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Upper Limit of the Age of the Universe with Cosmological Constant

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ABSTRACT

The Friedmann, Robertson-Walker universe is based on the assumption that the universe is exactly homogeneous and isotropic. This model expresses that there is an all encompassing big bang singularity in the past as the origin from which the universe emerges in a very hot phase and continues its expansion as it cools. Here homogeneous and isotropic assumptions of the observed universe are not strictly followed to calculate the present age of the universe. Einstein equation plays an important role in cosmology to determine the present age of the universe. The determination of present age and density of the universe are two very important issues in cosmology, as they determine the future evolution and the nature of the universe. An attempt has been taken here to find the upper limit of the age of the universe with cosmological constant.

PACs: 98.80-Bp: Origin and formation of the universe
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1 INTRODUCTION

The Friedmann, Robertson-Walker (FRW) model indicates that the universe is exactly homogeneous and isotropic around us. Even though the universe is

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clearly inhomogeneous at the local scales of stars and cluster of stars, it is generally argued that an overall homogeneity will be achieved only at a large enough scale of about 14 billion light years or so (undetermined scale), in a statistical sense only. Here homogeneous means the universe is roughly same at all spatial points and the matter is uniformly distributed all over the space i.e., no part of the universe can be distinguished from any other, and isotropy means that all spatial directions are equivalent. But there is no fundamental physical justification that isotropy and homogeneity are strictly obeyed in all regions of space and at all epochs of time. The global hyperbolicity space-time gives homogeneous, inhomogeneous, isotropic and anisotropic universe (Mohajan 2013d).

The development in modern astrophysics states that dark matter surrounds the bright stars and galaxies, and constitutes the dominant material content of our universe. At present time, the evidence of the dark matter seems to be (a) low mass, faint stars (b) massive black holes and (c) massive neutrinos, axions or particles predicted by super symmetry. Of these axions are the most popular candidate as it seems to fit best the astrophysical requirements. Axions could have been produced in the early universe, and if the axions have small, non-zero rest masses, and then they would be gravitationally dominant today. The existence of axions was originally invoked when Pecei and Quinn (1977) explained the property of C–P (charge and parity) conservation of strong interactions.

To discuss cosmological models we need the knowledge of general relativity and we have discussed briefly which portion of it is related to our study. We have highlighted on Schwarzschild geometry and FRW model and Raychaudhury equation (Mohajan 2013a, b, c, d, e). In this paper we have determined the age of the universe with a cosmological constant.

2 A BRIEF DISCUSSION OF GENERAL RELATIVITY

A contravariant vector $A^{\mu}\left(\mu=0,1,2,3\right)$ and a covariant vector A_{μ} in coordinates x^{μ} to x'^{μ} transform as follows:

$$A'^{\mu}=\frac{\partial x'^{\mu}}{\partial x^{\nu}}A^{\nu}, \qquad A'_{\mu}=\frac{\partial x'^{\nu}}{\partial x^{\mu}}A_{\nu}. \tag{1}$$

Similarly a mixed tensor of rank three can be transformed as,

$$A'^{\mu}_{\nu\lambda}=\frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}\frac{\partial x^{\gamma}}{\partial x'^{\lambda}}A^{\alpha}_{\beta\gamma} \tag{2}$$

where we have used summation convention.

A metric is defined as;

$$ds^{2}=g_{\mu\nu}dx^{\mu}dx^{\nu} \tag{3}$$

where $g_{\mu\nu}$ is an indefinite metric in the sense that the magnitude of non-zero vector could be either positive, negative or zero. Then any vector $X\in T_{p}$ is called timelike, null or spacelike if (Joshi 1996);

$$g\left(X,X\right)<0,\ g\left(X,X\right)=0,\ g\left(X,X\right)>0. \tag{4}$$

An indefinite metric divides the vectors in T_{p} into three disjoint classes, namely the timelike, null or spacelike vectors. The null vectors form a cone in the tangent space T_{p} which separates the timelike vectors from spaelike vectors.

The covariant differentiations of vectors are defined as;

$$A^{\mu}_{;\nu}=A^{\mu}_{,\nu}+\Gamma^{\mu}_{\nu\lambda}A^{\lambda} \tag{5a}$$

$$A_{\mu;\nu}=A_{\mu,\nu}-\Gamma^{\lambda}_{\mu\nu}A_{\lambda} \tag{5b}$$

where semi-colon denotes the covariant differentiation and coma denotes the partial differentiation.

By (5b) we can write;

$$A_{\mu;\nu;\sigma} - A_{\mu;\sigma;\nu} = R^{\alpha}_{\mu\nu\sigma} A_{\alpha}, \tag{6}$$

where $R^{\alpha}_{\mu\nu\sigma} = \Gamma^{\alpha}_{\mu\sigma;\nu} - \Gamma^{\alpha}_{\mu\nu;\sigma} + \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\sigma} - \Gamma^{\alpha}_{\beta\sigma}\Gamma^{\beta}_{\mu\nu}$ (6a)

is a tensor of rank four and called Riemann curvature tensor. From (6) we observe that the curvature tensor components are expressed in terms of the metric tensor and its second derivatives. From (6a) we get;

$$R^{\alpha}_{[\mu\nu\sigma]} = 0. \tag{7}$$

Taking inner product of both sides of (6a) with $g_{\rho\alpha}$ one gets covariant curvature tensor;

