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Graceful Labelling of Supersubdivision of Ladder

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Abstract : In this paper we prove that supersubdivisions of ladders are graceful.

Keywords : graceful labelling, subdivision of graphs, supersubdivision of graphs.

1 Introduction

Let G be a graph with q edges. A graceful labelling of G is an injection from the set of its vertices to the set $\{0, 1, 2, \ldots, q\}$ such that the values of the edges are all integers from 1 to q, the value of an edge being the absolute value of the difference between the integers attributed to its end vertices.

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Recently G. Sethuraman and P. Selvaraju [5] have introduced a new method of construction called supersubdivision of a graph and showed that arbitrary supersubdivisions of paths are graceful. They have posed two open problems:

Problem 1.1. Is there any graph different from paths whose arbitrary supersubdivisions are graceful?

Problem 1.2. Is it true that every connected graph has at least one supersubdivision which is graceful?

We work on these problems and find that supersubdivisions of ladders are graceful. The ladder graph L_n is defined by $L_n = P_n \times K_2$ where P_n is a path with $\times$ denotes the cartesian product. L_n has $2n$ vertices and $3n-2$ edges. In the complete bipartite graph $K_{2,m}$ we call the part consisting of two vertices, the 2-vertices part of $K_{2,m}$ and the part consisting of m vertices the m-vertices part of $K_{2,m}$.

Let G be a graph with n vertices and t edges. A graph H is said to be a subdivision of G if H is obtained by subdividing every edge of G exactly once. H is denoted by $S(G)$. A graph H is said to be a supersubdivision of G if H is obtained by replacing every edge e_i of G by the complete bipartite graph $K_{2,m}$ for some positive integer m in such a way that the ends of e_i are merged with the two vertices part of $K_{2,m}$ after removing the edge e_i from G.

A supersubdivision H of a graph G is said to be an arbitrary supersubdivision of the graph G if every edge of G is replaced by an arbitrary $K_{2,m}$ (m may vary for each edge arbitrarily). In this paper we prove that supersubdivisions of ladders are graceful.

2 Main Results

Let L_n be a ladder. A supersubdivision of L_n is denoted by $SS(L_n)$.

Theorem 2.1. *$SS(L_n)$ with each edge replaced by $K_{2,m}$ is graceful.*

Proof. Let $G = SS(L_n)$ where every edge of L_n is replaced by $K_{2,m}$. G has $2n + m(3n-2)$ vertices and $2m(3n-2)$ edges. Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be the vertices of L_n. Let $u_iu_{i+1}, i = 1, 2, \ldots, n-1$. $v_iv_{i+1}, i = 1, 2, \ldots, n-1$ and $u_iv_i, i = 1, 2, \ldots, n$ be the edges of L_n.

Let $u_{i,i+1}^{(k)}, k = 1, 2, \ldots, m$ be the vertices of the m vertices part of the bipartite graph $K_{2,m}$ merged with the edge $u_iu_{i+1}, i = 1, 2, \ldots, n-1$. Let $v_{i,i+1}^{(k)}, k = 1, 2, \ldots, m$ be the vertices of the m vertices part of the bipartite graph $K_{2,m}$ merged with the edge $v_iv_{i+1}, i = 1, 2, \ldots, n-1$.

Let $w_i^{(k)}, k = 1, 2, \ldots, m$ be the vertices of the m vertices part of $K_{2,m}$ merged with the edge $u_iv_i, i = 1, 2, \ldots, n$. Naming of the vertices is as shown in Figure 1.

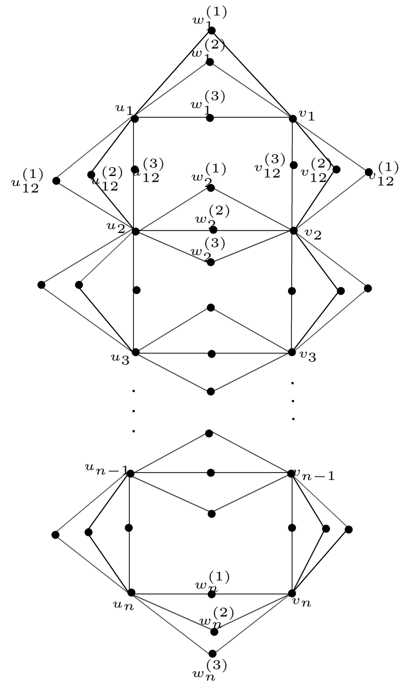

Figure 1: $SS(L_n)$

Define the following functions $\eta : N \to N$ by

$$\eta(i) = \begin{cases} 2i-1 & \text{if } i \text{ is even} \\ 2(i-1) & \text{if } i \text{ is odd} \end{cases}$$

