Bazı pell denklemlerinin temel çözümleri

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ÖZET

a, b pozitif tamsayılar olsun. Makalede, $d = a^2b^2 + 2b, a^2b^2 + b, a^2 \pm 2, a^2 \pm a$ olmak üzere $\sqrt{\mathrm{d}}'$ nin sürekli kesir açılımı bulundu. $d = a^2b^2 + 2b, a^2b^2 + b, a^2 \pm 2, a^2 \pm a$ olmak üzere $\sqrt{\mathrm{d}}'$ nin sürekli kesir yaklaşımları kullanılarak $x^2 - dy^2 = \pm 1$ denklemlerinin fundamental çözümleri elde edildi.

Anahtar Kelimeler: Diofant Denklemleri, Pell Denklemleri, Sürekli Kesirler.

Fundamental solutions to some pell equations

ABSTRACT

Let $\boldsymbol{a}, \boldsymbol{b}$ be positive integers. In this paper, we find continued fraction expansion of $\sqrt{\mathbf{d}}$ when $\boldsymbol{d} = \boldsymbol{a}^2\boldsymbol{b}^2 + 2\boldsymbol{b}, \boldsymbol{a}^2\boldsymbol{b}^2 + \boldsymbol{b}, \boldsymbol{a}^2 \pm 2, \boldsymbol{a}^2 \pm \boldsymbol{a}$. We will use continued fraction expansion of $\sqrt{\mathbf{d}}$ in order to get the fundamental solutions of the equations $\boldsymbol{x}^2 - \boldsymbol{d}\boldsymbol{y}^2 = \pm \mathbf{1}$ when $\boldsymbol{d} = \boldsymbol{a}^2\boldsymbol{b}^2 + 2\boldsymbol{b}, \boldsymbol{a}^2\boldsymbol{b}^2 + \boldsymbol{b}, \boldsymbol{a}^2 \pm 2, \boldsymbol{a}^2 \pm \boldsymbol{a}$.

Keywords: Diophantine Equations, Pell Equations, Continued Fractions.

1. INTRODUCTİON

Let d be a positive integer which is not a perfect square and N be any nonzero fixed integer. Then the equation $x^2 - dy^2 = N$ is known as Pell equation. For $N = \pm 1$, the equations $x^2 - dy^2 = 1$ and $x^2 - dy^2 = -1$ are known as classical Pell equations. If $a^2 - db^2 = N$, we say that (a, b) is a solution to the Pell equation $x^2 - dy^2 = N$. We use the notations (a, b) and $a + b\sqrt{d}$ interchangeably to denote solutions of the equation $x^2 - dy^2 = N$. Also, if a and b are both positive, then $a + b\sqrt{\mathrm{d}}$ is a positive solution to the equation $x^2 - dy^2 = N$.

The Pell equation $x^2 - dy^2 = 1$ has always positive integer solutions. When $N \neq 1$, the Pell equation $x^2 - dy^2 = N$ may not have any positive integer solutions. It can be seen that the equations $x^2 - 3y^2 = -1$ and $x^2 - 7y^2 = -4$ have no positive integer solutions. Whether or not there exists a positive integer solution to the equation $x^2 - dy^2 = -1$ depends on the period length of the continued fraction expansion of $\sqrt{\mathrm{d}}$ (See section 2 for more detailed information).

In the next section, we give some well known theorems and then we give main theorems in the third section.

2. PRELIMINARIES

If we know fundamental solution to the equations $x^2 - dy^2 = \pm 1$, then we can give all positive integer solutions to these equations. Our theorems are as follows. For more

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information about Pell equation, one can consult [1], [2] and [3].

Let $x_1+y_1\sqrt{d}$ be a positive solution to the equation $x^2-dy^2=N$. We say that $x_1+y_1\sqrt{d}$ is the fundamental solution to the equation $x^2-dy^2=N$, if $x_2+y_2\sqrt{d}$ is a different solution to the equation $x^2-dy^2=N$, then $x_1+y_1\sqrt{d}<x_2+y_2\sqrt{d}$. Recall that if $a+b\sqrt{\mathrm{d}}$ and $r+s\sqrt{\mathrm{d}}$ are two solutions to the equation $x^2-dy^2=N$, then $a=r$ if and only if $b=s$, and $a+b\sqrt{\mathrm{d}}<r+s\sqrt{\mathrm{d}}$ if and only if $a<r$ and $b<s$.

