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Edge Mean Graph

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Abstract. Let $G = (V,E)$ be a finite simple undirected graph of order p and size q having no isolated vertices. Let $L = \{1,2,\ldots,q\}$ except for graphs having a tree as one component in which case $L' = \{0,1,2,\ldots,q\}$. Let $f : E \to L(L')$ be an injection. For every v in V, let $f^*(v) = \left\lceil \frac{x}{d(v)} \right\rceil$ where $x = \sum f(e)$, the summation being taken over all edges e incident on v and $\lceil y \rceil$ denotes the smallest integer greater than or equal to y. If $f^*(v)$ are all distinct and belong to $L(L')$, we call f an edge mean labeling of G and a graph G that admits an edge mean labeling is called an edge mean graph. In other words f is an edge mean labeling of G if f induces an injection $f^* : V \to L(L')$. In this article, we investigate certain classes of graphs that admit edge mean labeling. We also show that cycles, complete graphs on 4 vertices and complete bipartite graph $K_{2,3}$ are not edge mean graphs.

Keywords: Edge mean labeling, edge mean graph

AMS Subject Classification: 05C78

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1 Introduction

A graph G is an ordered pair of sets $G = (V, E)$ where the elements of V are called points or vertices and the elements of E are called lines or edges. Labeling methods trace their origin to one introduced by Rosa in 1967. Labeling is a fast growing research area in Graph Theory. There are a number of graph labelings such as graceful labeling, harmonious labeling, cordial labeling, arithmetic labeling, magic-type labeling, anti-magic labeling, prime labeling, mean labeling etc.

Definition: Let $G = (V, E)$ be a finite simple undirected graph of order p and size q having no isolated vertices. Let $L = \{1, 2, \ldots, q\}$ except for graphs having a tree as one component in which case $L' = \{0, 1, 2, \ldots, q\}$. Let $f : E \to L(L')$ be an injection. For every v in V, let $f^*(v) = \left\lceil \frac{x}{d(v)} \right\rceil$, where $x = \sum f(e)$, the summation being taken over all edges e incident on v and $\lceil y \rceil$ denotes the smallest integer greater than or equal to y. If $f^*(v)$ are all distinct and belong to $L(L')$, we call f an edge mean labeling of G and a graph G that admits an edge mean labeling is called an edge mean graph. In other words, f is an edge mean labeling of G if f induces an injection $f^* : V \to L(L')$. Some edge mean graphs are given in Fig. 1.1.

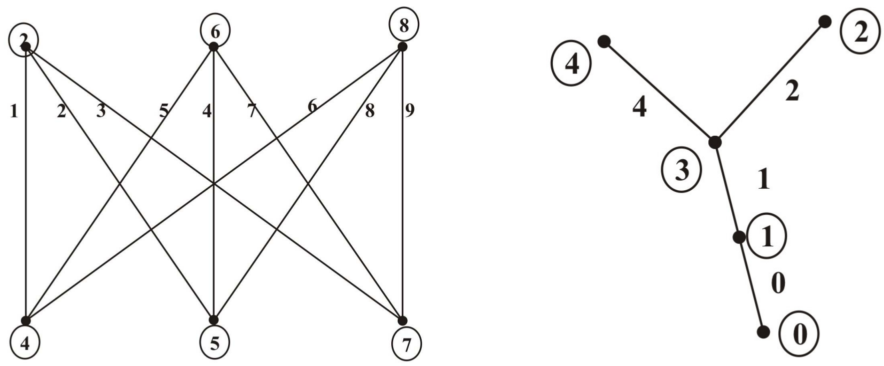

Figure 1.1: Some edge mean graphs.

In [1], Acharya and Hegde defined (k,d)-arithmetic graphs. They proved that if G is a (k,d)-arithmetic graph with k odd and d even then G is bipartite. They also proved that any (1,1)-arithmetic or (2,2)-arithmetic graph is either a star or has a triangle. In [5], Ponraj has defined mean graphs. A graph $G = (V,E)$ with p vertices and q edges is called a mean graph if it is possible to label the vertices $v \in V$ with distinct elements $f(v)$ from 0, 1, $\dots$,q in such a way that when edge $e = uv$ is labeled with $[f(u)+f(v)]/2$ if $[f(u)+f(v)]$ is even and $[f(u)+f(v)+1]/2$ if $[f(u)+f(v)]$ is odd, the resulting edge labels are distinct. f is called a mean labeling of G. He has showed that combs, cycles are mean graphs while the complete graph k_n $(n > 3)$, the wheel W_n $(n > 4)$ are not mean graphs. Similar concepts can be found in [2, 6]. A detailed account of various labeling problems can be found in the survey [3]. In this paper, we investigate certain classes of graphs that admit edge mean labeling and certain graphs which are not edge mean graphs. For terminology and symbols we refer to [4].