$\gamma : N \to N$ by

$$\gamma(i) = \begin{cases} 2(i-1) & \text{if } i \text{ is even} \\ 2i-1 & \text{if } i \text{ is odd} \end{cases}$$

$\alpha : N \to N$ by

$$\alpha(k) = \begin{cases} 6r & \text{if } k = 3r+1 \\ 6r+1 & \text{if } k = 3r+2 \\ 6r+2 & \text{if } k = 3r+3 \end{cases}$$

Case (i) $m \equiv 0 (mod\ 3)$.

Let $m = 3p$, where p is a positive integer. Define $f : V \to \{0, 1, 2, \ldots, q\}$ where $q = 2m(3n-2)$ as follows.

$$f(u_i) = \eta(i), i = 1, 2, \ldots, n$$

$$f(v_i) = \gamma(i), i = 1, 2, \ldots, n.$$

For $k = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, n-1$

$$\text{define } f(u^{(k)}_{i,i+1}) = \begin{cases} 2m(3n-2) - (i-1)(18p-2) & \\ +\alpha(k) - (12p-4) & \text{if } i \text{ is odd} \\ 2m(3n-2) - (i-2)(18p-2) & \\ -2(m-k) - (30p-3) & \text{if } i \text{ is even.} \end{cases}$$

For $k = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, n-2$ define $f(v^{(k)}_{i,i+1}) = f(u^{(k)}_{i+1,i+2}) + (18p-2)$.

For $k = 1, 2, \ldots, m$ define $f(v^{(k)}_{n-1,n}) = f(u^{(k)}_{n-2,n-1}) - (18p-2)$.

For $k = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, n-1$ define $f(w^{(k)}_i) = 2m(3n-2) - (i-1)(18p-2) - 2(m-k)$.

For $k = 1, 2, \ldots, m$ define $f(w^{(k)}_n) = 2(n+k-1)$.

Example 2.1. Graceful labelling of $SS(L_4)$ where each edge of L_4 is replaced by $K_{2,6}$.

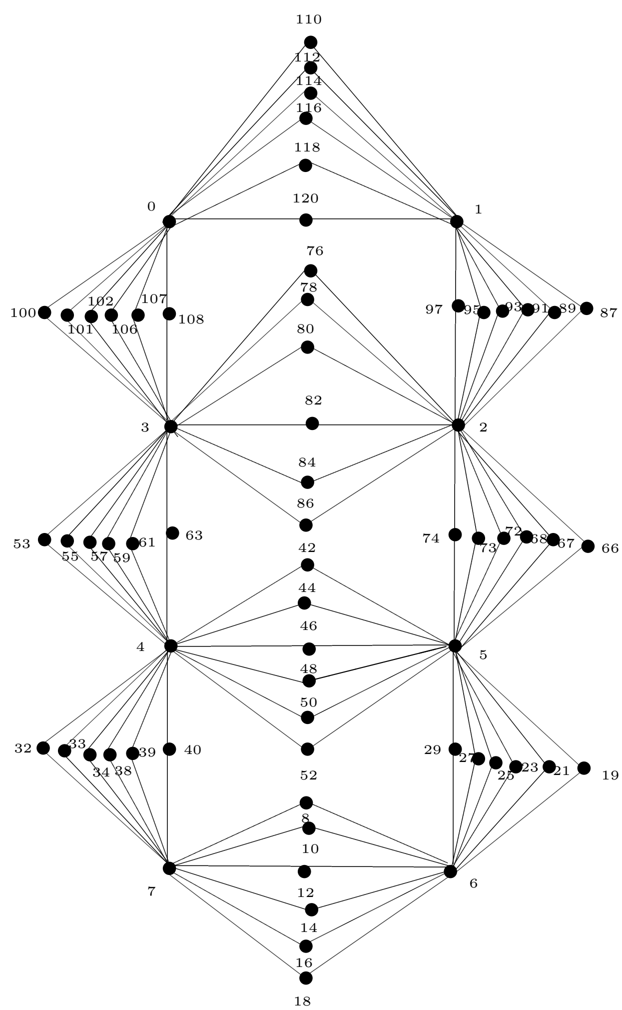

Figure 2

Case ii $m \equiv 1(mod\ 3)$.

