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An alternative block bootstrap in time series with weak dependence

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Abstract : When the observations are dependent, like in time series, Kunsch introduced the bootstrap with blocks forming by a fix number of consecutive observations. Different versions of block bootstrap has been formulated. In this paper we have proposed a bootstrap estimation with blocks formed from recalculated values of a statistic. We call it bootstrap with re-blocks. We have shown that this bootstrap works in time series strictly stationary α-mixing or m-dependent under some conditions. We have done simulations to compare the bootstrap with re-blocks with other block bootstrap methods.

Keywords : bootstrap, α-mixing, time series, weak dependence, m-dependent.

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1 Introduction

The bootstrap technique [6, 7] was introduced to provide nonparametric estimates of bias and standard error. The bootstrap is biased on repeated analyses of pseudo-data created by resampling the actual

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data. However, the bootstrap requires independence, which is rarely to get in time series.

Adoptions of the basic bootstrap to time series data work by resampling sets of consecutive observations to capture the process autocorrelation structure. Kunsch [13] and Liu and Singh [19] independently introduced the moving block bootstrap (MBB), which work by randomly overlapping blocks of fixed size with replacement. Liu and Singh [19] established that MBB estimate of the sample mean is biased in finite samples but asymptotically unbiased. They also showed the consistence of MBB estimate of the variance of the sample mean.

Politis and Romano [21] proposed another scheme for stationary time series, the stationary bootstrap (SB). In the SB, the data are resampled by concatenating blocks whose starting point is chosen at random and whose length is geometrically distributed with some chosen mean. Politis and Romano established that the SB estimate of the sample mean is unbiased and the SB estimate of the variance of the sample mean is consistent.

Different versions of block bootstrap methods has been formulated by Carlstein [2], Carlstein et al. [3], Kunsch and Carlstein [14], Hall [9], Politis and Romano [20], Ekonomi and Butka [8]. Properties of block bootstrap methods have been investigated by Davison and Hall [4], Shao and Yu [24], Lahiri [15, 16, 18] and others. The important problem of choosing optimal block length has been addressed by Buhlman and Kunsch [1], Hall et al [12] and Lahiri [17].

Let us suppose that the unknown parameter μ is a parameter of the joint distribution of the strictly stationary α-mixing or m-dependent time series $X_t, t \in Z$ and $\overset{\wedge}{\theta}$ is an estimator of μ based on the observations $X_1, ..., X_N$ from this time series.

In this paper we have proposed the bootstrap with re-blocks in time series strictly stationary α-mixing or m-dependent for estimating the parameter μ. In Section 2 we have shown the idea of this bootstrap estimation. At the beginning we have formed blocks compounded by s consecutive observations from a given time series. Then we have calculated the statistic of interest $\overset{\wedge}{\theta}$ for every block to estimate the unknown parameter μ. We have considered these calculated values like the observations of a new time series and with them we have formed blocks with length b. If we have r of them, we choose randomly k blocks with the same probability $\frac{1}{r}$. We concatenate these k blocks and we have construct the bootstrap sample compounded by $m = k \times b$ observations. In Section 3 we have shown that this bootstrap method works for time series strictly stationary α-mixing or m-dependent under some conditions. In Section 4 we have shown the simulation results in various ARMA models. From the results we see that the bootstrap with re-blocks perform shorter confidence intervals than the other block bootstrap methods. This bootstrap gives similar results with other bootstrap methods regarding coverage probability, bias and root mean square error (RMSE).

2 Bootstrap with re-blocks

In many time series problems the goal is to estimate a parameter of the joint distribution. Let suppose that μ is a unknown parameter of the joint distribution of $X_t, t \in Z$. The objective is to obtain confidence intervals for μ based on some observations from time series. We will focus on estimators of μ that are in the form of an average of functions defined on the observations.

It is given the time series $X_t, t \in Z$. Suppose that we have the observations $X_1, X_2, ..., X_N$ from this time series. We create blocks compounded by s consecutive observations in the form $S_1 = \{X_1, ..., X_s\}, S_2 = \{X_2, ..., X_{s+1}\}, ..., S_{N-s+1} = \{X_{N-s+1}, ..., X_N\}$. If we denote $n = N - s + 1$, we have formed the blocks $S_1, S_2, ..., S_n$. These blocks are moving blocks.