2 Main results

We note that from the definition, $K_2 = P_2$ is not an edge mean graph. Copies of K_2 are also not edge mean graphs.

Theorem 2.1. *Let T be a tree of order p and size q. If T is an edge mean graph, then 0 must be the label of a pendant edge.*

Proof. Since T is a tree, $p = q+1$. Therefore all the numbers 0, 1, 2, $\dots$, q must appear as vertex labels. If 0 is the label of an intermediate edge, there will be no vertex with label 0. □

Theorem 2.2. *Let T be a tree of order p and size q and be an edge mean graph. If v is a vertex of degree ≥ 3 such that there is at least one non-pendant edge incident on v.*

Then the following cannot happen. Label of a non-pendant edge incident on v is q and the label of any other edge incident on v is $q-1$ *simultaneously.*

Proof. Let f be an edge mean labeling of T. Suppose the above statement is true, then $f^*(v) = \left\lceil \frac{q+(q-1)+\cdots}{d(v)} \right\rceil \leq q-1$ so that there cannot be any vertex with label q as q is the label of a non-pendant edge. Hence the theorem. □

Theorem 2.3. *Let G be a graph with an edge mean labeling f and u be a pendant vertex of G. Let the vertex v adjacent to u be of degree 2. If* $f(uv) = n$ $(1 \leq n \leq q)$*, then the label of the other edge incident on v cannot be* $n-1$.

Proof. Let the other edge incident on v be e. Suppose $f(e) = n-1$, then $f^*(v) = \left\lceil \frac{f(e)+f(uv)}{2} \right\rceil = \left\lceil \frac{n-1+n}{2} \right\rceil = n = f^*(u)$ which is a contradiction. □

Theorem 2.4. *Let* $G = (p,q)$ *be an edge mean graph with an edge mean labeling f which is not a tree and let* $\delta(G) \geq p-2$. *Then for every v in V,* $f^*(v) \geq (p-1)/2$ *or* $p/2$ *according as p is odd or even.*

Proof. Since $\delta(G) \geq p-2$, there are at least $p-2$ edges incident on any $v \in V$. Hence $f^*(v) \geq \left\lceil \frac{1+2+\cdots+p-2}{p-2} \right\rceil = \frac{p-1}{2}$ or $\frac{p}{2}$ according as p is odd or even. □

3 Edge mean labeling of some trees

In this section, we investigate certain trees for edge mean labeling.

Theorem 3.1. *Any path* P_n $(n > 2)$ *is an edge mean graph.*

Proof. Let P_n be the path $u_1u_2\cdots u_{n-1}u_n$. Define $f : E(P_n) \to \{0,1,2,\ldots,n-1\}$ by

$$f(u_iu_{i+1}) = \begin{cases} i-1, & 1 \leq i \leq n-2 \\ n-1, & i = n-1. \end{cases}$$

Then $f^*(u_i) = i-1$, $1 \le i \le n$. □

Theorem 3.2. *The star graph $K_{1,n}$ is an edge mean graph.*

Proof. $K_{1,2}$ is the path P_3 and hence an edge mean graph. Consider $K_{1,n}$ $(n \ge 3)$ with central vertex u and pendant vertices u_i $(1 \le i \le n)$.

Define $f : E(K_{1,n}) \to \{0,1,2,\dots,n\}$ by

$$f(uu_i) = \begin{cases} i-1, & 1 \le i \le \lceil \frac{n}{2} \rceil \\ i, & \lceil \frac{n}{2} \rceil + 1 \le i \le n. \end{cases}$$

Then $f^*(u_i) = f(uu_i)$ and $f^*(u) = \lceil \frac{n}{2} \rceil$. □

Theorem 3.3. *Let f be any edge mean labeling of $K_{1,n}$ $(n \ge 3)$. Then 1 and n must occur as edge labels.*

Proof. Let u be the central vertex of $K_{1,n}$.

Suppose 1 is not an edge label.

$$\text{Then} \quad f^*(u) = \left\lceil \frac{0+2+\cdots+n}{n} \right\rceil = \left\lceil \frac{n+1}{2} \right\rceil \ge 2, \quad \text{since } n \ge 3.$$

Therefore there is no vertex with label 1.

Similarly, if n is not an edge label, there is no vertex with label n contradicting that f is an edge mean labeling. □

Theorem 3.4. *The labeling in Theorem 3.2 is the only edge mean labeling of $K_{1,n}$.*

Proof. Let f be an edge mean labeling of $K_{1,n}$ and let r be the number which we are not using in labeling $K_{1,n}$. Then by Theorem 3.3, $1 < r < n$ and $f^*(u) = \lceil \frac{0+1+\cdots+n-r}{n} \rceil = \lceil \frac{n+1}{2} - \frac{r}{n} \rceil$.

