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# Packing Chromatic Number of Enhanced Hypercubes

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**Abstract :** The packing chromatic number $\chi_\rho(G)$ of a graph $G$ is the smallest integer $k$ for which there exists a mapping $\pi : V(G) \longrightarrow \{1, 2, ..., k\}$ such that any two vertices of color $i$ are at distance at least $i+1$. In this paper, we compute the packing chromatic number for enhanced hypercubes.

**Keywords :** Packing chromatic number, Hypercube, Enhanced hypercube.

## 1 Introduction and Terminology

Let $G$ be a connected graph and let $k$ be an integer, $k \geq 1$. A packing $k$-coloring of a graph $G$ is a mapping $\pi : V(G) \longrightarrow \{1, 2, ..., k\}$ such that any two vertices of color $i$ are at distance at least $i+1$. The packing chromatic number $\chi_\rho(G)$ of $G$ is the smallest integer $k$ for which $G$ has packing $k$-coloring. The concept of packing coloring comes from the area of frequency assignment in wireless networks and was introduced by Goddard et al. in [2] under the name Broadcast coloring. It also has several applications,

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such as, resource placement and biological diversity. The term packing chromatic number was introduced by Brešar in [1].

Goddard et al. [2] proved that packing chromatic number problem is NP-complete for general graphs and computed $\chi_\rho(Q_r)$ for $1 \leq r \leq 5$. Recently, Torres et al. [4] computed $\chi_\rho(Q_r)$ for $6 \leq r \leq 8$. Hypercube network topology has become the most popular message-passing architectures, and several multicomputer configurations based on this topology have been designed and even marketed. There are different variations of hypercubes, for example folded hypercubes, crossed hypercubes, fibonacci cubes, augmented cubes and enhanced hypercubes. In this paper, we study the packing chromatic number of various classes of enhanced hypercubes [3].

**Definition 1.1** ([3])**.** *For $r \geq 1$, let $Q_r$ denote the graph of the $r$- dimensional hypercube. The vertex set $V(Q_r) = \{x_0x_1 \cdots x_{r-1} : x_i = 0 \text{ or } 1,\ 0 \leq i \leq r-1\}$. Two vertices $x_0x_1 \cdots x_{r-1}$ and $y_0y_1 \cdots y_{r-1}$ are adjacent if and only if they differ in exactly one position.*

**Definition 1.2** ([3])**.** *The enhanced hypercube $Q_{r,k}$, $0 \leq k \leq r-1$, is a graph with vertex set $V(Q_{r,k}) = V(Q_r)$ and edge set $E(Q_{r,k}) = E(Q_r) \cup \{x_0x_1...x_{k-2}x_{k-1}x_k...x_{r-1},$ $x_0x_1...x_{k-2}\overline{x}_{k-1}\overline{x}_k...\overline{x}_{r-1}) : x_i = 0 \text{ or } 1,\ 0 \leq i \leq r-1\}$. The edges of $Q_r$ in $Q_{r,k}$ are hypercube edges and the remaining edges of $Q_{r,k}$ are called complementary edges.*

**Remark 1.3** ([3])**.** *The set $\{(x_0x_1...x_{k-2}x_{k-1}x_k...x_{r-1}, x_0x_1...x_{k-2}\overline{x}_{k-1}\overline{x}_k...\overline{x}_{r-1})\}$ is empty when $k = 0$. Hence $Q_{r,0}$ reduces to the $r$- dimensional hypercube.*

## 2 Enhanced Hypercube $Q_{r,k}$, $k = r-3$ and $r \geq 4$

In this section, we consider the class of enhanced hypercubes $Q_{r,k}$, $0 \leq k \leq r-1$ where $k = r-3$ and $r \geq 4$. For $r \geq 5$, $Q_{r,r-3}$ is constructed by taking two copies of $Q_{r-1,r-4}$ and adding the hypercube edges of $Q_r$.

Giving color 1 to every vertex in a maximum independent set and distinct color to every other vertex of a graph $G$ of diameter 2, we get an exact packing chromatic number for $G$ [2]. In the following, $\beta_0$ denotes the independence number.

