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Weight of a graph

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Abstract. A total labeling of a graph $G(V,E)$ with n vertices and e edges is a bijection $\lambda : V \cup E \to \{1,2,\ldots,n+e\}$. In this paper we introduce the concept of weight of a graph associated with a total labeling and find the lower and upper bounds for the same for an arbitrary graph. The concept of weight magic-graph is also introduced.

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1 Introduction

All graphs in this paper are finite, simple and undirected. The graph $G(V,E)$ has vertex set $V = V(G)$ and edge set $E = E(G)$. A total labeling of a graph $G(V,E)$ with n vertices and e edges is a bijection $\lambda : V \cup E \to \{1,2,\ldots,n+e\}$. The weight of a vertex with respect to λ is defined by $\mathrm{wt}(v) = \lambda(v) + \sum \lambda(vw)$ where the summation is taken over all edges incident to v. The weight of an edge vw is defined by $\mathrm{wt}(vw) = \lambda(v) + \lambda(vw) + \lambda(w)$. The weight of G with respect to a total labeling λ is defined as the sum of the weight of all its elements (vertices and edges) and is denoted by $\mathrm{wt}_\lambda(G)$. That is, $\mathrm{wt}(G) = \sum \mathrm{wt}(v) + \sum \mathrm{wt}(e)$.

In this paper, we find the lower bound and upper bound for the weight of an arbitrary graph. Also, we introduce the concept of weight-magic graphs.

2 Main results

2.1 Bounds for the weight of a graph

Each vertex label contributes $d(v)+1$ times to $\mathrm{wt}_\lambda(G)$ and each edge label is counted thrice towards $\mathrm{wt}_\lambda(G)$. Let L be the set of all total labelings of G. We define $\mathrm{wt}_\star(G) = \min_{\lambda\in L} \mathrm{wt}_\lambda(G)\}$ and $\mathrm{wt}^\star(G) = \max_{\lambda\in L} \mathrm{wt}_\lambda(G)\}$. $\mathrm{wt}_\star(G)$ and $\mathrm{wt}^\star(G)$ are called the lower and upper bounds of $\mathrm{wt}(G)$ and we call the corresponding labeling, a minimal and maximal labeling of G.

First, we will find the bounds for some simple graphs and then extend the idea for any finite arbitrary graph. The following lemma is useful for this purpose.

Lemma 2.1. *Let a and b be positive integers and $a < b$. For any positive integers x and y we have $ax+by \leq ay+bx$ iff $x \geq y$.*

Proof. $(ax+by)-(ay+bx) \leq 0$ iff $(a-b)(x-y) \leq 0$, which proves the lemma. ∎

Since vertices of lower degree contribute less to $\mathrm{wt}(G)$, as an application of the above lemma, we assign smaller numbers to vertices of lower degree and larger numbers to vertices of higher degree to determine $\mathrm{wt}^*(G)$ and other way for $\mathrm{wt}_*(G)$.

Proposition 2.2. *For $n \geq 3$, $6n^2-7n+3 \leq \mathrm{wt}(P_n) \leq 6n^2-3n-3$.*

Proof. Let λ be a total labeling that assigns $2n-1$, $2n-2$ to the end vertices and $2n-3, 2n-4, \ldots, 2, 1$ to the remaining elements of P_n in any order. By Lemma 2.1, this will be a minimal labeling of P_n and $\mathrm{wt}_*(P_n) = 6n^2-7n+3$. A maximal total labeling of P_n assumes 1 and 2 to the end vertices and 3, 4,..., $2n-2$, $2n-1$ to the intermediate vertices and edges of P_n and hence $\mathrm{wt}^*(P_n) = 6n^2-3n-3$. ∎

Corollary 2.3. $\mathrm{wt}_*(P_n) + 2nr + r(r-1)/2 \leq \mathrm{wt}(P_n U r K_1) \leq \mathrm{wt}^*(P_n) + r(r+1)/2 + r(6n-5), r \geq 1$.

Proposition 2.4. $14n^2 + 16n + 1 \leq \mathrm{wt}(W_n) \leq 19n^2 + 9n + 1$.