Let suppose that $\overset{\wedge}{\theta}$ is a statistic based on the observations $X_1, X_2, ..., X_N$. We calculate $Y_1 = \overset{\wedge}{\theta}(S_1), Y_2 = \overset{\wedge}{\theta}(S_2), ..., Y_n = \overset{\wedge}{\theta}(S_n)$ and assume that $Y_1, Y_2, ..., Y_n$ are observations from a new time series $Y_t, t \in Z$. Their mean is $\overline{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$. With these observations, in the same way, we can construct blocks compounded by b consecutive observations in the form $B_1 = \{Y_1, ..., Y_b\}, B_2 = \{Y_2, ..., Y_{b+1}\}, ..., B_{n-b+1} = \{Y_{n-b+1}, ..., Y_n\}$ and we denote $r = n - b + 1$. These blocks we call re-blocks.

To construct the bootstrap sample we choose randomly k blocks with probability $\frac{1}{r}$. We sign that blocks B_1^*, B_2^*, ..., B_k^*. If we concatenate these blocks consecutively, we have performed the observations $Y_1^*, Y_2^*, ..., Y_m^*$, where $m = b \times k$. This is the bootstrap sample. Their mean is

$$\overline{Y}_m^* = \frac{1}{m}\sum_{i=1}^{m} Y_i^* \tag{2.1}$$

If the parameter of interest μ is the mean of the time series $X_t, t \in Z$, then (2.1) will be the bootstrap estimator with re-blocks (BRB) for μ. To show that this bootstrap method works we will prove that the bootstrap distribution of $\sqrt{m}(\overline{Y}_m^* - \overline{Y}_n)$ approximate the distribution of $\sqrt{n}(\overline{Y}_n - \mu)$.

3 The validity of the re-blocks bootstrap

Let we do the following assumptions:

A1) $\{X_t, t \in Z\}$ is strictly stationary and α-mixing time series,

A2) $E\left|Y_1\right|^{2p+\delta} < c$, where p is integer, $p > 2, 0 < \delta \leq 2$ and $c > 0$,

A3) $EY_1 = \mu + o(\frac{1}{\sqrt{n}})$,where μ is a parameter of the joint distribution of $X_t, t \in Z$,

A4) $\sqrt{n}(\overline{Y}_n - E\overline{Y}_n) \xrightarrow{d} N(0, \sigma_\infty^2)$, where $0 < \sigma_\infty^2 < \infty$.

A5) $b = o(n)$ and if $n \longrightarrow \infty$, then $m \longrightarrow \infty$ reciprocally,

A6) $\sum_{k=1}^{\infty} k^{p-1}(\alpha_X(k))^{\frac{\delta}{2p+\delta}}$, for p integer, $p > 2$ and $0 < \delta \leq 2$.

Assumption A3 shows that the asymptotic order of the bias of $\overline{Y}_n$ is smaller than the asymptotic order of its standard deviations. We note that the assumptions A3 and A4 allows us to consider confidence intervals for $E\overline{Y}_n$ as confidence intervals for μ asymptotically, since by Slutsky's theorem, we have

$$\sqrt{n}(\overline{Y}_n - \mu) \xrightarrow{d} N(0, \sigma_\infty^2).$$

This asymptotic normal distribution can be used to yield confidence intervals for μ. However, the variance of σ_∞^2 must be estimated, but in many cases it is not feasible. In addition, a different estimate of the sampling distribution of $\sqrt{n}(\overline{Y}_n - \mu)$ might gives a better approximation, thus giving confidence intervals that are more accurate. It is the role that the bootstrap is usually called to play.

In the following treatments all the limits are calculated when $N \to \infty$ and $m \to \infty$. In this case, form the relevant relations, we will understand that $n \to \infty$ and $r \to \infty$ and conversely.

Now we see some lemma and theorems that will help us to prove the validity of the proposed bootstrap.