$$R_{\rho\mu\nu\sigma} = \frac{1}{2}\left(\frac{\partial^2 g_{\rho\sigma}}{\partial x^{\nu}\partial x^{\mu}} + \frac{\partial^2 g_{\mu\nu}}{\partial x^{\sigma}\partial x^{\rho}} - \frac{\partial^2 g_{\mu\sigma}}{\partial x^{\nu}\partial x^{\rho}} - \frac{\partial^2 g_{\rho\nu}}{\partial x^{\sigma}\partial x^{\mu}} \right) + g_{\alpha\lambda}\left(\Gamma^{\alpha}_{\mu\nu}\Gamma^{\lambda}_{\rho\sigma} - \Gamma^{\alpha}_{\mu\sigma}\Gamma^{\lambda}_{\rho\nu}\right). \tag{8}$$

Contraction of curvature tensor (6) gives Ricci tensor;

$$R_{\mu\nu} = g^{\lambda\sigma} R_{\lambda\mu\sigma\nu}. \tag{9}$$

Further contraction of (9) gives Ricci scalar;

$$\hat{R} = g^{\lambda\sigma} R_{\lambda\sigma}. \tag{10}$$

From which one gets Einstein tensor as;

$$G^{\mu}_{\nu} = R^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu} R \tag{11}$$

where $div\left(G^{\mu}_{\nu}\right) = G^{\mu}_{\nu;\mu} = 0$.

The space-time (M, g) is said to have a flat connection iff;

$$R^{\mu}_{\nu\lambda\sigma} = 0. \tag{12}$$

This is necessary and sufficient condition for a vector at a point p to remain unaltered after parallel transported along an arbitrary closed curve through p. This is because all such curves can be shrunk to zero, in which case the space-time is simply connected (Hawking and Ellis 1973).

The energy momentum tensor $T^{\mu\nu}$ is defined as;

$$T^{\mu\nu} = \rho_0 u^\mu u^\nu \tag{13}$$

where ρ_0 is the proper density of matter, and if there is no pressure. A perfect fluid is characterized by pressure $p = p(x^\mu)$, then;

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}. \tag{14}$$

The principle of local conservation of energy and momentum states that;

$$T^{\mu\nu}_{;\nu} = 0. \tag{15}$$

According to the Newton's law of gravitation, the field equations in the presence of matter are;

$$\nabla^2 \phi = 4\pi G\rho \tag{16}$$

where ϕ is the gravitational potential, ρ is the scalar density of matter, G is the gravitational constant.

If classical equation (16) is generalized for the relative theory of gravitation then this must be expressed as a tensor equation satisfying following conditions;

- the tensor equation should not contain derivatives of $g_{\mu\nu}$ higher than the second order,
- it must be linear in the second differential coefficients, and
- its covariant divergence must vanish identically.

The most appropriate tensor of the form required is the Einstein's tensor which is given by (16); then Einstein's field equation can be written as;

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T^{\mu\nu}. \tag{17}$$

where $G = 6.673 \times 10^{-11} \mathrm{m}^3 \mathrm{kg}^{-1} \mathrm{s}^{-2}$ is the gravitational constant and $c = 10^8\, m/s$ is the velocity of light. Einstein introduced a cosmological constant $\Lambda(\approx 0)$ for static universe solutions as;

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} T^{\mu\nu}. \tag{18}$$

In relativistic unit $G = c = 1$, hence in relativistic units (18) becomes;

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi T_{\mu\nu}. \tag{19}$$

It is clear that divergence of both sides of (18) and (19) is zero. For empty space $T_{\mu\nu} = 0$ then $R_{\mu\nu} = \Lambda g_{\mu\nu}$, then;

$$R_{\mu\nu} = 0 \quad \text{for } \Lambda = 0 \tag{20}$$

which is Einstein's law of gravitation for empty space.

3 Schwarzschild Space-Time Manifold

The Schwarzschild metric which represents the outside metric for a star is given by;

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \tag{21}$$

If r_0 is the boundary of a star then $r > r_0$ gives the outside metric as in (21). If there is no surface, (21) represents a highly collapsed object viz. a black hole of mass m (will be discussed later). The metric (21) has singularities at $r = 0$ and $r = 2m$, so it represents patches $0 < r < 2m$ or $2m < r < \infty$. If we consider the patches $0 < r < 2m$ then it is seen that as r tends to zero, the curvature scalar,

$$R^{\mu\nu\sigma\lambda} R_{\mu\nu\sigma\lambda} = \frac{48 m^2}{r^6}$$

tends to ∞ and it follows that $r = 0$ is a genuine curvature singularity i.e., space-time curvature components tend to infinity (Mohajan 2013a, e).

