move a pebble to x_{34} from x_{44}. Now, we move two pebbles to G_3, one from G_1 and another from G_2. If not, suppose $\text{Þ}_{34} \geq 4$. Then we move two pebbles to x_{44}. Also, we move two pebbles to G_4, one from G_1 and another from G_2. Now, we place two pebbles on x_{41}. Otherwise, $\text{Þ}_{34} \leq 3$. We move two pebbles to G_3, one from G_1 and another from G_2 and we pebble the target. If $\text{Þ}_1 = 9$ then $\text{Þ}_2 = 6$. If $(x_{44}) = (4)$, we move two pebbles to x_{34} and we move one more pebble to G_3 form G_1. Or, if a pebble is moved to G_1 from G_4 then we get ten pebbles on G_1 and so we are done. Or, if a pebble is moved to G_2 from G_4 then we move a pebble to G_1 from G_2. As there are ten pebbles on G_1, we are done. Or, if either $(x_{16}) = (2)$ or $(x_{13}) = (4)$, we move either one pebble to x_{36} or two pebbles to x_{33}. Otherwise, if $(x_{34}) = (4)$ we move two pebbles to x_{44}. Also, either $(x_{11}) = (2)$ or $(x_{12}) = (4)$ by Fact 9 and we move these pebbles to G_4. Now, we place two pebbles on x_{41}. If not, then $\text{Þ}_{34} \leq 3$ and if $(x_{44}) = (2)$ we move one pebble to x_{34}. And, we move one more pebble to G_3 from G_1 and so we pebble the target. If all the above fail, then $\text{Þ}_{4j} = 1$ for every j, $1 \leq j \leq 6$. So, we move a pebble to G_4 from G_1 and we place two pebbles on x_{41}.

Let $\text{Þ}_4 = 5$. Then $\text{Þ}_1 \geq 8$ and $\text{Þ}_2 \leq 8$. Let $\text{Þ}_1 = \text{Þ}_2 = 8$. If $\text{Þ}_{34} \leq 3$ we move two pebbles to G_3, one from G_1 and another from G_2. Now, clearly we pebble the target. If not, then $\text{Þ}_{34} \geq 4$. Suppose $(x_{44}) = (2)$, we move a pebble to x_{34} and we move two pebbles to G_3, one from G_1 and another from G_2. Suppose not. If either $(x_{16}) = (2)$ or $(x_{13}) = (4)$ or $(x_{21}) = (2)$ or $(x_{26}) = (4)$ we are done. Otherwise, either $(x_{11}) = (2)$ or $(x_{12}) = (4)$ and either $(x_{22}) = (2)$ or $(x_{21}) = (4)$ by Fact 9. Now, we move the above pebbles to G_4. Also we move two pebbles to x_{44} from x_{34} and in the resulting distribution, we place two pebbles on x_{41}. If $\text{Þ}_1 = 9$ then $\text{Þ}_2 = 7$. we move a pebble to G_1 from G_2. As there are ten pebbles on G_1, we proceed as before. If $\text{Þ}_1 \geq 10$, we are done.

Next, let $Þ_4 \le 4$. Therefore $Þ_1 \ge 9$ and $Þ_2 \le 9$. That is, either $Þ_1 \ge 10$ or the number of pebbles on G_1 can be made ten after a pebbling move from G_2 and so we are done.

Step 7. $Þ_3 = 6$.

If we move either two pebbles to G_3 or one pebble to either x_{34} or x_{35}, using the pebbles from G_i, $i \neq 3$, we can pebble the target.

Suppose there are two copies among G_1, G_2 and G_4, each with at least seven pebbles then we move two pebbles to G_3. Or, if there exists an i, i 6= 3, such that $Þ_i \ge 10$ then we are done by Fact 2.

Suppose not. Let $Þ_4 = 9$. Then $Þ_1 \ge 6$. If $Þ_1 \ge 7$, we move two pebbles to G_3, one from G_1 and another from G_4. If $Þ_1 = 6$ then $Þ_2 = 5$. If a pebble is moved to G_4 from G_1 then we get ten pebbles on G_4 and so we are done. Or, if a pebble is moved to G_3 from G_1, we move one more pebble to G_3 from G_4. Or, if a pebble is moved to either G_1 or G_3 or G_4 from G_2, we are done. If not, then $Þ_{2j} = 1$ except for one j, $1 \le j \le 6$. Suppose we move a pebble to G_2 from G_1, then we move one more pebble to G_1 from G_4 and we place two pebbles on x_{23} and so we move a pebble to x_{33}. Suppose not. Then $Þ_{1j} = 1$ for every j, $1 \le j \le 6$. So, we move a pebble to G_1 from G_4 and clearly we place two pebbles on x_{16}. So, we move a pebble to x_{36}.

Let $Þ_4 = 8$. Then $Þ_1 \ge 6$ and $Þ_2 \le 6$. If $Þ_1 \ge 7$. We move two pebbles to G_3, one from G_1 and another from G_4. If $Þ_1 = 6$, then $Þ_2 = 6$. If we move a pebble to either G_3 or G_2 from G_1 or if we move a pebble to either G_3 or G_1 from G_2, we are done. Or, if we move two pebbles to G_4 either from G_1 or from G_2 or one from G_1, another from G_2, we are done. If not, then $Þ_{ij} = 1$, for i = 1 or i = 2 or i = 1, 2 and for every j, $1 \le j \le 6$. Without loss of generality, we assume $Þ_{1j} = 1$, $1 \le j \le 6$. Now, we move a pebble to G_1 from G_4 and we place two pebbles on x_{16}. So, we move a pebble to x_{36}.

Let $p_4 = 7$. Then $p_1 \geq 7$. So, we move two pebbles to G_3, one from G_4 and another from G_1.

Let $p_4 = 6$. Then $p_1 \geq 7$ and $p_2 \leq 7$. If $p_1 = p_2 = 7$, we are done. If $p_1 = 8$ then $p_2 = 6$. We proceed as in the case if $p_4 = 8$, $p_1 = p_2 = 6$. If $p_1 = 9$ then $p_2 = 5$. We proceed as in the case if $p_4 = 9$, $p_1 = 6$ and $p_2 = 5$. If $p_1 \geq 10$ then we are done.

Let $p_4 = 5$. Then $p_1 \geq 8$ and $p_2 \leq 7$. If $p_1 = 8$ and $p_2 = 7$ then we move two pebbles to G_3, one from G_1 and another from G_2. If $p_1 = 9$ then $p_2 = 6$. We proceed as before. If $p_1 \geq 10$ then we are done.

Let $p_4 \leq 4$. Then $p_1 \geq 8$ and $p_2 \leq 8$. If $p_1 \geq 7$ and $p_2 \geq 7$ then we move two pebbles to G_3. Otherwise, $p_1 \geq 10$ and so we are done.

Step 8. $p_3 = 7$

Clearly, there exists at least one i, $i \neq 3$ such that $p_i \geq 7$. So, we move a pebble to G_3 from G_i.

4.Conclusion and open problem

We have found the pebbling number of S_4. Computation of (i) pebbling number (ii) t-pebbling number and (iii) cover pebbling number of S_n will be another interesting area of research.

Conjecture 4.1 The pebbling number of S_n is $f(S_n) = n! + 2$.

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