Lemma 3.1. *If the conditions A1-A6 are true, then*

$$\frac{1}{r}\sum_{i=1}^{r} Z_i \xrightarrow{p} EZ_1, \frac{1}{r}\sum_{i=1}^{r} |Z_i| \xrightarrow{p} E|Z_1|, \frac{1}{r}\sum_{i=1}^{r} |Z_i|^3 \xrightarrow{p} E|Z_1|^3,$$

where $Z_i = \frac{1}{\sqrt{b}}\sum_{j=i}^{i+b-1} Y_j, i = 1, 2, ..., r.$

Proof. Since the time series $X_t, t \in Z$ is strictly stationary and α-mixing time series, also the time series $Y_t, t \in Z$ is strictly stationary and α-mixing with $\alpha_Y(t) \leq \alpha_X(t - s + 1)$. We have $\frac{1}{r}\sum_{i=1}^{r} Z_i - EZ_1 = \frac{1}{r}\sum_{i=1}^{r}(Z_i - EZ_i)$. Then, based on some moment inequalities for mixing sequences [22, 23, 25, 26], we have

$$var(\frac{1}{r}\sum_{i=1}^{r}(Z_i - EZ_i)) =$$

$$= \frac{1}{r}var(Z_1 - EZ_1) + \frac{2}{r^2}\sum_{i=1}^{r-1}(r-i)cov(Z_1 - EZ_1, Z_{i+1} - EZ_{i+1}) \leq$$

$$\leq \frac{10}{r}(E|Z_1 - EZ_1|^p)^{\frac{2}{p}}(\frac{1}{2})^{\frac{p-2}{p}} + \frac{20}{r^2}\sum_{i=1}^{r-1}(r-i)(E|Z_1 - EZ_1|^p)^{\frac{2}{p}}(\alpha_Y(i))^{\frac{p-2}{p}} \leq$$

$$\leq \frac{10}{r}(K(E|Y_1|^{2p+\delta})^{\frac{2}{2p+\delta}}(\frac{1}{2})^{\frac{p-2}{p}} + \frac{20}{r^2}\sum_{i=1}^{r-1}(r-i)(KE|Y_1|^{2p+\delta})^{\frac{2}{2p+\delta}}(\alpha_Y(i))^{\frac{p-2}{p}} =$$

$$= O(\frac{1}{r} + \frac{20}{r^2}\sum_{i=1}^{r-1}(r-i)(\alpha_Y(i))^{\frac{p-2}{p}})$$

From A6, the series $\sum_{k=1}^{\infty} k\alpha_X(k)^{\frac{p-2}{p}}$ and $\sum_{k=1}^{\infty}\alpha_X(k)^{\frac{p-2}{p}}$ are convergent.

Thus we have $var(\frac{1}{r}\sum_{i=1}^{r}(Z_i - EZ_i)) \xrightarrow{p} 0$. So, by the Chebyshev theorem we have $\frac{1}{r}\sum_{i=1}^{r} Z_i \xrightarrow{p} EZ_1$. Then, from the properties of the convergence in probability we have $\frac{1}{r}\sum_{i=1}^{r}|Z_i| \xrightarrow{p} E|Z_1|$ and $\frac{1}{r}\sum_{i=1}^{r}|Z_i|^3 \xrightarrow{p} E|Z_1|^3$. □

Lemma 3.2. *If the conditions A1-A6 are true, then* $\frac{1}{r}\sum_{i=1}^{r}(Z_i-\sqrt{b}\overline{Y}_n)^2 \xrightarrow{p} \sigma_\infty^2$.

Proof. We have

$$\frac{1}{r}\sum_{i=1}^{r}(Z_i-\sqrt{b}\overline{Y}_n)^2 = \frac{1}{r}\sum_{i=1}^{r}(Z_i-EZ_i-\sqrt{b}(\overline{Y}_n-\frac{1}{\sqrt{b}}EZ_i))^2 =$$

$$= A_n - 2C_n + D_n,$$

where

$$A_n = \frac{1}{r}\sum_{i=1}^{r}(Z_i-EZ_i)^2,$$

$$C_n = \frac{1}{r}\sum_{i=1}^{r}\sqrt{b}(Z_i-EZ_i)(\overline{Y}_n-\frac{1}{\sqrt{b}}EZ_i),$$

and

$$D_n = \frac{1}{r}\sum_{i=1}^{r}b(\overline{Y}_n-\frac{1}{\sqrt{b}}EZ_i)^2.$$

Let we analyze these terms

1) In similar way with Lemma 3.1, we have

$$varA_n \le \frac{10}{r}(E\left|Z_1-EZ_1\right|^{2p})^{\frac{2}{p}}(\frac{1}{2})^{\frac{p-2}{p}}+$$