**Lemma 2.1.** *The number of vertices in $Q_{r,r-3}$, $r \geq 4$ that receives a unique color in a packing chromatic number is $2^r - (\sum_{i=1}^{r-2} 2^i - 2^{r-3})$.*

*Proof.* We prove the lemma by induction on $r$. When $r = 4$, $Q_{4,1}$ is of diameter 2. Since $\beta_0 = 4$, at most four vertices are colored 1 and remaining vertices receive distinct colors greater than 1. Therefore, the number of vertices that receives a unique color is $2^4 - 4 = 2^4 - (\sum_{i=1}^{2} 2^i - 2^1)$. See Figure 1(a).

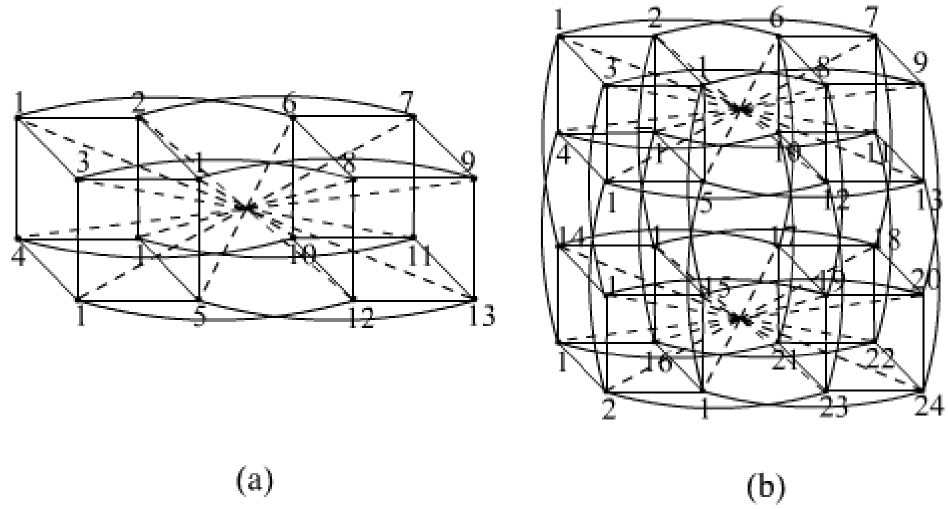



Figure 1 (a): $\chi_\rho(Q_{4,1}) = 13$ and (b): $\chi_\rho(Q_{5,2}) = 24$.

We assume the result is true for $r = k-1$. Therefore, the number of vertices that receives a unique color is $2^{k-1} - (\sum_{i=1}^{k-3} 2^i - 2^{k-4})$.
Let $r = k$. The diameter of $Q_{k,k-3}$ is $k-2$. Further, $Q_{k,k-3}$ consists of two copies of $Q_{k-1,k-4}$. Therefore, $2(\sum_{i=1}^{k-3} 2^i - 2^{k-4})$ vertices receive color 1,...,$k-4$, two vertices receive color $k-3$ and remaining vertices receive distinct color greater than $k-3$. Hence, the number of vertices that receives a unique color is $2^k - (2(\sum_{i=1}^{k-3} 2^i - 2^{k-4}) + 2) = 2^k - (\sum_{i=1}^{k-2} 2^i - 2^{k-3})$.

□

The colors less than the diameter of $Q_{r,r-3}$, $r \geq 4$ can be repeated in the packing coloring of $Q_{r,r-3}$. Therefore, the number of colors already used is $diam(Q_{r,r-3}) - 1 = (r-2) - 1$. To find the packing chromatic number for $Q_{r,r-3}$, $r \geq 4$, we add the colors already used in $Q_{r,r-3}$ with $2^r - (\sum_{i=1}^{r-2} 2^i - 2^{r-3})$. Thus, the following theorem is an easy consequence of Lemma (2.1). The packing coloring of $Q_{5,2}$ is shown in Figure 1(b).

**Theorem 2.2.** *Let $Q_{r,r-3}$,$r \geq 4$ be an $r$- dimensional enhanced hypercube. Then $\chi_\rho(Q_{r,r-3}) = (2^r - (\sum_{i=1}^{r-2} 2^i - 2^{r-3})) + r-3$.*

# 3 Enhanced Hypercube $Q_{r,r-2}$,$r \geq 4$

In this section, we consider the class of enhanced hypercubes $Q_{r,k}$, $0 \leq k \leq r-1$ where $k = r-2$ and $r \geq 4$. For $r \geq 4$, $Q_{r,r-2}$ is constructed by taking two copies of $Q_{r-1,r-3}$ and adding the hypercube edges of $Q_r$.