Theorem 2.1: Let d be a positive integer that is not a perfect square. Then there is a continued fraction expansion of $\sqrt{\mathrm{d}}$ such that

$$\sqrt{d}=[a_0,\overline{a_1,a_2,..a_{n-1},2a_0}\ ]$$

where l is the period length and for $0\le n\le n-1,\ a_j$ is given by the recussion formulas;

$$\alpha_0=\sqrt{\mathrm{d}},a_k=[\![\alpha_k]\!] \text{ and } \alpha_{k+1}=\frac{1}{\alpha_k-a_k}\ ,$$
$$k=0,1,2,3,\ldots$$

Recall that $a_l=2a_0$ and $a_{l+k}=a_k$ for $k\ge 1$. The n^{th} convergence of $\sqrt{d}$ for $n\ge 0$ is given by

$$\frac{p_n}{q_n}=[a_0,a_1,\ldots,a_n]=a_0+\cfrac{1}{a_1+\cfrac{1}{\ddots\, 1+\cfrac{1}{a_n}}}.$$

By means of the k^{th} convergence of $\sqrt{\mathrm{d}}$, we can give the fundamental solution to the equations $x^2-dy^2=1$ and $x^2-dy^2=-1$.

Now we give the fundamental solution to the equations $x^2-dy^2=\pm 1$ by means of the period length of the continued fraction expansion of $\sqrt{d}$.

Lemma 2.2 Let l be the period length of continued fraction expansion of $\sqrt{\mathrm{d}}$. If l is even, then the fundamental solution to the equation $x^2-dy^2=1$ is given by

$$x_1+y_1\sqrt{d}=p_{l-1}+q_{l-1}\sqrt{\mathrm{d}}$$

and the equation $x^2-dy^2=-1$ has no positive integer solutions. If l is odd, then the fundamental solution to the equation $x^2-dy^2=1$ is given by

$$x_1+y_1\sqrt{d}=p_{2l-1}+q_{2l-1}\sqrt{d}$$

and the fundamental solution to the equation $x^2-dy^2=-1$ is given by

$$x_1+y_1\sqrt{d}=p_{l-1}+q_{l-1}\sqrt{d}.$$

Theorem 2.3: Let $x_1+y_1\sqrt{d}$ be the fundamental solution to the equation $x^2-dy^2=1$. Then all positive integer solutions of the equation $x^2-dy^2=1$ are given by

$$x_n+y_n\sqrt{d}=(x_1+y_1\sqrt{d})^n$$

with $n\ge 1$.

Theorem 2.4: Let $x_1+y_1\sqrt{d}$ be the fundamental solution to the equation $x^2-dy^2=-1$. Then all positive integer solutions of the equation $x^2-dy^2=-1$ are given by

$$x_n+y_n\sqrt{d}=(x_1+y_1\sqrt{d})^{2n-1}$$

with $n\ge 1$.

3. MAIN THEOREMS

From now on, we will assume that a and b are positive integers. We give continued fraction expansion of $\sqrt{d}$ for $d=a^2b^2+2b, a^2b^2+b, a^2\pm 2, a^2\pm a$.

Theorem 3.1: Let $d=a^2b^2+2b$. Then

$$\sqrt{d}=[ab,\overline{a,2ab}].$$

Proof: Let $\alpha_0=a^2b^2+2b$. It can be seen that

$$(ab)^2<a^2b^2+2b<(ab+1)^2.$$

Then, by Theorem 2.1, we get

$$a_0=\left[\!\!\left[\sqrt{a^2b^2+2b}\right]\!\!\right]=ab$$

and therefore

$$\alpha_1=\frac{1}{\sqrt{a^2b^2+2b}-ab}=\frac{\sqrt{a^2b^2+2b}+ab}{2b}.$$

On the other hand, since $ab<\sqrt{a^2b^2+2b}$, it follows that

$$\frac{ab+ab}{2b}=a<\frac{\sqrt{a^2b^2+2b}+ab}{2b}<a+1.$$

Then, by Theorem 2.1, we get

$$a_1=[\![\alpha_1]\!]=\left[\!\!\left[\frac{\sqrt{a^2b^2+2b}+ab}{2b}\right]\!\!\right]=a.$$