$$+\frac{20}{r^2}\sum_{i=1}^{r-1}(r-i)(E\left|Z_1-EZ_1\right|^{2p})^{\frac{2}{p}}(\alpha_Y(i))^{\frac{p-2}{p}}$$

or

$$varA_n \le \frac{10}{r}K(E\left|Y_1\right|^{2p+\delta})^{\frac{4}{2p+\delta}}+$$

$$+\frac{20}{r^2}\sum_{i=1}^{r-1}(r-i)K(E\left|Y_1\right|^{2p+\delta})^{\frac{4}{2p+\delta}}(\alpha_Y(i))^{\frac{p-2}{p}} =$$

$$= O(\frac{1}{r}+\frac{20}{r^2}\sum_{i=1}^{r-1}(r-i)(\alpha_Y(i))^{\frac{p-2}{p}}).$$

We see that $varA_n \xrightarrow{p} 0$. Now, from the condition A3, we have

$$EA_n = varZ_1 = var(\frac{1}{\sqrt{b}}\sum_{j=1}^{b}Y_j) = var(\sqrt{b}(\frac{1}{b}\sum_{j=1}^{b}Y_j)) \xrightarrow{p} \sigma_\infty^2.$$

Based on the Chebyshev theorem we see that $A_n \xrightarrow{p} \sigma_\infty^2$.

2) Let we consider the term C_n.

$$C_n = \sqrt{b}(\overline{Y}_n-\frac{1}{\sqrt{b}}EZ_1)\frac{1}{r}\sum_{i=1}^{r}\sqrt{b}(Z_i-EZ_1),$$

In Lemma 3.1 we have shown that $var(\frac{1}{r}\sum_{i=1}^{r}(Z_i - EZ_i)) \xrightarrow{p} 0$ and $E(\frac{1}{r}\sum_{i=1}^{r}(Z_i - EZ_i)) \xrightarrow{p} 0$. By the conditions A2, A3 and A4 we have $\sqrt{b}(\overline{Y}_n - \frac{1}{\sqrt{b}}EZ_1) = o_p(1)$. Based on Chebyshev theorem we have $C_n \xrightarrow{p} 0$.

3) We have

$$D_n = \frac{1}{r}\sum_{i=1}^{r} b(\overline{Y}_n - \frac{1}{\sqrt{b}}EZ_i)^2 =$$

$$= \frac{1}{r}\sum_{i=1}^{r} b(\overline{Y}_n - \mu)^2 + b(\mu - \frac{1}{\sqrt{b}}EZ_i)^2 + 2b(\overline{Y}_n - \mu)(\mu - \frac{1}{\sqrt{b}}EZ_i).$$

Let analyze separately the three terms:

a) From the conditions A2 and A3 we have

$$\sqrt{b}(\overline{Y}_n - \mu) = \sqrt{b}(\overline{Y}_n - E\overline{Y}_n) + \sqrt{b}E(\overline{Y}_n - \mu) =$$

$$= \frac{\sqrt{b}}{\sqrt{n}}\sqrt{n}(\overline{Y}_n - E\overline{Y}_n) + \sqrt{b}o(n^{-\frac{1}{2}}) \xrightarrow{p} 0.$$

Then

$$b(\overline{Y}_n - \mu)^2 = \frac{b}{n}(\sqrt{n}(\overline{Y}_n - \mu))^2 \xrightarrow{p} 0.$$

b)

$$b(\overline{Y}_n - \mu)(\mu - \frac{1}{\sqrt{b}}EZ_i) = b(\overline{Y}_n - \mu)(\mu - EY_1) = o_p(bn^{-1}) \xrightarrow{p} 0.$$

c)

$$b(\mu - \frac{1}{\sqrt{b}}EZ_i)^2 = o(bn^{-1}) \xrightarrow{p} 0.$$

Then $D_N \xrightarrow{p} 0$. Joining the convergence results of A_n, C_n, D_n we see the truth of lemma. □

Theorem 3.1. *If the conditions A1, A2 and A5 are fulfilled, then* $E^*\overline{Y}^*_m = \overline{Y}_n + o_p(1)$.