**Lemma 3.1.** *The number of vertices in $Q_{r,r-2}$, $r \geq 4$ that receives a unique color in a packing chromatic number is* $2^r - (\sum_{i=1}^{r-2} 2^i + 2^{r-3})$.

*Proof.* We prove the lemma by induction on $r$. When $r = 4$, $Q_{4,2}$ is of diameter 3. Since $\beta_0 = 8$, at most eight vertices can be colored 1. Further, at most two vertices can be colored 2; but, if eight vertices are colored 1 then remaining vertices receive distinct colors starting from 2. Therefore, the number of vertices that receives a unique color is $2^4 - 8 = 2^4 - (\sum_{i=1}^{2} 2^i + 2^1)$. See Figure 2(a).

We assume the result is true for $r = k - 1$. Therefore, the number of vertices that receive a unique color is $2^{k-1} - (\sum_{i=1}^{k-3} 2^i + 2^{k-4})$.

Let $r = k$. $Q_{k,k-2}$ is of diameter $k - 1$. $Q_{k,k-2}$ consists of two copies of $Q_{k-1,k-3}$. Therefore, $2(\sum_{i=1}^{k-3} 2^i + 2^{k-4})$ vertices receive color 1,...,$k-4$ and two vertices receive color $k-3$. Hence, the number of vertices that receives a unique color is $2^k - (2(\sum_{i=1}^{k-3} 2^i + 2^{k-4}) + 2) = 2^k - (\sum_{i=1}^{k-2} 2^i + 2^{k-3})$.

□

To find the packing chromatic number for $Q_{r,r-2}$, $r \geq 4$, we add the number of colors already used in $Q_{r,r-2}$ with $2^r - (\sum_{i=1}^{r-2} 2^i + 2^{r-3})$. Thus, the following theorem is an easy consequence of Lemma (3.1). The packing coloring of $Q_{5,3}$ is shown in Figure 2(b).

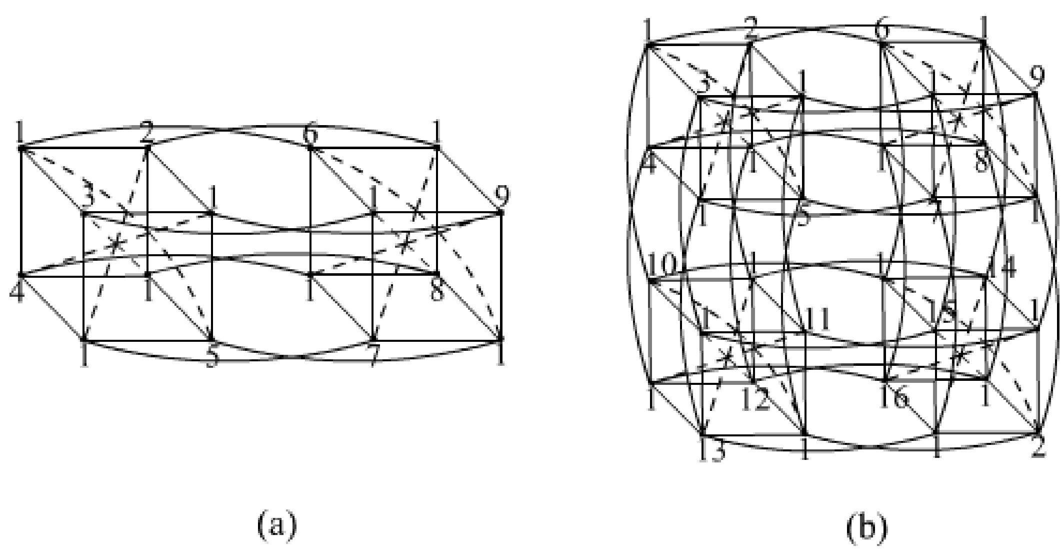



Figure 2 (a): $\chi_\rho(Q_{4,2}) = 9$ and (b): $\chi_\rho(Q_{5,3}) = 16$.

**Theorem 3.2.** *Let $Q_{r,r-2}$,$r \geq 4$ be an $r$- dimensional enhanced hypercube. Then* $\chi_\rho(Q_{r,r-2}) = (2^r - (\sum_{i=1}^{r-2} 2^i + 2^{r-3})) + r-3$.