Case (i): n is odd, say, $n = 2m+1$. Then $f^*(u) = \lceil \frac{2m+2}{2} - \frac{r}{n} \rceil = m+1$ since $\frac{r}{n} < 1$. Therefore we cannot use $m+1$ as an edge label. Hence $r = m+1$.

Case (ii): n is even, say, $n = 2m$.

$f^*(u) = \lceil \frac{2m+1}{2} - \frac{r}{2n} \rceil = \lceil m + \frac{1}{2} - \frac{r}{2m} \rceil$.

Subcase (i): Let $r < m$. Then $\frac{r}{2m} < \frac{1}{2}$ and hence $f^*(u) = m+1 = f^*(u_i)$ for some i.

Subcase (ii): Let $r > m$. Then $\frac{r}{2m} > \frac{1}{2}$. Also $\frac{r}{2m} < 1$. Therefore $f^*(u) = m = f^*(u_i)$ for some i.

Thus, the edges of $K_{1,n}$ should be labeled by $0,1,2,\ldots,m,m+2,\ldots,n$ if $n = 2m+1$ and by $0,1,2,\ldots,m-1,m+1,\ldots,n$ if $n = 2m$. Hence the theorem. □

Theorem 3.5. *The bistar $B_{n,n}$ is an edge mean graph.*

Proof. $B_{1,1}$ is P_4 and hence an edge mean graph. $B_{2,2}$ is an edge mean graph with the given labeling. An edge mean labeling of $B_{2,2}$ is given in Fig. 3.1.

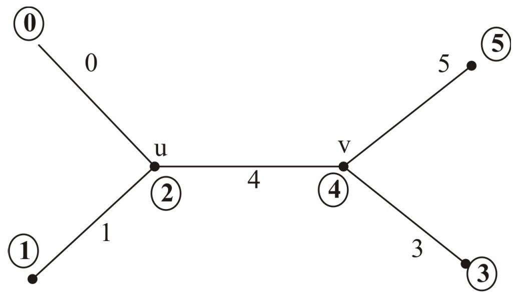

Figure 3.1: An edge mean labeling of $B_{2,2}$.

Let n$\geq$ 3. Let u and v be the central vertices and u_i, v_i $(1 \leq i \leq n)$ be the pendant vertices of $B_{n,n}$.

Case (i): Let $n = 2m+1$. Define $f : E(B_{n,n}) \to \{0,1,2,\ldots,2n+1\}$ by

$$f(uu_i) = \begin{cases} i-1, & 1 \leq i \leq m+1 \\ i, & m+2 \leq i \leq n \end{cases}$$

$$f(vv_i) = \begin{cases} n+i, & 1 \leq i \leq m \\ n+i+1, & m+1 \leq i \leq n \end{cases}$$

and $f(uv) = m+1$.

Then $f^*(u_i) = f(uu_i)$ and $f^*(v_i) = f(vv_i)$. Also, $f^*(u) = m+1$ and $f^*(v) = n+m+1$.

Case (ii): Let $n = 2m$. Define $f : E(B_{n,n}) \to \{0,1,2,\ldots,2n+1\}$ as in Case (i) except for $f(uv) = n+m+1$. It can be verified that $f^*(u) = m+1$ and $f^*(v) = n+m+1$. □

Corollary 3.6. *Let u and v be the central vertices of $B_{n,n}$. There always exists an edge mean labeling such that $f^*(v) = n + f^*(u)$.*

Proof. The labeling f given in Theorem 3.5 is one such labeling. □

Theorem 3.7. *For any $n \geq 2, 1$ and $2n+1$ cannot be the label of the intermediate edge of $B_{n,n}$.*

Proof. Let u and v be the central vertices and u_i, v_i $(1 \leq i \leq n)$ be the pendant vertices of $B_{n,n}$.

(1). Let $f(uv) = 1$.

Case (i): Let $n = 2$. Let u_1, u_2 and v_1, v_2 be the vertices adjacent to u and v respectively. Let $f(uu_1) = 0$. To get the vertex label 1, the only choice is $f(uu_2) = 2$.

Hence, $f(vv_1), f(vv_2) \in \{3,4,5\}$ and $f(vv_1) \neq f(vv_2)$.

But, for any such choice of $f(vv_1)$ and $f(vv_2)$, the induced map f^* cannot be an injection.

Case (ii): $n \geq 3$. Then $\min f^*(u)$ or $\min f^*(v) = \left\lceil \frac{(0+2+3+\cdots+n)+1}{n+1} \right\rceil \geq 2$.

Hence there is no vertex with label 1.

Hence $f(uv) \neq 1$.