Proof. We have

$$E^*\overline{Y}^*_m = E^*\frac{1}{k}\sum_{i=1}^{k}\frac{1}{b}\sum_{j=(i-1)b+1}^{ib} Y^*_j = E^*\frac{1}{k}\sum_{i=1}^{k}\frac{1}{\sqrt{b}}Z^*_i = E^*\frac{1}{\sqrt{b}}Z^*_1,$$

where $Z^*_i = \frac{1}{\sqrt{b}}\sum_{j=(i-1)b+1}^{ib} Y^*_j$.

From the other hand

$$E^*Z^*_1 = \frac{1}{r}\sum_{i=1}^{r} Z_i = \frac{1}{r}\sum_{i=1}^{r}\frac{1}{\sqrt{b}}\sum_{j=i}^{i+b-1} Y_j = \frac{1}{\sqrt{b}}(b\overline{Y}_n - \frac{1}{r}\sum_{i=1}^{b-1}(b-i)(Y_i + Y_{n-i+1})).$$

Now

$$E^*\overline{Y}^*_m = \frac{1}{b}(b\overline{Y}_n - \frac{1}{r}\sum_{i=1}^{b-1}(b-i)(Y_i + Y_{n-i+1}).$$

From the condition A2 and the Chebychev theorem we see that $Y_i, i = 1, 2, ..., n$ are bounded in probability. To finish the proof of the theorem we show that

$$\left|\frac{1}{rb}\sum_{i=1}^{b-1} i(Y_i + Y_{n-i+1})\right| \leq \frac{1}{rb}\sum_{i=1}^{b-1}(b-i)(|Y_i| + |Y_{n-i+1}|) \leq \frac{2c_1(b-1)}{r} = o_p(1).$$

□

Theorem 3.2. *If the conditions A1-A6 are true, then* $m \cdot var^*(\overline{Y}^*_m) \xrightarrow{p} \sigma^2_\infty$.

Proof. Z^*_i for $i = 1, 2, ..., k$ are independent. We see that

$$m \cdot var^*(\overline{Y}^*_m) = m \cdot var^*(\frac{1}{m}\sum_{i=1}^{m} Y^*_i) = m \cdot var^*(\sum_{i=1}^{k}\frac{1}{k}\sum_{j=(i-1)b+1}^{ib} Y^*_i) =$$

$$= m \cdot var^*(\frac{1}{k}\sum_{i=1}^{k}\frac{1}{\sqrt{b}} Z^*_i) = var^*(Z^*_1).$$

But

$$var^*(Z^*_1) = \frac{1}{r}\sum_{i=1}^{r}(Z_i - E^*Z^*_i)^2 = \frac{1}{r}\sum_{i=1}^{r}(Z_i - \sqrt{b}\overline{Y}_n + o_p(1))^2.$$

This expression has the same limit with the expression $\frac{1}{r}\sum_{i=1}^{r}(Z_i - \sqrt{b}\overline{Y}_n)^2$ that converges in probability to σ^2_∞ based on the result of Lemma 3.2. Now it is clear the truth of the theorem. □

Theorem 3.3. *If the conditions A1-A6 are fulfilled, then*
$\sup_x \left|p^*\{\frac{\overline{Y}^*_m - E^*\overline{Y}^*_m}{\sqrt{var^*_m \overline{Y}^*}} \leq x\} - \Phi(x)\right| \xrightarrow{p} 0.$

Proof. We have

$$\overline{Y}^*_m = \frac{1}{k}\sum_{i=1}^{k}\frac{1}{b}\sum_{j=(i-1)b+1}^{ib} Y^*_j = \frac{1}{k\sqrt{b}}\sum_{i=1}^{k} Z^*_i.$$

From this we have

$$E^*\overline{Y}^*_m = \frac{1}{k\sqrt{b}}\sum_{i=1}^{k} E^*Z^*_i$$

and

$$\sqrt{var^*\overline{Y}^*_m} = \frac{1}{\sqrt{b}}\sqrt{\frac{1}{k}\sum_{i=1}^{k} var^*Z^*_i}.$$

So

$$\frac{\overline{Y}^*_m - E^*\overline{Y}^*_m}{\sqrt{var^*\overline{Y}^*_m}} = \frac{\frac{1}{k}\sum_{i=1}^{k} Z^*_i - \frac{1}{k}\sum_{i=1}^{k} EZ^*_i}{\sqrt{\frac{1}{k}\sum_{i=1}^{k} var^*Z^*_i}}.$$