# 4 Enhanced Hypercube $Q_{r,r-1}$,$r \geq 4$

In this chapter, we consider the class of enhanced hypercubes $Q_{r,k}$, $0 \leq k \leq r-1$ where $k = r-1$ and $r \geq 4$. For $r \geq 4$, $Q_{r,r-1}$ is constructed by taking two copies of $Q_{r-1,r-2}$ and adding the hypercube edges of $Q_r$.

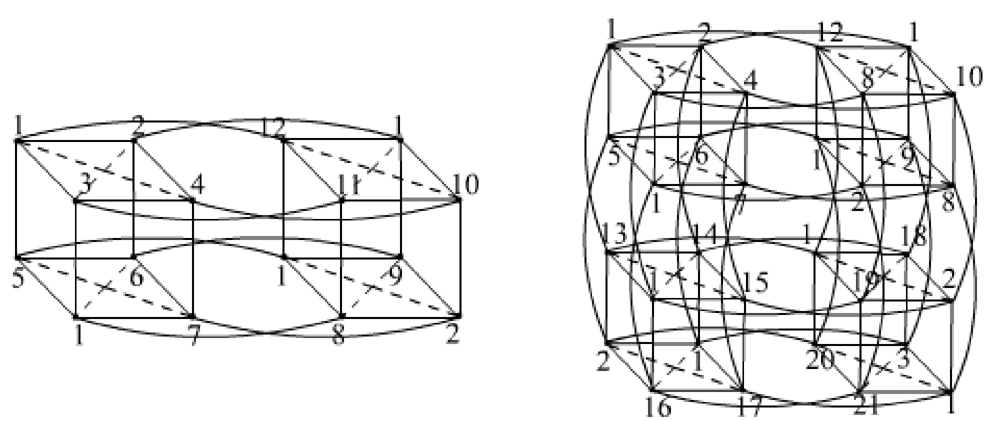
**Lemma 4.1.** *The number of vertices in $Q_{r,r-1}$, $r \geq 4$ that receives a unique color in a packing chromatic number is $2^r - \sum\limits_{i=1}^{r-2} 2^i$.*

*Proof.* We prove the lemma by induction on $r$. When $r = 4$, $Q_{4,3}$ is of diameter 3. It is clear that there are four copies of $K_4$ in $Q_{4,3}$. We denote them as $K_4^{(1)}$, $K_4^{(2)}$, $K_4^{(3)}$ and $K_4^{(4)}$. Giving color 1 to any $K_4$ in $Q_{4,3}$, at most four vertices of $Q_{4,3}$ can be colored 1. Further, giving color 2 to any $K_4$ in $Q_{4,3}$, at most two vertices can be colored 2 and remaining vertices receive distinct colors greater than two. Therefore, the vertices that receive a unique color is $2^4 - 6 = 2^4 - \sum\limits_{i=1}^{2} 2^i$. See Figure 3(a).

We assume the result is true for $r = k-1$. Therefore, the vertices that receive a unique color is $2^{k-1} - \sum\limits_{i=1}^{k-3} 2^i$. Let $r = k$. $Q_{k,k-1}$ is of diameter $k-1$. Further, $Q_{k,k-1}$ consists of two copies of $Q_{k-1,k-2}$. Therefore, $2\sum\limits_{i=1}^{k-3} 2^i$ vertices receive color 1,...,$k-3$ and two vertices receive color $k-2$. Hence, the total number of vertices that receives a unique color is $2^k - (2\sum\limits_{i=1}^{k-3} 2^i + 2) = 2^k - \sum\limits_{i=1}^{k-2} 2^i$.

□

To find the packing chromatic number for $Q_{r,r-1}$, $r \geq 4$, we add the number of colors already used in $Q_{r,r-1}$ with $2^r - \sum\limits_{i=1}^{r-2} 2^i$. Thus, the following theorem is an easy consequence of Lemma (4.1). The packing coloring of $Q_{5,4}$ is shown in Figure 3(b).



(a) (b)

Figure 3 (a): $\chi_\rho(Q_{4,3}) = 12$ and (b): $\chi_\rho(Q_{5,4}) = 21$.

**Theorem 4.2.** *Let $Q_{r,r-1}$,$r \geq 4$ be an $r$- dimensional enhanced hypercube. Then* $\chi_\rho(Q_{r,r-1}) = (2^r - \sum_{i=1}^{r-2} 2^i) + r - 2$.

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