At $r = 2m$ the curvature scalars are well behaved at this point, so it is a singularity due to inappropriate choice of coordinates. The maximal extension of the manifold (21) with $2m < r < \infty$ was obtained by Kruskal (1960) and Szekeres (1960). We now discuss about this, which uses suitably defined advanced and retarded null coordinates. For null geodesics (21) takes the form,

$$\left(1-\frac{2m}{r}\right)dt^2 = \left(1-\frac{2m}{r}\right)^{-1} dr^2$$

$$t = \pm\int \frac{r}{r-2m}dr$$

$$= \pm\left[r + 2m\log\left(\frac{r}{2m}-1\right)\right] + \text{constant}$$

$$t = r^* + \text{constant}. \tag{22}$$

The null coordinates u and v are defined by;

$$u = t - r^*, \qquad v = t + r^*$$

$$r^* = \frac{v-u}{2} \tag{23}$$

Now;

$$\left(1-\frac{2m}{r}\right)dt^2 + \left(1-\frac{2m}{r}\right)^{-1} dr^2$$

$$= -\frac{2m}{r}\left(\frac{r}{2m}-1\right)\left(dt^2 - dr^{*2}\right)$$

$$= -\frac{2m}{r} e^{-r/2m} e^{(v-u)/2m} dudv$$

$$ds^2 = -\frac{2m}{r} e^{-r/2m} e^{(v-u)/2m} dudv + r^2 d\Omega^2 . \tag{24}$$

Here ;

$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$.

As $r \to 2m$ corresponds to $u \to \infty$ or $v \to \infty$, we define new coordinates U and V by;

$U = -e^{u/4m}$, $V = e^{v/4m}$

$$dUdV = \frac{1}{16m^2} e^{(v-u)/4m}\, dudv.$$

Hence (24) becomes;

$$ds^2 = -\frac{32m^3}{r} e^{-r/2m}\, dUdV + r^2 d\Omega^2. \tag{25}$$

Hence there is no singularity at $U = 0$ or $V = 0$ which corresponds to the value at $r = 2m$.

Let us take a final transformation by;

$T = \frac{U+V}{2}$ and $X = \frac{V-U}{2}$, then (25) becomes;

$$ds^2 = \frac{32m^3}{r} e^{-r/2m}\left(-dT^2 + dX^2\right) + r^2 d\Omega^2 \tag{26}$$

which is Kruskal-Szekeres form of Schwarzschild metric. Then transformation (t, r) to (T, X) becomes;

$$X^2 - T^2 = -UV = e^{(v-u)/2m} = e^{r/2m}\left(\frac{r}{2m} - 1\right) \tag{27}$$

$$\frac{T}{X} = \tanh\frac{t}{4m} \quad \Rightarrow t = 4m \tanh^{-1}\frac{T}{X}. \tag{28}$$

From (28), $r > 0$ gives $X^2 - T^2 > -1$. The physical singularity at $r = 0$ gives $X = \pm\left(T^2 - 1\right)^{1/2}$, and we observe that there is no singularity now at $r = 2m$.

4 FRIEDMANN, ROBERTSON–WALKER (FRW) MODEL

The FRW model plays an important role in Cosmology. This model is established on the basis of the homogeneity and isotropy of the universe as described above. The current observations give a strong motivation for the adoption of the cosmological principle stating that at large scales the universe is homogeneous and isotropic and, hence, its large-scale structure is well described by the FRW metric. The FRW geometries are related to the high symmetry of these backgrounds. Due this symmetry numerous physical problems are exactly solvable and a better understanding of physical effects in FRW models could serve as a handle to deal with more complicated geometries.

In (t, r, θ, ϕ) coordinates the Robertson-Walker line element is given by;

$$ds^2 = -dt^2 + S^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right] \tag{29}$$

where k is a constant which denotes the spatial curvature of the three-space and could be normalized to the values +1, 0, –1. When k = 0 the three-space is flat and (29) is called Einstein de-Sitter static model, when k = +1 and k = –1 the three-space are of positive and negative constant curvature; these incorporate the closed and open Friedmann models respectively (Figure 1).

Let us assume the matter content of the universe as a perfect fluid then by (14) and (15), solving (29) we get;

$$\frac{3\ddot{S}}{S} + 4\pi(\rho + 3p) = 0 \tag{30}$$

, and

$$\frac{3\dot{S}^2}{S^2} - \left(8\pi\rho - \frac{3k}{S^2}\right) = 0 \tag{31}$$

where we have considered $\Lambda = 0$. If $\rho > 0$ and $p \geq 0$ then $\ddot{S} < 0$. So $\dot{S}$ = constant and $\dot{S} > 0$ indicates the universe must be expanding, and $\dot{S} < 0$ indicates contracting universe. The observations by Hubble of the red-shifts of the galaxies were interpreted by him as implying that all of them are receding from us with a velocity proportional to their distances from us that is why the universe is expanding. For expanding universe $\dot{S} > 0$, so by (30) and (31) we get $\ddot{S} < 0$. Hence $\dot{S}$ is a decreasing function and at earlier times the universe must be expanding at a faster rate as compared to the present rate of expansion. But if the expansion be constant rate as like the present expansion rate at all times then,

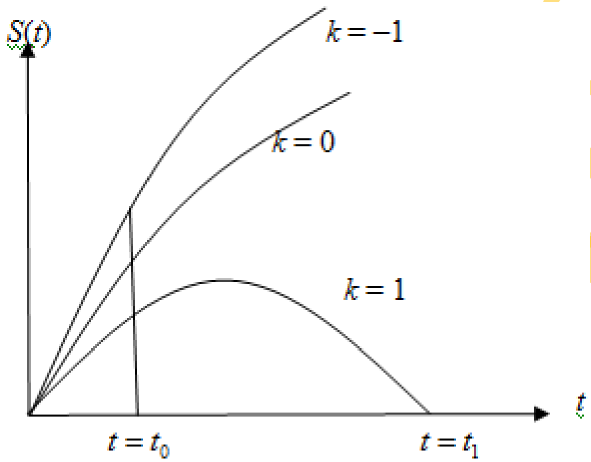

Figure 1: The behavior of the curve $S(t)$ for the three values k = –1, 0, +1; the time $t = t_0$ is the present time and $t = t_1$ is the time when $S(t)$ reaches zero again for k = +1 .