It can be seen that

$$\alpha_2=\frac{1}{\frac{\sqrt{a^2b^2+2b}+ab}{2b}-a}=\sqrt{a^2b^2+2b}+ab$$

and therefore

$$a_2=[\![\alpha_2]\!]=\left[\!\!\left[\sqrt{a^2b^2+2b}+ab\right]\!\!\right]=2ab$$
$$=2a_0.$$

Thus, by Theorem 2.1, it follows that

$$\sqrt{a^2b^2+2b}=\left[ab,\overline{a,2ab}\right].$$

Then the proof follows.

Theorem 3.2: Let $d=a^2b^2+b$. Then

$\sqrt{d}=\left[ab,\overline{2a,2ab}\right]$.

Proof: Let $\alpha_0=a^2b^2+b$. It can be seen that

$$(ab)^2<a^2b^2+b<(ab+1)^2.$$

Then by Theorem 2.1, we get

$$a_0=\left[\!\left[\sqrt{a^2b^2+b}\right]\!\right]=ab,$$

and therefore

$$\alpha_1=\frac{1}{\sqrt{a^2b^2+b}-ab}=\frac{\sqrt{a^2b^2+b}+ab}{b}.$$

On the other hand, since $ab<\sqrt{a^2b^2+b}$, it follows that

$$\frac{ab+ab}{b}=2a<\frac{\sqrt{a^2b^2+b}+ab}{b}<2a+1.$$

Then, by Theorem 2.1, we get

$$a_1=[\![\alpha_1]\!]=\left[\!\left[\frac{\sqrt{a^2b^2+b}+ab}{b}\right]\!\right]=2a$$

and therefore

$$\alpha_2=\frac{1}{\frac{\sqrt{a^2b^2+b}+ab}{b}-2a}$$

$$=\sqrt{a^2b^2+b}+ab.$$

Since $2ab<\sqrt{a^2b^2+b}+ab<2ab+1$, it follows that

$$a_2=[\![\alpha_2]\!]=\left[\!\left[\sqrt{a^2b^2+b}+ab\right]\!\right]=2ab=2a_0.$$

Thus, by Theorem 2.1, we get.

$$\sqrt{a^2b^2+b}=\left[ab,\overline{2a,2ab}\right]$$

This completes the proof.

Theorem 3.3: Let $d=a^2+a$. Then

$\sqrt{d}=[a,\overline{2,2a}]$.

Proof: Let $\alpha_0=a^2+a$. Since $a^2<a^2+a<(a+1)^2$, it follows that

$$a_0=[\![\alpha_0]\!]=\left[\!\left[\sqrt{a^2+a}\right]\!\right]=a,$$

and therefore

$$\alpha_1=\frac{1}{\sqrt{a^2+a}-a}=\frac{\sqrt{a^2+a}+a}{a}.$$

Since $\frac{a+a}{a}=2<\frac{\sqrt{a^2+a}+a}{a}<3$, it follows that

$$a_1=\left[\!\left[\frac{\sqrt{a^2+a}+a}{a}\right]\!\right]=2$$

and therefore

$$\alpha_2=\frac{1}{\frac{\sqrt{a^2+a}+a}{a}-2}=\sqrt{a^2+a}+a.$$

Since $2a<\sqrt{a^2+a}+a<2a+1$, we get

$$a_2=\left[\!\left[\sqrt{a^2+a}+a\right]\!\right]=2a=2a_0.$$

Thus, by Theorem 2.1, we get

$$\sqrt{a^2+a}=[a,\overline{2,2a}].$$

This completes the proof.