Let $m = 3p+1$ where p is a nonnegative integer. Define $f : V \to \{0, 1, 2, \ldots, q\}$ where $q = 2m(3n-2)$ as follows.

$$f(u_i) = \eta(i), i = 1, 2, \ldots, n$$
$$f(v_i) = \gamma(i), i = 1, 2, \ldots, n.$$

When i is even, define $f(u_{i,i+1}^{(k)}) = 2m(3n - 2) - (i - 2)(18p + 4) - 2(m - k) - (30p + 7)$ for $k = 1, 2, 3, \ldots, m$. When i is odd, define

$$f(u_{i,i+1}^{(k)}) = \begin{cases} 2m(3n-2) - (i-1)(18p+4) & \\ +\alpha(k) - 12p & \text{if } k = 1, 2, \ldots, m-1 \\ 2m(3n-2) - (i-1)(18p+4) & \\ -6p & \text{if } k = m \end{cases}$$

For $i = 1, 2, \ldots, n - 2$, define $f(v_{i,i+1}^{(k)}) = f(u_{i+1,i+2}^{(k)}) + (18p + 4)$ for $k = 1, 2, \ldots, m$.

Define $f(v_{n-1,n}^{(k)}) = f(u_{n-2,n-1}^{(k)}) - (18p + 4)$ for $k = 1, 2, \ldots, m$.

For $i = 1, 2, 3, \ldots, n-1$, define

$$f(w_i^{(k)}) = \begin{cases} 2m(3n-2) - (i-1)(18p+4) & \\ -(6p+1) & \text{if } k = 1 \\ 2m(3n-2) - (i-1)(18p+4) & \\ -2(m-k) & \text{if } k = 2, 3, \ldots, m \end{cases}$$

For $k = 1, 2, \ldots, m$, define $f(w_n^{(k)}) = 2(n + k - 1)$.

Example 2.2. Graceful labelling of $SS(L_5)$ where each edge of L_5 is replaced by $K_{2,4}$.

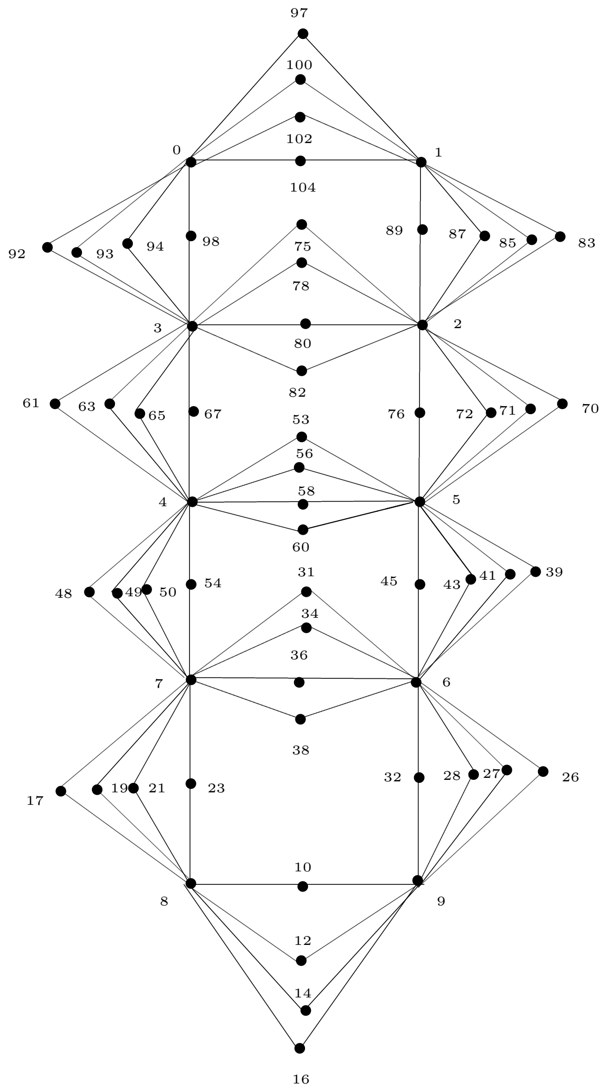

Figure 3

Note. In the above, taking $p = 0$ we obtain subdivision of a ladder and we have a graceful labelling of the subdivision of ladders, as a deduction from our labelling of vertices. This gives another graceful labelling for subdivision of ladders established by KM. Kathiresan [3].