$$\left(\frac{\dot{S}}{S}\right)_{t=t_0} \equiv H_0 . \qquad (32)$$

Now H_0^{-1} implies a global upper limit for the age of any type of Friedmann models. So the age of the universe will be less than H_0^{-1}. The quantity H_0 is called Hubble constant and at any given epoch it measures the rate of expansion of the universe. By observation H_0 has a value somewhere in the range of 50 to 120 kms^{-1}Mpc^{-1}.

At S = 0, the entire three-surface shrinks to zero volume and the densities and curvatures grow to infinity. Hence by FRW models there is a singularity at a finite time in the past. This curvature singularity is called the big bang. Now we have a basic qualitative difference between the Schwarzschild singularity and that occurring in FRW models. The Schwarzschild singularity could be the final result of a gravitationally collapsing of massive star. However FRW singularity must be interpreted as the catastrophic event from which the entire universe emerged and where all the known physical laws breakdown in such a way that we cannot know what was before this singularity. The existence of a strong curvature singularity at t = 0 indicated by the FRW models imply the existence of a very hot, dense and radiation dominated region in the very early phase of the evolution of the universe (Islam 2002, Hawking and Ellis 1973).

5 Raychaudhuri Equation and Gravitational Focusing

Now let us consider the Raychaudhuri equation (Raychaudhuri 1955), (for null case similar equation holds with $\frac{1}{3}$ is replaced by $\frac{1}{2}$)

$$\frac{d\theta}{dt} = -R_{\mu\nu}V^{\mu}V^{\nu} - \frac{1}{3}\theta^2 - 2\sigma^2 + 2\omega^2 \tag{33}$$

which describes the rate of change of the volume expansion as one moves along the timelike geodesic curves in the congruence (Mohajan 2013a). Here $\theta > 0$ is expansion, $\sigma > 0$ is shear and ω is rotation tensors which are defined as follows:

$$\theta_{\mu\nu} = V_{(\alpha;\beta)} h^{\alpha}_{\mu} h^{\beta}_{\nu}$$

$$\sigma_{\mu\nu} = \theta_{\mu\nu} - \frac{1}{3} h_{\mu\nu}\theta$$

$$\omega_{\mu\nu} = h^{\alpha}_{\mu} h^{\beta}_{\nu} V_{[\alpha;\beta]}.$$

By Einstein equation (19) we can write (Joshi 1996, Kar and SenGupta 2007);

$$R_{\mu\nu}V^{\mu}V^{\nu} = 8\pi\left(T_{\mu\nu}V^{\mu}V^{\nu} + \frac{1}{2}T\right). \qquad (34)$$

The term $T_{\mu\nu}V^{\mu}V^{\nu}$ is the energy density measured by a timelike observer with the unit tangent four velocity of the observer, V^{μ}. In classical physics;

$$T_{\mu\nu}V^{\mu}V^{\nu} \geq 0. \qquad (35)$$

Such an assumption is called the weak energy condition (the matter density observed by the corresponding observers is always non-negative i.e. $\rho \geq 0$ and $\rho + p \geq 0$). Now let us consider (Joshi 2013);

$$T_{\mu\nu}V^{\mu}V^{\nu} \geq -\frac{1}{2}T. \qquad (36)$$

Such an assumption is called the strong energy condition (the trace of the tidal tensor measured by the corresponding observers is always non-negative i.e., $\rho + p \geq 0$ and $\rho + 3p \geq 0$) which implies from (34) for all timelike vectors V^{μ},

$$R_{\mu\nu}V^{\mu}V^{\nu} \geq 0. \qquad (37)$$

Both the strong and weak energy condition will be valid for perfect fluid provided energy density $\rho \geq 0$ and there are no large negative pressures.

An additional energy condition required often by the singularity theorems is the dominant energy condition which states that in addition to the weak energy condition, the pressure of the medium must not exceed the energy density (i.e., $\rho \geq |p|$). The dominant energy condition also states that $T^{\mu}_{\nu}V^{\nu}$ is non-spacelike and future-directed. Equation (37) implies that the effect of matter on space-time curvature causes a focusing effect in the congruence of timelike geodesics due to gravitational attraction.

Let us suppose γ is a timelike geodesic. Then two points p and q along γ are called conjugate points if there exists Jacobi field along γ which is not identically zero but vanishes at p and q. If infinitesimally nearby null geodesics of the congruence meet again at some other point q in future, then p and q are called conjugate to each other, where $\theta \to -\infty$ at q (Figure 2). We can define conjugate point another way as follows (Mohajan 2013c):

Let S be a smooth spacelike hypersurface in M which is an embedded three dimensional sub-manifold. Consider a congruence of timelike geodesics orthogonal to S. Then a point p along a timelike geodesic γ of the congruence is called conjugate to S along γ if there exists a Jacobi vector field along γ which is non-zero at S but vanishes at p, which means that there are two infinitesimally nearby geodesics orthogonal to S which intersect at p (Figure 3). Again we face