(2). Suppose, $f(uv) = 2n+1$.

Then $\max f^*(u)$ or $\max f^*(v) = \left\lceil \frac{(n+1)+(n+2)+\cdots+(n+n)+2n+1}{n+1} \right\rceil \leq 2n$.

Therefore, there is no vertex with label $2n+1$. Hence the theorem. □

Theorem 3.8. *Combs are edge mean graphs.*

Proof. Let G_n be the comb obtained from a path $P_n : u_1u_2\cdots u_{n-1}u_n$ by joining a vertex v_i to u_i $(1 \leq i \leq n)$. Define $f : E(G_n) \to \{0,1,2,\ldots,2n-1\}$ by

$$f(u_iu_{i+1}) = \begin{cases} 1, & i = 1 \\ 2(i-1), & 2 \leq i \leq n-1 \end{cases}$$

$$f(u_iv_i) = \begin{cases} 0, & i = 1 \\ 2i-1, & 2 \leq i \leq n. \end{cases}$$

Then $f^*(v_i) = f(u_iv_i)$, for $1 \leq i \leq n$.

$f^*(u_1) = 1$, $f^*(u_2) = 2$, $f^*(u_i) = 2i-2$, $3 \leq i \leq n-1$, $f^*(u_n) = 2n-2$.

Therefore f is an edge mean labeling of G_n. □

4 Edge mean labeling of some graphs other than trees

Definition 4.1. *The graph G^2 of a graph G has $V(G^2) = V(G)$ with u,v adjacent in G^2 whenever $d(u,v) \leq 2$ in G. The powers G^3, G^4 … of G are similarly defined.*

Theorem 4.2. *P_n^k where $k = \min\{n/2, 5\}$ is an edge mean graph.*

Proof. Let P_n be the path $u_1u_2\cdots u_n$.

P_n^k has n vertices and $q = kn - \frac{k(k+1)}{2}$ edges.

$E(P_n^k) = \{u_iu_{i+r:}\ 1 \leq r \leq k$ and $1 \leq i \leq n-r\}$.

Define $f : E(P_n^k) \to \{0,1,2,\ldots,q\}$ by

$f(u_iu_{i+r)} = ki-(k-r)$, $1 \leq r \leq k-1$ and $1 \leq i \leq n-k+1$.

$f(u_iu_{i+k}) = ki$, $1 \leq i \leq n-k$.

$f(u_{n-k+2}u_{n-k+3}) = kn-k(k-1) = A$ (say).

$f(u_{n-k+s}u_{n-k+s+1}) = A+(k-2)+(k-3)+\cdots+(k-s+1)$, $3 \leq s \leq k-1$,

$f(u_{n-k+s}u_{n-k+s+t}) = f(u_{n-k+s}u_{n-k+s+t-1})+1$, $2 \leq s \leq k-t$ and $2 \leq t \leq k-2$.

It can be verified that

(i) For $1 \le i \le k$, $f^*(u_i) = \left\lceil \frac{x}{d(u_i)} \right\rceil$ where $d(u_i) = k+i-1$ and $x = \sum_{r=1}^{k} [ki-(k-r)] + \sum_{r=2}^{i} [(i-r)k+r-1]$.

(ii) For $1 \le i \le n-2k$, $f^*(u_{k+i}) = \left\lceil \frac{x}{d(u_{k+i})} \right\rceil$ where $d(u_{k+i}) = 2k$ and $x = \sum_{r=1}^{k} [k(k+i)-(k-r)] + \sum_{r=2}^{k+1} [(k+i-r)k+r-1]$.

To determine $f^*(u_{n-k+1}), f^*(u_{n-k+2}), \ldots, f^*(u_n)$.

(i) $k = 2$.

$f^*(u_{n-1}) = 2n-4$; $f^*(u_n) = 2n-3$.

(ii) $k = 3$.

$f^*(u_{n-2}) = 3n-10$; $f^*(u_{n-1}) = 3n-9$, $f^*(u_n) = 3n-7$.

(iii) $k = 4$.

$f^*(u_{n-3}) = 4n-19$; $f^*(u_{n-2}) = 4n-16$, $f^*(u_{n-1}) = 4n-14$, $f^*(u_n) = 4n-12$.

(iv) $k = 5$.

$f^*(u_{n-4}) = 5n-30$; $f^*(u_{n-3}) = 5n-27$, $f^*(u_{n-2}) = 5n-24$, $f^*(u_{n-1}) = 5n-21$, $f^*(u_n) = 5n-19$.

Hence the theorem.

An edge mean labeling of P_{12}^5 is given in Fig. 4.1. □

Theorem 4.3. *The complete graph k_n ($n \ge 5$) is an edge mean graph.*

Proof. **Case (i)**: Let $n = 5$. An edge mean labeling of K_5 is given in Fig. 4.2.