Theorem 3.4: Let $d=a^2-a$. Then

$\sqrt{a^2-a}=\left[a-1,\overline{2,2(a-1)}\right]$.

Proof: Let $\alpha_0=a^2-a$. It can be seen that

$$(a-1)^2<(a^2-a)<a^2.$$

Then, by Theorem 2.1, we get

$$a_0=\left[\!\left[\sqrt{a^2-a}\right]\!\right]=a-1$$

and therefore

$$\alpha_1=\frac{1}{\sqrt{a^2-a}-(a-1)}=\frac{\sqrt{a^2-a}+(a-1)}{a-1}.$$

Since $\frac{a-1+a-1}{a-1}=2<\frac{\sqrt{a^2-a}+a-1}{a-1}<3$, it follows that

$$a_1=\left[\!\left[\frac{\sqrt{a^2-a}+a-1}{a-1}\right]\!\right]=2$$

and therefore

$$\alpha_2 = \frac{1}{\frac{\sqrt{a^2-a}+a-1}{a-1} - 2}$$

$$= \sqrt{a^2-a} + (a-1).$$

Since $2(a-1) < \sqrt{a^2-a} + a - 1 < 2a-1,$ it follows that

$$a_2 = [\![\sqrt{a^2-a} + a - 1]\!]$$
$$= 2(a-1) = 2a_0.$$

Thus, by Theorem 2.1, we get

$$\sqrt{a^2-a} = [a-1, \overline{2, 2(a-1)}].$$

This completes the proof.

Theorem 3.5: Let $d = a^2 + 2.$ Then

$\sqrt{a^2+2} = [a, \overline{a, 2a}].$

Proof: Let $\alpha_0 = a^2 + 2$. It can be seen that

$$a^2 < (a^2+2) < (a+1)^2 .$$

Then, by Theorem 2.1, we get

$$a_0 = [\![\sqrt{a^2+2}]\!] = a$$

and therefore

$$\alpha_1 = \frac{1}{\sqrt{a^2+2}-a} = \frac{\sqrt{a^2+2}+a}{2}.$$

Since $a < \frac{\sqrt{a^2+2}+a}{2} < a+1$, it follows that

$$a_1 = \left[\!\!\left[\frac{\sqrt{a^2+2}+a}{2}\right]\!\!\right] = a$$

and therefore

$$\alpha_2 = \frac{1}{\frac{\sqrt{a^2+2}+a}{2} - a} = \sqrt{a^2+2} + a.$$

Thus $a_2 = [\![\sqrt{a^2+2} + a]\!] = 2a = 2a_0.$
Then, by Theorem 2.1, it follows that

$$\sqrt{a^2+2} = [a, \overline{a, 2a}].$$

This completes the proof.

Theorem 3.6: Let $d = a^2 - 2.$ Then

$$\sqrt{a^2-2} = [a-1, \overline{1, a-2, 1, 2(a-1)}].$$

Proof: Let $\alpha_0 = a^2 - 2$. It can be seen that

$$(a-1)^2 < (a^2-2) < a^2.$$

Then, by Theorem 2.1, we get

$$a_0 = [\![\sqrt{a^2-2}]\!] = a - 1$$

and therefore

$$\alpha_1 = \frac{1}{\sqrt{a^2-2}-(a-1)} = \frac{\sqrt{a-2}+(a-1)}{2a-3}.$$

Since $1 + \frac{1}{2a-3} < \frac{\sqrt{a^2-2}+(a-1)}{2a-3} < 1 + \frac{2}{2a-3}$, it follows that

$$a_1 = \left[\!\!\left[\frac{\sqrt{a^2-2}+(a-1)}{2a-3}\right]\!\!\right] = 1$$

and therefore

$$\alpha_2 = \frac{1}{\frac{\sqrt{a^2-2}+(a-1)}{2a-3} - 1}$$

$$= \frac{\sqrt{a^2-2}+(a-2)}{2}.$$

Since $a - 2 + \frac{1}{2} < \frac{\sqrt{a^2-2}+(a-2)}{2} < a - 1$, it follows that

$$a_2 = \left[\!\!\left[\frac{\sqrt{a^2-2}+a-2}{2}\right]\!\!\right] = a - 2$$

and therefore

$$\alpha_3 = \frac{1}{\frac{\sqrt{a^2-2}+(a-2)}{2} - (a-2)}$$
$$= \frac{\sqrt{a^2-2}+(a-2)}{2a-3}.$$