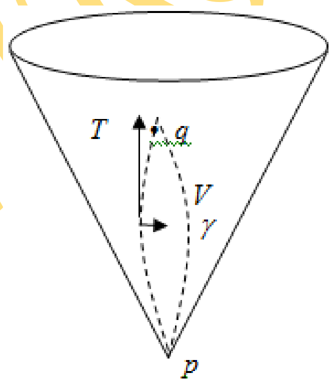

Figure 2: Infinitesimally separated null geodesics cross at p and q, which are conjugate points along the curve γ.

equivalent condition that the expansion for the congruence orthogonal to S tends to $\theta \to -\infty$ at p. If V^{μ} denotes the normal to S, then the extrinsic curvature $\chi_{\mu\nu}$ of S is defined as;

$$\chi_{\mu\nu} = \nabla_\mu V_\nu \tag{38}$$

which is evaluated at S.

So, $\chi_{\mu\nu} V^\mu = \chi_{\mu\nu} V^\nu = 0$. Since S is orthogonal to the congruence this implies $\omega_{\mu\nu} = 0$, hence $\chi_{\mu\nu} = \chi_{\nu\mu}$.

The trace of the extrinsic curvature, is denoted by χ, and is given by;

$$\chi = \chi^\mu_\mu = h^{\mu\nu} \chi_{\mu\nu} = \theta \tag{39}$$

where θ is expansion of the congruence orthogonal to S.

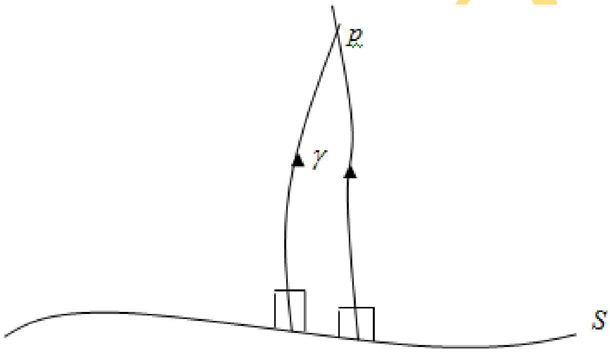

Figure 3: A point p conjugate to the spacelike hypersurface S. The timelike geodesic is orthogonal to S, which is intersected by another infinitesimally nearby timelike geodesic.

Let us consider the situation when the space-time satisfies the strong energy condition and the congruence of timelike geodesics is hypersurface orthogonal, then $\omega_{\mu\nu} = 0$ implies $\omega^2 = 0$ then (33) gives;

$$\frac{d\theta}{d\tau} \le -\frac{\theta^2}{3} \tag{40}$$

which means that the volume expansion parameter must be necessarily decreasing along the timelike geodesics. Let us denote θ_0 as initial expansion then integrating (40) we get;

$$\frac{1}{\theta} \geq \frac{\tau}{3} + c \quad (41)$$

Initially $\theta = \theta_0$ then (41) becomes;

$$\frac{1}{\theta} \geq \frac{\tau}{3} + \frac{1}{\theta_0} \quad (42)$$

By (42) we confirm that if the congruence is initially converging and θ_0 is negative then $\theta \to -\infty$ within a proper time distance $\tau \leq \frac{3}{\theta_0}$, provided γ can be extended to that value of the proper time.

Now suppose $\theta = \chi = 0$, and further, it is bounded above by a negative value $\theta_{\max}$, so all the timelike curves of the congruence orthogonal to S will contain a point conjugate to S within a proper time distance $\tau \leq \frac{3}{|\theta_{\max}|}$, provided the geodesics can be extended to that value of the proper time.

By the above results the existence of space-time singularities in the form of geodesic incompleteness. Now we introduce the gravitational focusing effect for the congruence of null geodesics orthogonal to a spacelike two surfaces as follows (Joshi 1996):

Let M be a space-time satisfying $R_{\mu\nu}K^{\mu}K^{\nu} \geq 0$ for all null vectors K^{μ} and γ be a null geodesic of the congruence. If the convergence θ of null geodesic from some point p is $\theta = \theta_0 < 0$ at some point q along γ, then within an affine distance less than or equal to $\frac{2}{|\theta_0|}$ from q the null geodesic γ will contain a point conjugate to p, provided that it can be extended to that affine distance.

6 Upper Limit of the Age of the Universe

6.1 Mathematical Formulation

For $\omega = 0$ (33) becomes;

$$\frac{d\theta}{dt} = -R_{\mu\nu}V^{\mu}V^{\nu} - \frac{1}{3}\theta^{2} - 2\sigma^{2}. \tag{43}$$

A timelike geodesic $\gamma(t)$ will be orthogonal to S_0 provided the expansion θ along $\gamma(t)$ satisfies $\theta = \chi_i^i$ at S_0, where χ_{ij} is the second fundamental form of the spacelike hypersurface. Let $\theta \equiv \frac{1}{z}.\frac{dz}{dt}$ with $z = x^3$ then (43) becomes (Joshi 1996);

$$\frac{d^2x}{ds^2} + H(t)x = 0 \tag{44}$$

where $H(t) = \frac{1}{3}\left(R_{\mu\nu}V^{\mu}V^{\nu} + 2\sigma^{2}\right)$.