Case (ii): Let $n \ge 6$. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$.

Then $E(K_n) = \{v_i v_j : 1 \le i \le n-1 \quad \text{and } i+1 \le j \le n\}$ and $q = \frac{n(n-1)}{2}$.

Define $f : E(K_n) \to \{1, 2, \ldots, q\}$ by $f(v_1 v_j) = q-(j-2)$, $2 \le j \le n$.

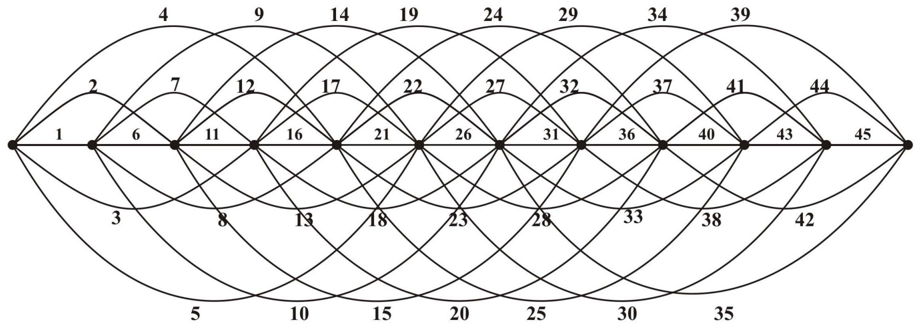

Figure 4.1: An edge mean labeling of P_{12}^5.

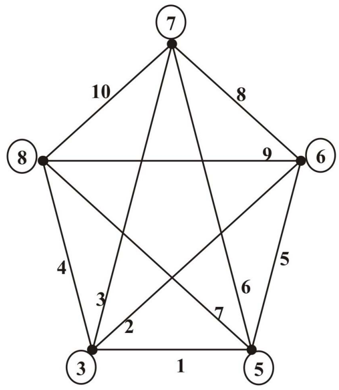

Figure 4.2: An edge mean labeling of K_5.

$f(v_iv_j) = f(v_{i-1}v_n) - (j-i)$, $2 \le i \le n-1$ and $i+1 \le j \le n$.
It can be verified that $f^*(v_1) = \left\lceil \frac{n^2-2n+2}{2} \right\rceil$. For $2 \le r \le n$, $f^*(v_r) = \left\lceil \frac{x}{n-1} \right\rceil$ where $x = (r-1)q' + (n-r)q'' - (r-2)(n-2) - (r-3)(n-3) - \cdots - 1(n-r+1) - \frac{(n-r-1)(n-r)}{2}$.
Here $q' = q - r + 2$ and $q'' = q - (r-1)n + \frac{(r-1)r}{2}$.
Therefore f is an edge mean labeling of K_n.