We can apply Berry-Esseen theorem and we have

$$\sup_x \left| p^*\{\frac{\overline{Y}^*_m - E^*\overline{Y}^*_m}{\sqrt{var^*\overline{Y}^*_m}} \leq x\} - \Phi(x) \right| = \sup_x \left| p^*\{\frac{\frac{1}{k}\sum\limits_{i=1}^{k} Z_i^* - \frac{1}{k}\sum\limits_{i=1}^{k} EZ_i^*}{\sqrt{\frac{1}{k}\sum\limits_{i=1}^{k} var^* Z_i^*}} \leq x\} - \Phi(x) \right| \leq$$

$$\leq \frac{E^*\left|Z_1^* - EZ_1^*\right|^3}{\sqrt{var^* Z_1^*}}.$$

But

$$E^*\left|Z_1^* - EZ_1^*\right|^3 = \frac{1}{r}\sum_{i=1}^{r}\left|Z_i - \frac{1}{r}\sum_{j=1}^{r} Z_j\right|^3.$$

We can apply the Minkowski inequality and we have

$$\left(\sum_{i=1}^{r}\left|Z_i - \frac{1}{r}\sum_{j=1}^{r} Z_j\right|^3\right)^{\frac{1}{3}} \leq \left(\sum_{i=1}^{r}|Z_i|^3\right)^{\frac{1}{3}} + \left(\sum_{i=1}^{r}\left|\frac{1}{r}\sum_{j=1}^{r} Z_j\right|^3\right)^{\frac{1}{3}} =$$

$$= \left(\sum_{i=1}^{r}|Z_i|^3\right)^{\frac{1}{3}} + \frac{1}{r^{\frac{2}{3}}}\left|\sum_{j=1}^{r} Z_j\right|.$$

Then

$$E^*\left|Z_1^* - EZ_1^*\right|^3 \leq \frac{1}{r}\left(\left(\sum_{i=1}^{r}|Z_i|^3\right)^{\frac{1}{3}} + \frac{1}{r^{\frac{2}{3}}}\left|\sum_{j=1}^{r} Z_j\right|\right)^3 =$$

$$= \left(\left(\frac{1}{r}\sum_{i=1}^{r}|Z_i|^3\right)^{\frac{1}{3}} + \left|\frac{1}{r}\sum_{j=1}^{r} Z_j\right|\right)^3.$$

From Lemma 3.1 we take

$$\left(\left(\frac{1}{r}\sum_{i=1}^{r}|Z_i|^3\right)^{\frac{1}{3}} + \left|\frac{1}{r}\sum_{j=1}^{r} Z_j\right|\right)^3 \xrightarrow{p} \left(\left(E\left|Z_1\right|^3\right)^{\frac{1}{3}} + |EZ_1|\right)^3.$$

From this convergence we conclude that $E^*\left|Z_1^* - EZ_1^*\right|^3$ is bounded in probability. It is clear enough that the proof is complete. □

Now we prove the main result that shows the validity of the proposed bootstrap. This result allows us to justify the construction of confidence intervals for μ based on the distribution of the bootstrap with re-blocks.

Theorem 3.4. *If the conditions A1-A6 are fulfilled, then*

$$\sup_x \left| p^*\{\frac{\overline{Y}_m^* - E^*\overline{Y}_m^*}{\sqrt{var^*\overline{Y}_m^*}} \leq x\} - p\{\sqrt{n}(\overline{Y}_n - \mu) \leq x\} \right| \xrightarrow{p} 0.$$

Proof. From assumptions A3 and A4 we have

$$\sup_x \left| p\{\sqrt{n}(\overline{Y}_n - \mu) \leq x\} - \Phi(\frac{x}{\sigma_\infty}) \right| \xrightarrow{p} 0. \tag{3.1}$$