Now we have to find a point p conjugate to S_0 along $\gamma(t)$, that is to find a solution $x(t)$ to equation (44) which vanishes at p. So that initially for some constant α;

$x(0) = \alpha$, and

$$\frac{dx}{dt} = \frac{1}{3}\theta\, x \;, \quad \left.\frac{dx}{dt}\right|_{t=0} = \frac{1}{3}\alpha\theta = \alpha\chi_{\mu}^{\mu} \tag{45}$$

which vanishes at p.

To solve equation (44) we use Sturn comparison theorem which compares the distribution zeros of the solutions $u(t)$ and $v(t)$ of the equations (Joshi 1996) ;

$$\frac{d^2u}{dt^2} + G_1(t)\,u = 0 \;, \tag{46}$$

$$\frac{d^2v}{dt^2} + G_2(t)\,v = 0$$

where $G_1 \le G_2$ in an interval (*a*, *b*). The theorem then shows that if $u(t)$ has m zeros in $a < t < b$ then $v(t)$ has at least m zeros in the same interval and the k^{th} zero of $v(t)$ is must be earlier than the k^{th} zero of $u(t)$.

Now let,

$$A^2 = \min H(t) = \min \frac{1}{3}\left(R_{\mu\nu} V^\mu V^\nu + 2\sigma^2\right)$$

and consider the equation;

$$\frac{d^2 x}{dt^2} + A^2 x = 0 \qquad (47)$$

If we apply the Sturm theorem to equations (44) and (47) we observe that if the solution to equation (47) satisfying the initial conditions (45) has a zero in the interval $0 < t < t_1$, then the solution of equation (24) defined by the same initial conditions must have a zero in the same interval, which must occur before the zero of the solution of equation (47). Now the general solution of (47) can be written as;

$$x = C_1 \sin\left(C_2 + At\right). \qquad (48)$$

Let us choose the initial condition as;

$$x(0) = \frac{1}{\left(\left(\chi_\mu^\mu\right)^2 + A^2\right)^{1/2}},$$

$$\left.\frac{dx}{dt}\right|_{t=0} = \frac{\chi_\mu^\mu}{\left(\left(\chi_\mu^\mu\right)^2 + A^2\right)^{1/2}}. \qquad (49)$$

Since the universe is expanding everywhere so $\chi_\mu^\mu < 0$ on S_0. The universe may contract or it may expand at some places and may contract in some other places, but we shall not consider such possibilities here, instead we consider only expanding behavior. Using initial conditions (49), solution (48) can be written as;

$$x = \frac{1}{A}\sin(\theta - At) \tag{50}$$

with

$$\theta = \sin^{-1}\left\{\frac{A}{\left(\left(\chi_{\mu}^{\mu}\right)^{2} + A^{2}\right)^{1/2}}\right\}.$$

We have $0 < \theta < \frac{\pi}{2}$ and a zero for x must occur within the interval;

$$0 \le At \le \frac{\pi}{2} \quad \Rightarrow 0 \le t \le \frac{\pi}{2A}, \tag{51}$$

i.e., if $\gamma(t)$ is any timelike curve geodesic orthogonal to S_0, then there must be a point p on $\gamma(t)$, conjugate to S_0, within the above interval where $A^2 = \min H(t) > 0$.

No timelike curve from S_0 can be extended into the past beyond the proper time length $\frac{\pi}{2A}$. Let q be an event on S_0 and γ be a past directed, endless timelike curve from $q = \gamma(0)$. Let γ can be extended to arbitrary values of proper time in the past, then choose $p = \gamma\left(\frac{\pi}{2A}\right)$ to be an event on this trajectory. Then there exists a timelike γ' from p orthogonal to S_0 along which the proper time lengths of all non-spacelike curves from p to S_0 are maximized and further, γ' does not contain any conjugate point to S_0 between p and S_0. Again, we have shown that any timelike geodesic $\gamma(t)$

must contain a point conjugate to S_0 within the proper time length $\frac{\pi}{2A}$. But this is impossible and we can say that timelike curve form can be extended into the past beyond the proper time length $\frac{\pi}{2A}$ i.e., $t_{\max} = \frac{\pi}{2A}$.

Now the above results can be applied to obtain general upper bounds to the age of a globally hyperbolic universe in the following manner.

By using (14) and (18) we can write,

$$R_{\mu\nu}V^{\mu}V^{\nu} = 8\pi\, G\left(T_{\mu\nu}V^{\mu}V^{\nu} + \frac{1}{2}T\right)$$

$$R_{\mu\nu}V^{\mu}V^{\nu} = 8\pi\, G(\rho + P)(V^{4})^{2} - 4\pi\, G\rho + 4\pi\, GP$$

$$\Rightarrow R_{\mu\nu}V^{\mu}V^{\nu} \geq 4\pi\, G(\rho + 3P). \qquad (52)$$

6.2 Limit of the Age of the Universe

We assume that energy density of the present universe is mainly contributed by the non-relativistic free gas of neutrinos for which $P << \rho$ then (52) becomes;

$$R_{\mu\nu}V^{\mu}V^{\nu} \geq 4\pi\, G\rho$$

$$A^{2} = \min \frac{1}{3}\left(R_{\mu\nu}V^{\mu}V^{\nu} + 2\sigma^{2}\right) \geq \frac{4}{3}\pi\rho\, G$$

where ρ is the present density of the universe. Hence the maximum possible age of the universe $t_{\max}$, is given by;

$$t_{\max} = \frac{\pi}{2A} = \frac{\pi}{2}\left(\frac{3}{4\pi\rho\, G}\right)^{1/2} = \pi\left(\frac{3}{16\pi\rho\, G}\right)^{1/2} \qquad (53)$$

with the basis of general globally hyperbolic space-time (Figure 4).