Example 2.3. Graceful labelling of $S(L_6)$.

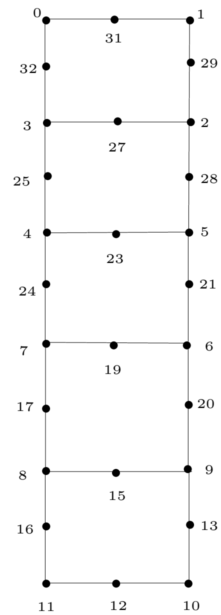

Figure 4

Case (iii) $m \equiv 2(mod\ 3)$.

Let $m = 3p+2$ where p is a nonnegative integer. Define $f : V \to \{0, 1, 2, \ldots, q\}$ where $q = 2m(3n-2)$ as follows.

$$f(u_i) = \eta(i), i = 1, 2, 3, \ldots, n$$

$$f(v_i) = \gamma(i), i = 1, 2, 3, \ldots, n.$$

When i is even, define $f(u^{(k)}_{i,i+1}) = 2m(3n-2) - (i-2)(18p+10) - 2(m-k) - (30p+17)$ for $k = 1, 2, \ldots, m$.

When i is odd, define $$f(u^{(k)}_{i,i+1}) = \begin{cases} 2m(3n-2) - (i-1)(18p+10) & \\ +\alpha(k) - 4(3p+1) & \text{if } k = 1, 2, \ldots, m-2 \\ 2m(3n-2) - (i-1)(18p+10) & \\ -2(3p+2) & \text{if } k = m-1 \\ 2m(3n-2) - (i-1)(18p+10) & \\ -6p & \text{if } k = m \end{cases}$$

For $i = 1, 2, \ldots, n-2$, define $f(v^{(k)}_{i,i+1}) = f(u^{(k)}_{i+1,i+2}) + (18p+10)$.

For $k = 1, 2, \ldots, m$, define $f(v^{(k)}_{n-1,n}) = f(u^{(k)}_{n-2,n-1}) - (18p+10)$.

For $k = 1, 2, \ldots, m$, define $f(w^{(k)}_n) = 2(n+k-1)$.

For $k = 1, i = 1, 2, \ldots, n-1$, define $f(w^{(k)}_i) = 2m(3n-2) - (i-1)(18p+10) - (6p+5)$.

For $k = 2, i = 1, 2, \ldots, n-1$, define $f(w^{(k)}_i) = 2m(3n-2) - (i-1)(18p+10) - (6p+1)$.

For $k = 3, 4, \ldots, m$ and $i = 1, 2, \ldots, n-1$, define $f(w^{(k)}_i) = 2m(3n-2) - (i-1)(18p+10) - 2(m-k)$.

Example 2.4. Graceful labelling of $SS(L_5)$ where each edge of L_5 is replaced by $K_{2,5}$.

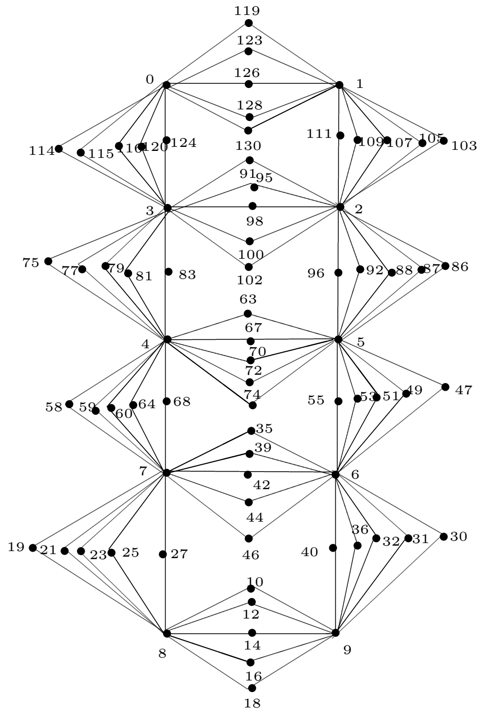

Figure 5

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