Since $1 < \frac{\sqrt{a^2-2}+(a-2)}{2a-3} < 1 + \frac{1}{2a-3}$, we get

$$a_3 = \left[\!\!\left[\frac{\sqrt{a-2}+(a-2)}{2a-3}\right]\!\!\right] = 1$$

and therefore

$$\alpha_3 = \sqrt{a^2-2} + (a-1).$$

Since $2(a-1) < \sqrt{a^2-2} + (a-1) < 2a-1$, it follows that

$$a_3 = [\![\sqrt{a^2-2} + (a-1)]\!] = 2(a-1) = 2a_o$$

Thus, by Theorem 2.1, we get

$$\sqrt{a^2-2} = [a-1, \overline{1, a-2, 1, 2(a-1)}].$$

This completes the proof.

Now we give the fundamental solution to the equation $x^2 - dy^2 = 1$ when $d \in \{a^2b^2 + 2b, a^2b^2 + b, a^2 \pm 2, a^2 \pm a\}$.

Corollary 1: Let $d = a^2b^2 + 2b$. Then the fundamental solution to the equation $x^2 - dy^2 = 1$ is

$$x_1 + y_1\sqrt{d} = a^2b + 1 + a\sqrt{d}.$$

Proof: The period of length of continued fraction of $\sqrt{a^2b^2 + 2b}$ is 2 by Theorem 3.1. Therefore the fundamental solution to the equation $x^2 - dy^2 = 1$ is $p_1 + q_1\sqrt{d}$ by Lemma 2.2. Since

$$\frac{p_1}{q_1} = a_0 + \frac{1}{a_1} = ab + \frac{1}{a} = \frac{a^2b+1}{a},$$

the proof follows.
Since the proofs of the following corollaries are similar, we omit them.

Corollary 2: Let $d = a^2b^2 + b$. Then the fundamental solution to the equation $x^2 - dy^2 = 1$ is

$$x_1 + y_1\sqrt{d} = 2a^2b + 1 + 2a\sqrt{d}.$$

Corollary 3: Let $d = a^2 + 2$. Then the fundamental solution to the equation $x^2 - dy^2 = 1$ is

$$x_1 + y_1\sqrt{d} = a^2 + 1 + a\sqrt{d}.$$

Corollary 4: Let $d = a^2 + a$. Then the fundamental solution to the equation $x^2 - dy^2 = 1$ is

$$x_1 + y_1\sqrt{d} = 2a + 1 + 2\sqrt{d}.$$

Corollary 5: Let $d = a^2 - a$. Then the fundamental solution to the equation $x^2 - dy^2 = 1$ is

$$x_1 + y_1\sqrt{d} = 2a - 1 + 2\sqrt{d}.$$

Corollary 6: Let $d = a^2 - 2$. Then the fundamental solution to the equation $x^2 - dy^2 = 1$ is

$$x_1 + y_1\sqrt{d} = a^2 - 1 + a\sqrt{d}.$$

Proof: The period of length of continued fraction of $\sqrt{a^2-2}$ is 4 by Theorem 3.6. Therefore the fundamental solution to the equation $x^2 - dy^2 = 1$ is $p_3 + q_3\sqrt{d}$ by Lemma 2.2. Since

$$\frac{p_3}{q_3} = (a-1) + \frac{1}{1+\frac{1}{(a-2)+\frac{1}{1}}} = \frac{a^2-1}{a},$$

the proof follows.

From Lemma 2.2, we can give the following corollary.

Corollary 7: Let $d \in \{a^2b^2 + 2b, a^2b^2 + b, a^2 \pm 2, a^2 \pm a\}$. Then the equation $x^2 - dy^2 = -1$ has no integer solutions.

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