In radiation dominated models we can write $P = \frac{1}{3}\rho$, then (52) becomes;

$$R_{\mu\nu}V^{\mu}V^{\nu} \geq 8\pi\rho\, G. \quad (54)$$

Then (33) becomes;

$$t_{\max} = \pi\left(\frac{3}{32\pi\rho\, G}\right)^{1/2}. \quad (55)$$

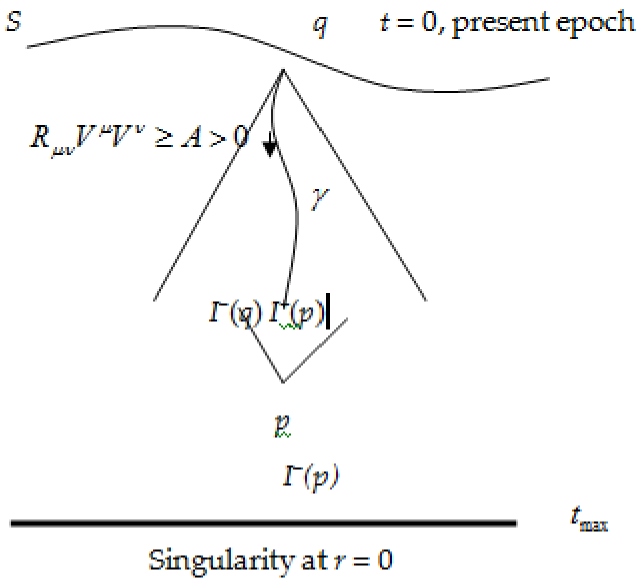

Figure 4: No timelike curve γ from the surface S extends the maximum limit $t_{\max}$ in the past and must encounter a space-time singularity before this epoch.

Both (53) and (55) give upper limits to the age of the universe irrespective of whether or not the distribution of whether on S_0 is isotropic and homogeneous which does not assume.

The average mass density as indicated by the visible galaxies is about 10^{-30} gm cm^{-3}. The X-ray observations strongly favor the existence of a hot ionized intergalactic gas within the cluster of galaxies whereas weakly interacting massive neutrinos could be another source. If the microwave background radiation (MBR) as having some kind of global origin, then ρ_{MBR} provides a firm lower limit of the $\rho_{\min}$ sought for and general upper limit to the age of the universe as given by (55) is;

$$t_{\max} = \pi\left(\frac{3}{32\pi\rho\, G}\right)^{1/2} = 3.2\times 10^{12} \text{ years,} \qquad (56)$$

$\rho_{MBR} = 4.4\times 10^{-3}$ gm cm–3.

The relationships (54) and (56) provide upper limits to the age even when allowing for departures from homogeneity and isotropy. If we take the contribution by matter into account, we have to choose an entire range of densities as suggested by the above mentioned possibilities.

The average matter density arising from all possible sources is believed to be between 10^{-31} to 10^{-28} gm cm^{-3}.

For $\rho = 10^{-31}$ gm cm^{-3} in (56) we find;

$$t_{\max} = \pi\left(\frac{3}{32\pi\rho\, G}\right)^{1/2} = 9.43\times 10^{10} \text{ years,} \qquad (57)$$

and for $\rho = 10^{-28}$ gm cm^{-3} in (56) we find;

$$t_{\max} = \pi\left(\frac{3}{32\pi\rho\, G}\right)^{1/2} = 0.94\times 10^{10} \text{ years.} \qquad (58)$$

7 Lower Bounds on Axion Rest Mass

In a general cosmology when the densities may vary on S_0 and then $t_{\max}$ comes from observations. Thus if t_{ob} denotes observed age, we can write $t_{ob} < t_{\max}$, then (53) becomes;

$$\rho_0 \le \frac{3\pi}{16G}\cdot\frac{1}{t_{ob}^2} \qquad (59)$$

which implies that a prescribed lower limit on the observed age of the universe will provide an upper limit to the matter density. If ρ_0^F be the Friedmann density parameter then (59) becomes;

$$\rho_0^F \leq \frac{3\pi}{16G} \cdot \frac{1}{t_{ob}^2} . \tag{60}$$

In the Friedmann model, the density due to axions, which produced in the early universe, is a function of absolute temperature (Preskill et al. 1983);

$$\rho_0^F(T) \leq \frac{3m_a T^3 f_a^2}{m_{Pl}\Lambda_{QCD}} \tag{61}$$

where $m_a =$ axion mass, $m_{Pl} = (\hbar c / G)^{1/2}$ is the Plank mass and $\Lambda_{QCD}(\approx 200\, MeV)$ is the scale parameter in quantum chronomodynamics. Now axion mass m_a is related to the vacuum expectation value f_a of the scalar field that the spontaneously breaks the Peccei–Quinn symmetry invoked to explain the C–P invariance of strong interactions, and is given by (Weinberg 1978, Weilezek 1978)

$$m_a = 1.24 \times 10^{-5} eV \left(\frac{10^{12} GeV}{f_a} \right) . \tag{62}$$