From Theorem 3.1 we have

$$p^*\{\sqrt{m}(\overline{Y}_m^* - \overline{Y}_n) \leq x\} = p^*\{\frac{\overline{Y}_m^* - E^*\overline{Y}_m^*}{\sqrt{var^*\overline{Y}_m^*}} \leq o_p(1) + \frac{x}{\sqrt{m \cdot var^*\overline{Y}_m^*}}\},$$

and from Theorem 3.2 and 3.3 we have

$$\mathop{sup}_x \left| p^*\{\sqrt{m}(\overline{Y}_m^* - \overline{Y}_n) \leq x\} - \Phi(\frac{x}{\sigma_\infty}) \right| \xrightarrow{p} 0. \tag{3.2}$$

Now, from the inequality

$$\sup_x \left| p^*\{\frac{\overline{Y}_m^* - E^*\overline{Y}_m^*}{\sqrt{var^*\overline{Y}_m^*}} \leq x\} - p\{\sqrt{n}(\overline{Y}_n - \mu) \leq x\} \right| \leq$$

$$\leq \sup_x \left| p^*\{\sqrt{m}(\overline{Y}_m^* - \overline{Y}_n) \leq x\} - \Phi(\frac{x}{\sigma_\infty}) \right| +$$

$$+\sup_x \left| p\{\sqrt{n}(\overline{Y}_n - \mu) \leq x\} - \Phi(\frac{x}{\sigma_\infty}) \right|$$

and the relations (3.1) and (3.2) the truth of this theorem is evident. □

We showed with above theorem that the proposed bootstrap estimation works in the case of time series that are strictly stationary and α-mixing. Now let suppose that the time series $X_t, t \in Z$ is strictly stationary and m-dependent. In this case $\alpha_X(k) = 0$ for $k > m$. From this we take that the time series $Y_t, t \in Z$ is $m - s + 1$-dependent. If we analyze the assumption A6, we see that $\sum_{k=1}^{\infty} k^{p-1}(\alpha_X(k))^{\frac{\delta}{2p+\delta}}$ is a finite sum. So, the proposed above bootstrap estimator works in the case of time series strictly stationary and m-dependent.

4 Simulation study

This section investigates the performance of the bootstrap with re-blocks for the confidence intervals estimation, when the parameter of interest is the time series mean of an $ARMA$ process. We consider $AR(1)$, $AR(2)$, $MA(1)$, $MA(2)$ and $ARMA(1,1)$ processes. In order to get a comparison to other bootstrap methods were used a wide range of coefficient values for the $AR(1)$ and $MA(1)$ processes. Coefficient values were chosen such that satisfy the stationarity and invertibility conditions.

For each model we generated time series of length n from 100 to 1000 with increments of 100. We implemented the moving block bootstrap (MBB) [13], the non-overlapping block bootstrap (NBB) [2], stationary bootstrap (SP) [21] and bootstrap with cycling blocks (BCB) [8] in order to estimate and to construct confidence intervals for the mean of time series. We constructed the percentile bootstrap (PB) and the bias corrected bootstrap (CB) intervals for the mean [5, 10]. The nominal level of the intervals was chosen to be 0.95. We used $Q = 1000$ bootstrap replications [11]. The block length was chosen at order $O(n^{\frac{1}{3}})$ for all methods [17]. We used 500 Monte Carlo replications for each simulation case to calculate the bias and the RMSE for each bootstrap point estimator and the average interval length and the empirical coverage probability in percentage for each type of intervals. R software was used. Some of the simulation results are shown in following tables.

From Tables 1,2 and 3 we see that BRB give shorter confidence intervals for time series mean μ. The other simulations results are similar between them.

Size (n)	coeff.	BCB	MBB	SB	BRB
100	0.1	0.4159	0.4098	0.4067	0.3397
	0.4	0.5776	0.5506	0.5533	0.4889
	0.7	1.0587	0.8991	0.9425	0.8215
	0.85	1.8110	1.3361	1.4258	1.2569
300	0.1	0.2453	0.2488	0.2406	0.2159
	0.4	0.3558	0.3403	0.3414	0.3118
	0.7	0.6864	0.5664	0.6128	0.5497
	0.85	1.2977	0.8855	1.0105	0.8840
500	0.1	0.1924	0.1911	0.1885	0.1725
	0.4	0.2858	0.2697	0.2702	0.2521
	0.7	0.5497	0.4592	0.4856	0.4503
	0.85	1.0680	0.7268	0.8363	0.7352
700	0.1	0.1633	0.1914	0.1617	0.1486
	0.4	0.2417	0.2713	0.2304	0.2171
	0.7	0.4734	0.4019	0.4227	0.3937
	0.85	0.9097	0.7624	0.7333	0.6569
1000	0.1	0.1365	0.1363	0.1346	0.1249
	0.4	0.2042	0.1954	0.1946	0.1850
	0.7	0.4038	0.3512	0.3590	0.3401
	0.85	0.7824	0.5812	0.6354	0.5711

Table 1. The confidence interval length for 95% PB intervals in $AR(1)$ model for various values of coefficient φ.