Proof. $W_n = C_n + K_1$, $n \geq 3$, has $n+1$ vertices and $2n$ edges and hence has $3n+1$ elements. Let λ be any total labeling of W_n. In the calculation of $\mathrm{wt}(W_n)$, the label of the central vertex v_0 occurs $d(v_0) + 1 = n + 1 \geq 4$ times, label of the vertices v_i ($i = 1, 2, \ldots, n$) in the cycle C_n occurs $d(v_i) + 1 = 4$ times. Label the vertex v_0 by 1; v_i ($1 \leq i \leq n$) by 2, 3, ..., $n+1$ and the edges by $n+2$, $n+3$, ..., $3n+1$. Then $\mathrm{wt}_*(W_n) = 14n^2 + 16n + 1$. Label the edges by 1,2, ..., $2n$; v_i ($1 \leq i \leq n$) by $2n+1, \ldots, 3n$ and v_0 by $3n+1$. Then $\mathrm{wt}^*(W_n) = 19n^2 + 9n + 1$. ∎

Proposition 2.5. *Let G be any k-regular ($k \geq 2$) graph with n vertices. Then*

(i) $\mathrm{wt}_*(G) = \mathrm{wt}^*(G) = 3n(2n+1)$ *when* $k = 2$.

(ii) $1/2\{3(n+e)(n+e+1) + (k-2)n(n+1)\} \leq \mathrm{wt}(G) \leq 1/2\{(k+1)(n+e)(n+e+1) + (2-k)e(e+1)\}$ *when* $k \geq 3$.

Proof. Let G be a k-regular ($k \geq 2$) graph with n vertices and e edges. Then G has $n+e$ elements where $e = nk/2$. Let λ be a total labeling of G by the integers 1, 2..., $n+e$.

Case(i): When $k = 2$, G is the cycle C_n and $e = n$. In a cycle, the contribution of any vertex or edge to its weight is thrice its label. Hence the elements of G can be labeled by the integers 1, 2, ..., $2n$ in any order. Hence $\mathrm{wt}_*(G) = \mathrm{wt}^*(G) = \mathrm{wt}(G) = 3n(2n+1)$.

Case (ii): When $k \geq 3$, the contribution of the vertices towards $\mathrm{wt}(G)$ is $k+1 \geq 4$ times of its label. Assign the numbers 1, 2, ..., n to the vertices and $n+1, n+2, \ldots, n+e$ to the edges. Then $\mathrm{wt}_*(G) = 1/2\{3(n+e)(n+e+1) + (k-2)n(n+1)\}$.

Assign 1, 2, ..., e to the edges and $e+1, e+2, \ldots, e+n$ to the vertices. Then $\mathrm{wt}^*(G) = 1/2\{(k+1)(n+e)(n+e+1) + (2-k)e(e+1)\}$. ∎

Corollary 2.6.

$$\frac{1}{2}\{3(n+e)(n+e+1)+(n-3)n(n+1)\} \leq \mathrm{wt}(K_n)$$
$$\leq \frac{1}{2}\{n(n+e)(n+e+1)+(3-n)n(n+1)\}$$

where $e = n(n-1)/2$.

Theorem 2.7. *Let* G *be a graph with* n *vertices and* e *edges. Denote the number of vertices of degree* i *as* n_i, $i = 0, 1, 2, \ldots, r$. *Then*

$$\mathrm{wt}^*(G) = \frac{1}{2}\sum_{i=0}^{r}(i+1)n_i(n_i+1) + \sum_{i=1}^{r}(i+1)(n_0+n_1+\cdots+n_{i-1})n_i$$
$$+ (n+n_1+2n_0+2e)e + \frac{3}{2}e(e+1)$$

and $$\mathrm{wt}_*(G) = \frac{1}{2}\sum_{i=0}^{r}(i+1)n_i(n_i+1)$$
$$+ \sum_{i=1}^{r} i(n_i+n_{i+1}+\cdots+n_{r-1}+n_r)n_{i-1}$$
$$+ (3n-n_1-2n_0)e + \frac{3}{2}e(e+1).$$

Proof. Let λ be a total labeling of G that assigns i_0 $(1 \leq i_0 \leq n_0)$ to the isolated vertices; n_0+i_1 $(1 \leq i_1 \leq n_1)$ to the pendant vertices; $n_0+n_1+i_2$ $(1 \leq i_2 \leq n_2)$ to the vertices of degree 2; $n_0+n_1+n_2+i_e$ $(1 \leq i_e \leq e)$ to the edges; $n_0+n_1+n_2+e+i_3$ $(1 \leq i_3 \leq n_3)$ to the vertices of degree 3 and so on, finally, assigns $n_0+n_1+\cdots+n_{r-1}+e+i_r$ $(1 \leq i_r \leq n_r)$ to the vertices of degree r. By Lemma 2.1, this will be a maximal labeling of G.