An edge mean labeling of K_8 is given in Fig. 4.3. □

Theorem 4.4. *The wheel $W_n = C_n + k_1$ $(n > 3)$ is an edge mean graph.*

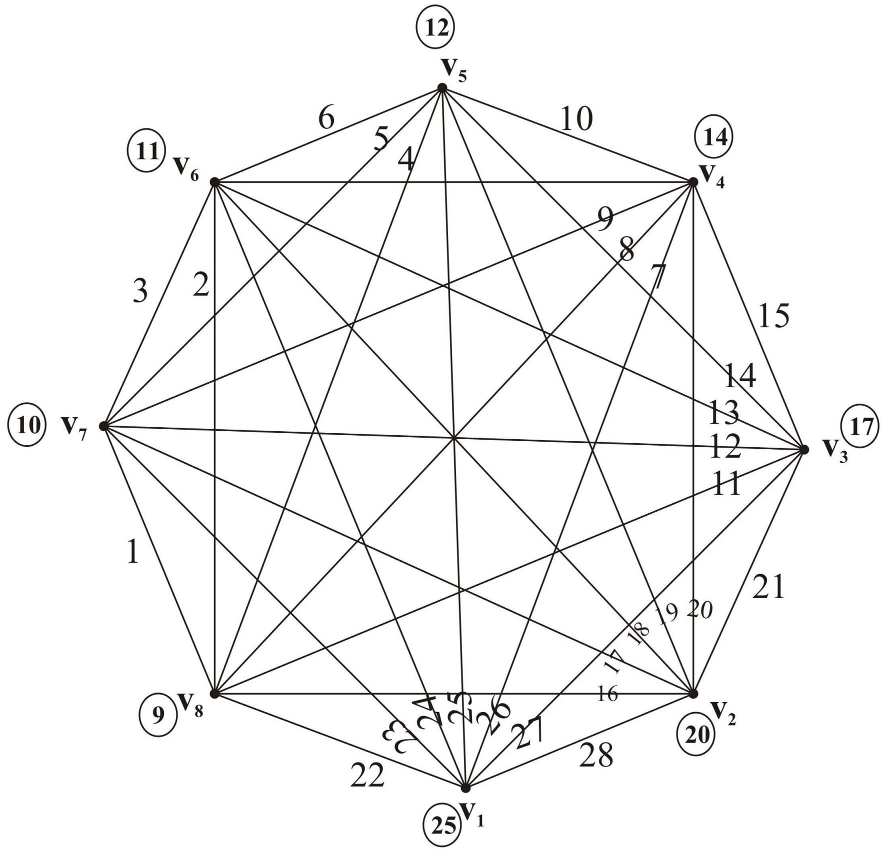

Figure 4.3: An edge mean labeling of K_8.

Proof. Let C_n be the cycle $u_1u_2\cdots u_nu_1$ and $k_1 = \{u\}$.

Then $E(W_n) = \{u_iu_{i+1}, u_iu,\ 1 \le i \le n\}$ and $L = \{1, 2, \ldots, 2n\}$.

Case (i): $n \equiv 0 \pmod 6$. That is, $n = 6r$, $r = 1, 2, 3 \ldots$.

Define $f : E(W_n) \to L$ by

$f(u_iu_{i+1}) = 2i$, $1 \le i \le n-3$;

$f(u_{n-2}u_{n-1}) = 2n$, $f(u_{n-1}u_n) = 2n-2$,

$f(u_nu_1) = 2n-4$.

$f(u_iu) = 2i-1$, $1 \leq i \leq n$. Then $f^*(u) = n$, $f^*(u_1) = 4r$, $f^*(u_i) = 2i-1$, $2 \leq i \leq n-3$.

$f^*(u_{n-2}) = 2n-3$, $f^*(u_{n-1}) = 2n-1$, $f^*(u_n) = 2n-2$.

Case (ii): $n \equiv 1 \pmod 6$. That is $n = 6r+1$, $r = 1,2,3\ldots$.

Define $f : E(W_n) \to L$ by $f(u_iu_{i+1}) = 2i-1$; $f(u_iu) = 2i$, $1 \leq i \leq n$.

Then $f^*(u) = n+1$, $f(u_1) = 4r+2$; $f^*(u_i) = 2i-1$, $2 \leq i \leq n$.

Case (iii): $n \equiv 2 \pmod 6$. That is $n = 6r+2$, $r = 1,2,\ldots$.

Define $f : E(W_n) \to L$ by $f(u_iu_{i+1}) = 2i$, $1 \leq i \leq n-2$; $f(u_{n-1}u_n) = 2n$; $f(u_nu_1) = 2n-2$.

$f(u_iu) = 2i-1$, $1 \leq i \leq n$.

Then $f^*(u) = n$; $f^*(u_1) = 4r+2$; $f^*(u_i) = 2i-1$, $2 \leq i \leq n-2$.

$f^*(u_{n-1}) = 2n-2$; $f^*(u_n) = 2n-1$.

Case (iv): $n \equiv 3 \pmod 6$. That is $n = 6r+3$, $r = 0,1,2,\ldots$.

Subcase (i): When $r = 0$, $n = 3$ and $W_3 = C_3 + K_1 = K_4$ which is not an edge mean graph by Theorem 5.2.

Subcase (ii): When $r = 1,2,3,\ldots$.

Define $f : E(W_n) \to L$ by

$f(u_iu_{i+1}) = 2i-1$, $1 \leq i \leq n-2$;

$f(u_{n-1}u_n) = 2n-1$; $f(u_nu_1) = 2n-3$; $f(u_iu) = 2i$, $1 \leq i \leq n$.

Then $f^*(u) = n+1$; $f^*(u_1) = 4r+2$, $f^*(u_i) = 2i-1$, $2 \leq i \leq n-2$.

$f^*(u_{n-1}) = 12r+4$; $f^*(u_n) = 12r+5$.

Case (v): $n \equiv 4 \pmod 6$. That is $n = 6r+4$, $r = 0,1,2,\ldots$.

Subcase (i): When $r = 0$, $n = 4$.

An edge mean labeling of W_4 is given in Fig. 4.4.

Subcase (ii): When $r = 1,2,3,\ldots$.

Define $f : E(W_n) \to L$ by

$f(u_iu_{i+1}) = 2i$, $f(u_iu) = 2i-1$, $1 \leq i \leq n$.