Particle physics does not specify the exact value of f_a; it can lie anywhere between the weak interaction scale and the mass scale of grand unification. If dark matter is made up entirely of axions, then since $\rho_a^F(T) \leq \rho_0^F$, so;

$$\rho_a^F(T) \leq \frac{3\pi}{16G} \cdot \frac{1}{t_0^2} . \tag{63}$$

By (61) and (63) becomes;

$$f_a \leq \frac{\pi}{16G} \cdot \frac{m_{Pl}\Lambda_{QCD}}{T^3} \cdot \frac{1}{t_0^2} \cdot \frac{1}{(1.24 \times 10^{-2} GeV)^2} \equiv (f_a)_{\max} . \tag{64}$$

If T= 2.73K be the present temperature of the universe and $\Lambda_{QCD} = 200\,MeV$ then;

$$m_a \geq 1.24\times10^{-5} eV\left(\frac{10^{12} GeV}{(f_a)_{\max}}\right) = (m_a)_{\min} \quad (65)$$

t_{ob} (in Gyr)	$(f_a)_{\max}$ (in $10^{12} GeV$)	$(m_a)_{\min}$ (in $10^{-5} eV$)
13	1.15	1.07
14	0.99	1.24
15	0.87	1.43
16	0.76	1.62
17	0.67	1.83
18	0.60	2.05
19	0.54	2.29
20	0.49	2.53
21	0.44	2.80
22	0.40	3.07
23	0.36	3.41
24	0.33	3.72
25	0.30	4.09

Table 1: Upper limits on f_a and lower limits on m_a for various values of the observed age of the universe

Observationally, the best lower limits for the age of the universe come from studies of globular clusters of stars in our galaxy, these are (13–19) Giga year (1Gyr = 10^9 yr), (18.8–24.8) Gyr. The results are shown in the following Table 1.

8 Upper Limit of the Age of the Universe with Cosmological Constant

If we include cosmological constant Λ then for $P << \rho$ then (52) takes the form (Joshi 1996);

$$R_{\mu\nu}V^{\mu}V^{\nu} \geq 8\pi\rho_m G - \Lambda c^2 \quad (66)$$

By this we obtain the maximum possible age $t_{\max}$;

$$t_{\max} = \frac{\pi}{2}\sqrt{\frac{3}{4\pi G\rho_m - \Lambda c^2}} = \frac{1}{4}\sqrt{\frac{\pi}{G}}\sqrt{\frac{3}{\rho_m - 2\rho_v}} \tag{67}$$

where ρ_m is the material energy density and ρ_v is the vacuum energy, so the critical energy density,

$$\rho_c = \rho_m + \rho_v = \rho \, . \tag{68}$$

For a dust full universe, $P = -\rho_v$. Defining $\Omega_v = \frac{\rho_v}{\rho_c}$, $\Omega_m = \frac{\rho_m}{\rho_c}$ we get;

$$t_{\max}^2 = \frac{3\pi}{16G}\cdot\frac{1}{\rho_c - 3\rho_v} = \frac{3\pi}{16G\rho_c}\cdot\frac{1}{1 - 3\frac{\rho_v}{\rho_c}} = \frac{3\pi}{16G\rho_c}\cdot\frac{1}{1 - 3\Omega_v} \tag{69}$$

$$1 - 3\Omega_v = \frac{3\pi}{16G\rho_c t_{\max}^2} = \frac{1}{2}\left(\frac{\pi}{H_0 t_{\max}}\right)^2 > 0 \tag{70}$$

$$\therefore \Omega_v < \frac{1}{3} \, . \tag{71}$$

Clearly $t_{\max} > t_u$, where t_u = age of the oldest known objects in the universe. Combining the above constraints we get;

$$\frac{1}{3}\left[1 - \frac{1}{2}\left(\frac{\pi}{H_0 t_u}\right)^2\right] < \Omega_v < \frac{1}{3} \, . \tag{72}$$

Taking $t_u = 1.5P \times 10^{10}$ yrs, $H_0 = 100\, h_0 kms^{-1}Mpc^{-1}$ (72) becomes;

$$\frac{1}{3}\left(1 - \frac{2}{h_0^2 p^2}\right) < \Omega_v < \frac{1}{3} \, . \tag{73}$$

Positive lower bound on Ω_v is obtained for $Ph_0 > \sqrt{2}$, hence for $h_0 = 1$ we get, t_u = 21 billion yrs; this implies a positive lower bound on Ω_v.

For radiation dominated age $P = \frac{1}{3}\rho_m - \rho_v$ then (69) becomes;

$$t_{\max}^2 = \frac{3\pi}{16G} \cdot \frac{1}{2\rho_m - 5\rho_v} \tag{74}$$

Which gives the bound on Ω_v when $Ph_0 > 1$. Thus it is possible to derive the required bounds in terms of the age of the oldest objects in the universe.

9 Conclusions

In this paper we have tried to describe the general upper limit of the age of the universe with cosmological constant. We have briefly described general relativity, Schwarzschild geometry, FRW model, and Raychaudhury equation to make the study easier to the common readers. The universe is homogeneous and isotropic around us about 14 billion light years. In our discussion we have not strictly followed the homogeneity and isotropy of the universe to determine the age of the universe. In our stud we have found that the age of the universe is around 10^{10} years. We have avoided difficult mathematical calculations and have displayed diagrams where necessary.

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