Hence

$$\mathrm{wt}^*(G) = 1/2\sum_{i=0}^{r}(i+1)n_i(n_i+1) + \sum_{i=1}^{r}(i+1)(n_0+n_1+\cdots+n_{i-1})n_i$$
$$+ (n+n_1+2n_0+2e)e + \frac{3}{2}e(e+1).$$

A minimal total labeling of G assigns the labels 1, 2, 3, ..., $n+e$ to the elements of G, starting from the vertices of degree r and going through the elements of G according to their contribution to $\mathrm{wt}(G)$ in descending order.

Hence

$$\mathrm{wt}_*(G) = 1/2\sum_{i=0}^{r}(i+1)n_i(n_i+1) + \sum_{i=1}^{r} i(n_i+n_{i+1}+\cdots+n_{r-1}+n_r)n_{i-1}$$
$$+(3n-n_1-2n_0)e+\frac{3}{2}e(e+1).$$

Hence the theorem. ∎

3 Weight magic graphs

Exoo et al. [2] call a function λ a totally magic labeling of a graph G if λ is both a vertex magic and edge magic total labeling of G. A total labeling λ of a graph $G(V,E)$ is called (i) vertex magic total labeling if for any vertex v in V, $\mathrm{wt}(v) = h$ for some constant h. (ii) Edge magic total labeling if for any edge e in E, $\mathrm{wt}(e) = k$ for some constant k.

We introduce the concept of weight magic graphs. A graph G is called weight magic if weight of G is the same for any total labeling λ of G.

Theorem 3.1. *Any cycle C_n is weight magic with constant weight $3n(2n+1)$.*

Proof. Let λ be any total labeling of C_n. All the three numbers which contribute to the weight of any vertex or edge are distinct. Then, $\mathrm{wt}(C_n) = 3n(2n+1)$ which is independent of λ. Hence C_n is weight magic. ∎

Lemma 3.2. *Let $x_1, x_2, \ldots, x_m$, k be positive integer such that for each permutation $a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n$ of labels 1, 2, …, $m+n$, $a_1x_1 + a_2x_2 + \cdots + a_mx_m + k(b_1 + b_2 + \cdots + b_n) = C$, a constant. Then $x_1 = x_2 = \cdots = x_m = k$.*

Proof. The numbers 1, 2, …, $m+n$ can be permuted to a_1, a_2, …, a_m, b_1, b_2, …, b_n in $(m+n)!$ ways. For each such permutation,

$$x_1a_1 + x_2a_2 + \cdots + x_ma_m + k(b_1+b_2+\cdots+b_n) = C \quad \text{(Constant)}. \tag{1}$$

Consider the permutation where a_1 and b_1 interchange their positions in (1). Then we have,

$$x_1b_1 + x_2a_2 + \cdots + x_ma_m + k(a_1 + b_2 + \cdots + b_n) = C. \quad (2)$$

(1)-(2) gives

$$x_1(a_1 - b_1) = k(a_1 - b_1)$$
$$x_1 = k.$$

Similarly considering the permutations where a_2, a_3, ..., a_m interchange their positions with b_1 in (1), we get,

$$x_2 = x_3 = \cdots = x_m = k. \qquad \blacksquare$$

Theorem 3.3. *A graph is weight magic iff it is 2-regular.*

Proof. Let G be a weight magic graph with constant weight k. Let v_1, v_2, ..., v_m and e_1, e_2, ..., e_n be the vertices and edges of G respectively. Let λ be a total labeling of G. Then $\mathrm{wt}(v_1) + \cdots + \mathrm{wt}(v_m) + \mathrm{wt}(e_1) + \cdots \mathrm{wt}(e_n) = k$. That is, $\lambda(v_1)(d(v_1)+1) + \cdots + \lambda(v_m)(d(v_m)+1) + 3(\lambda(e_1) + \cdots + \lambda(e_n)) = k$, which becomes $x_1\lambda(v_1) + \cdots + x_m\lambda(v_m) + 3(\lambda(e_1) + \cdots + \lambda(e_n)) = k$, where $x_i = d(v_i)+1$, $1 \le i \le m$, $\lambda(v_1), \ldots, \lambda(v_m)$, $\lambda(e_1), \ldots, \lambda(e_n)$ are all distinct. Therefore, $x_1 = \cdots x_m = 3$ (by Lemma 3.2). Hence, $d(v_i) = 2$, $1 \le i \le m$. which implies that G is 2-regular.

Conversely, let G be 2-regular. Let λ be a total labeling of G by the integers 1, 2, ..., $m+n$. In the calculation of $\mathrm{wt}_\lambda(G)$, the label of each vertex and each edge appears exactly three times. Hence $\mathrm{wt}_\lambda(G)$ is the same for any total labeling λ of G and hence G is weight magic. $\blacksquare$

Note: The above theorem says that only cycles and disjoint union of cycles are weight magic.

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