Size (n)	coeff.	BCB	MBB	SB	BRB
100	0.1	0.4043	0.4098	0.4004	0.3363
	0.4	0.5033	0.5098	0.4973	0.4265
	0.7	0.6026	0.6045	0.5898	0.5031
	0.85	0.6554	0.6597	0.6406	0.5592
300	0.1	0.2408	0.2422	0.2401	0.2151
	0.4	0.3052	0.3021	0.3003	0.2717
	0.7	0.3686	0.3648	0.3606	0.3304
	0.85	0.4074	0.3957	0.3945	0.3579
500	0.1	0.1895	0.1881	0.1869	0.1697
	0.4	0.2417	0.2380	0.2335	0.2137
	0.7	0.2916	0.2852	0.2852	0.2636
	0.85	0.31.84	0.3086	0.3066	0.2834
700	0.1	0.1600	0.1601	0.1588	0.1470
	0.4	0.2038	0.2013	0.1998	0.1850
	0.7	0.2465	0.2427	0.2420	0.2244
	0.85	0.2674	0.2639	0.2636	0.2458
1000	0.1	0.1362	0.1346	0.1344	0.1244
	0.4	0.1733	0.1954	0.1676	0.1575
	0.7	0.2082	0.2051	0.2029	0.1918
	0.85	0.2253	0.2227	0.2218	0.2080

Table 2. The interval length for 95% interval length in $MA(1)$ model with various values of coefficient θ.

Size (n)	$ARMA$ model	BCB	MBB	SB	BRB
100	$AR(2)$	0.8557	0.7903	0.7766	0.7013
	$MA(2)$	0.8176	0.7818	0.7615	0.6783
	$ARMA(1,1)$	1.1614	1.0566	1.0649	0.9361
300	$AR(2)$	0.5419	0.4906	0.4915	0.4597
	$MA(2)$	0.4905	0.4783	0.4719	0.4365
	$ARMA(1,1)$	0.7613	0.6805	0.6954	0.6385
500	$AR(2)$	0.4248	0.3921	0.3898	0.3646
	$MA(2)$	0.3881	0.3784	0.3781	0.3530
	$ARMA(1,1)$	0.6181	0.5406	0.5585	0.5148
700	$AR(2)$	0.3599	0.3361	0.3357	0.3160
	$MA(2)$	0.3351	0.3220	0.3223	0.3029
	$ARMA(1,1)$	0.5277	0.4690	0.4785	0.4510
1000	$AR(2)$	0.3062	0.2841	0.2840	0.2711
	$MA(2)$	0.2792	0.2711	0.2696	0.2565
	$ARMA(1,1)$	0.4461	0.4024	0.4069	0.3852

Table 3. The average interval length for 95% PB intervals for the mean with MBCB. The case: $AR(2)$ model with $\varphi_1 = 0.7, \varphi_2 = -0.1, MA(2)$ model with $\theta_1 = 0.8, \theta_2 = 0.5, ARMA(1,1)$ with $\varphi = 0.65$ and $\theta = 0.3$.

5 Conclusions

From our theoretical studies and simulation results we show that bootstrap with re-blocks (BRB) is a very good alternative for estimations in time series when the parameter of interests is a parameter of the joint distribution of a strictly stationary $\alpha-$mixing or $m-$dependent time series. We have shown that the bootstrap distribution of BRB approximates the distribution of $\sqrt{n}(\overline{X}_n - \mu)$.

So we can construct successfully confidence intervals for the unknown time series parameters estimated by a statistic $\hat{\theta}$ that satisfy some conditions. We have seen this fact from the simulation results. From the simulation results we see that BRB perform shorter confidence intervals than MBB, NBB, SP and BCB. The result for empirical coverage probability, bias and RMSE are similar with the other block bootstrap methods.

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