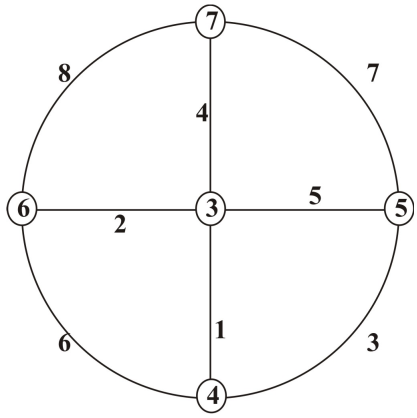

Figure 4.4: An edge mean labeling of W_4.

Then $f^*(u) = n$, $f^*(u_1) = 4r+4$, $f^*(u_i) = 2i-1$, $2 \leq i \leq n$.

Case (vi): $n \equiv 5 \pmod 6$. That is $n = 6r+5$, $r = 0, 1, 2, \ldots$.

Define $f : E(W_n) \to L$ by

$f(u_i u_{i+1}) = 2i-1$, $1 \leq i \leq n-2$.

$f(u_{n-1}u_n) = 2n-1$; $f(u_n u_1) = 2n-3$, $f(u_i u) = 2i$, $1 \leq i \leq n$.

Then $f^*(u) = n+1$, $f^*(u_1) = 4r+4$, $f^*(u_i) = 2i-1$, $2 \leq i \leq n-2$.

$f^*(u_{n-1}) = 12r+8$; $f^*(u_n) = 12r+9$.

Thus, W_n is an edge mean graph for $n > 3$.

An edge mean labeling of W_8 is given in Fig. 4.5. □

5 Some graphs which are not edge mean graphs

In this section we prove that the cycle C_n, the complete graph K_4 and the complete bipartite graph $K_{2,3}$ are not edge mean graphs.

Theorem 5.1. *The cycle C_n is not an edge mean graph.*

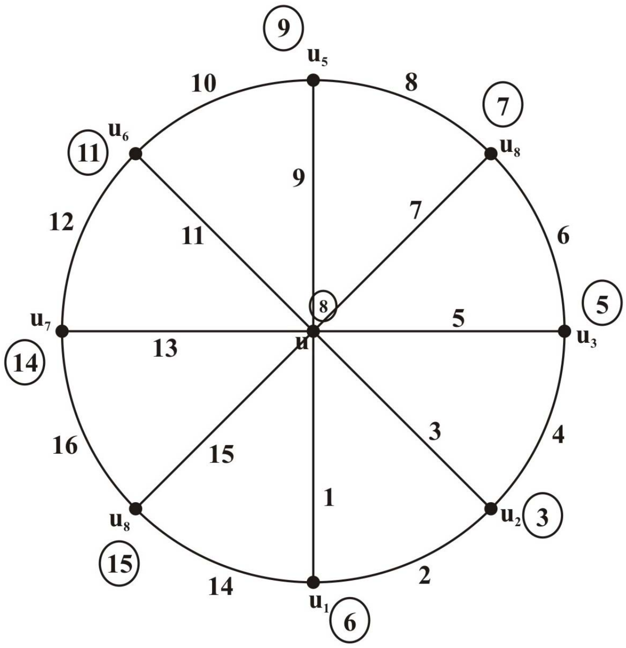

Figure 4.5: An edge mean labeling of W_8.

Proof. Let f be an edge mean labeling of C_n.

Since $q = p = n$ in C_n, all the numbers 1, 2, ..., n must appear as vertex label. Also, since $d(v) = 2$ for every vertex v in C_n, $\min f^*(v) = \lceil \frac{1+2}{2} \rceil = 2$.
Therefore, there will not be any vertex with label 1. Hence the theorem. □

Theorem 5.2. *The complete graph K_4 is not an edge mean graph.*

Proof. Let $V(K_4) = \{v_1, v_2, v_3, v_4\}$.
Then $E(K_4) = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4, v_3v_4\}$.
Let f be an edge mean labeling of k_4.
For any vertex v in K_4, $d(v) = 3$.
$\min f^*(v) = \lceil \frac{1+2+3}{3} \rceil = 2$ and $\max f^*(v) = \lceil \frac{4+5+6}{3} \rceil = 5$.
Hence the four vertices get the labels 2, 3, 4, 5 and are distinct. To get the label 2 all

the three edges incident on a vertex must be labeled 1, 2, 3.

Let $f(v_1v_2) = 1$, $f(v_1v_3) = 2$, $f(v_1v_4) = 3$ so that $f^*(v_1) = 2$.

Case (i): Let $f(v_2v_3) = 4$.

Then $f(v_2v_4), f(v_3v_4) \in \{5,6\}$ and $f(v_2v_4) \neq f(v_3v_4)$.

In both cases $f^*(v_2) = f^*(v_3) = 4$.

Case (ii): Let $f(v_2v_3) = 5$.

Then $f(v_2v_4) = 4$ and $f(v_3v_4) = 6$ give $f^*(v_3) = f^*(v_4) = 5$.

$f(v_2v_4) = 6$ and $f(v_3v_4) = 4$ give $f^*(v_2) = f^*(v_3) = 4$.

Case (iii): Let $f(v_2v_3) = 6$.

Then $f(v_2v_4), f(v_3v_4) \in \{4,5\}$ and $f(v_2v_4) \neq f(v_3v_4)$ give $f^*(v_2) = f^*(v_4) = 4$.

Therefore, f cannot be an edge mean labeling of K_4.

Hence K_4 is not an edge mean graph. □

Theorem 5.3. *$K_{2,3}$ is not an edge mean graph.*

Proof. Let $V = \{V_1, V_2\}$ where $V_1 = \{u_1, u_2\}$ and $V_2 = \{v_1, v_2, v_3\}$ be a bipartition of $V(K_{2,3})$.

Then $E(K_{2,3}) = \{u_1v_i, u_2v_i : \ 1 \leq i \leq 3\}$.

Suppose $f : E(K_{2,3}) \to \{1,2,3,4,5,6\}$ is an edge mean labeling of $K_{2,3}$.

Since for any $v \in V$, $2 \leq f^*(v) \leq 6$, all the labels 2, 3, 4, 5, 6 must be assumed by the vertices of $K_{2,3}$.

Now, to get the vertex label 2 all the edges incident on u_1 or u_2 must have the labels 1, 2, 3 (or) the edges incident on v_1 or v_2 or v_3 must have the label pair (1, 2) or (1, 3). In the first case, there is no possibility of getting the vertex label 6.

Case (ii): Let $f(u_1v_1) = 1$ and $f(u_2v_1) = 2$.

Then $f^*(v_1) = 2$. Now to get the label 6 we must have

(*) $f(u_1v_2), f(u_2v_2) \in \{5,6\}$ and $f(u_1v_2) \neq f(u_2v_2)$ or

(**) $f(u_1v_3), f(u_2v_3) \in \{5,6\}$ and $f(u_1v_3) \neq f(u_2v_3)$.

The two cases (*) and (**) are identical. So, we discuss only the case (*).

Subcase (i): Let $f(u_1v_2) = 5$ and $f(u_2v_2) = 6$.

Then $f(u_1v_3), f(u_2v_3) \in \{3,4\}$ and $f(u_1v_3) \neq f(u_2v_3)$ imply $f^*(u_2) = f^*(v_3) = 4$.

Subcase (ii): Let $f(u_1v_2) = 6$ and $f(u_2v_2) = 5$.

Then $f(u_1v_3)$, $f(u_2v_3) \in \{3,4\}$ and $f(u_1v_3) \neq f(u_2v_3)$.

In this case $f^*(u_1) = f^*(u_2) = f^*(v_3) = 4$.

Case (iii): Let $f(u_1v_1) = 1$ and $f(u_2v_1) = 3$. Then $f^*(v_1) = 2$. Proceed as in Case (ii).

Subcase (i): Let $f(u_1v_2) = 5$ and $f(u_2v_2) = 6$.

Then $f(u_1v_3) = 2$ and $f(u_2v_3) = 4$ imply $f^*(u_1) = f^*(v_3) = 3$.

$f(u_1v_3) = 4$ and $f(u_2v_3) = 2$ imply $f^*(u_1) = f^*(u_2) = 4$.

Subcase(ii): Let $f(u_1v_2) = 6$ and $f(u_2v_2) = 5$.

Then $f(u_1v_3) = 2$ and $f(u_2v_3) = 4$ imply $f^*(u_1) = f^*(v_3) = 3$.

$f^*(u_1v_3) = 4$ and $f(u_2v_3) = 2$ imply $f^*(u_1) = f^*(u_2) = 4$.

Therefore f is not an edge mean labeling of $K_{2,3}$.

Hence $K_{2,3}$ is not an edge mean graph. □

References

[1] B.D. Acharya and S.M. Hegde, *Arithmetic Graphs*, Journal of Graph Theory, **14(3)**, 275–299 (1989).

[2] L.W. Beineke and S.M. Hegde, *Strongly Multiplicative Graphs*, Discussions Mathematics—Graph Theory, **21**(2001) 63–75.

[3] J.A. Gallian, *A Dynamic survey of Graph Labeling*, The Electronic Journal of Combinatorics **14** (2007) #DS6.

[4] F. Harary, Graph Theory, Addison-Wesley, Reading MA (1969).

[5] R.M. Ponraj, Mean Graph, Ph.D. Thesis, Manonmaniam Sundaranar University, Tamil Nadu (2005).

[6] M. Sundaram, R. Ponraj and S. Somasundaram, *On a Prime Labeling conjecture*, ARS Combinatoria, **79** (2006), 205–209.