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# Recognition of the group $G_2(5)$ by the prime graph

P. Nosratpour[a,*] and M. R. Darafsheh [b]

[a] *Department of mathematics, ILam Branch, Islamic Azad university, Ilam, Iran*;
[b] *School of Mathematics, statistics and Computer Science, College of Science, University of Tehran, Tehran, Iran*

**Abstract.**Let $G$ be a finite group. The prime graph of $G$ is a graph $\Gamma(G)$ with vertex set $\pi(G)$, the set of all prime divisors of $|G|$, and two distinct vertices $p$ and $q$ are adjacent by an edge if $G$ has an element of order $pq$. In this paper we prove that if $\Gamma(G) = \Gamma(G_2(5))$, then $G$ has a normal subgroup $N$ such that $\pi(N) \subseteq \{2, 3, 5\}$ and $G/N \cong G_2(5)$.

**Keywords:** prime graph, recognition, linear group

## 1. Introduction

Let $G$ be a finite group. The spectrum $\omega(G)$ of $G$ is the set of orders of elements in $G$, where each possible order element occurs once in $\omega(G)$ regardless of how many elements of that order $G$ has. This set is closed and partially ordered by divisibility, hence it is uniquely determined by its maximal elements. The set of maximal elements of $\omega(G)$ is denoted by $\mu(G)$. The number of isomorphic classes of finite groups $H$ such that $\omega(G) = \omega(H)$ is denoted by $h(G)$. If $h(G) = k \geqslant 1$ is finite then the group $G$ is called a k-recognizable group by spectrum. If $h(G)$ is not finite, $G$ is called non-recognizable. A 1-recognizable group is usually called a recognizable group. The recognizability of finite groups by spectrum was first considered by W.J.Shi et.al. in [16]. A list of finite simple groups which are known to be or not to be recognizable by spectrum is given in [11].

For $n \in N$, let $\pi(n)$ denote the set of all the prime divisors of n, and for a finite group $G$ let us set $\pi(G) = \pi(|G|)$. The prime graph $\Gamma(G)$ of a finite group $G$ is a simple graph with vertex set $\pi(G)$ in which two distinct vertices $p$ and $q$ are joined by an edge if and only if $G$ has an element of order $pq$. It is clear that a knowledge of $w(G)$ determines $\Gamma(G)$ completely but not vise-versa in general. Given a finite group $G$, the number of non-isomorphic classes of finite groups $H$ with $\Gamma(G) = \Gamma(H)$ is denoted by $h_\Gamma(G)$. If $h_\Gamma(G) = 1$, then $G$ is said to be recognizable by prime graph. If $h_\Gamma(G) = k < \infty$, then $G$ is called k-recognizable by prime graph, in case $h_\Gamma(G) = \infty$ the group $G$ is called non-recognizable by prime graph. Obviously a group recognizable by spectra need not to be recognizable by prime graph, for example $A_5$ is recognizable by spectra but $\Gamma(A_5) = \Gamma(A_6)$.

*Corresponding author. Email: P.Nosratpour@ilam-iau.ac.ir

The number of connected components of $\Gamma(G)$ is denoted by $s(G)$. As a consequence of the classification of the finite simple groups it is proved in [19] and [9], that for any finite simple group $G$ we have $s(G) \leqslant 6$. Let $\pi_i = \pi_i(G)$, $1 \leqslant i \leqslant s$, be the connected components of $G$. For a group of even order we let $2 \in \pi_1$. Recognizability of groups by prime graph was first studied in [5] where some sporadic simple groups were characterized by prime graph. As another concept we say that a non-abelian simple group $G$ is quasi-recognizable by graph if every finite group whose prime graph is $\Gamma(G)$ has a unique non-abelian composition factor isomorphic to $G$.

It is proved in [20] that the simple groups $G_2(7)$and ${}^2G_2(q)$, $q = 3^{2m+1} > 3$, are recognizable by prime graph, where both groups have disconnected prime graphs. A series of interesting results concerning recognition of finite simple groups were obtained by B.Khosravi et.al. In particular they have stablished quasi-recognizability of the group $L_{10}(2)$ by graph and the recognizability of $L_{16}(2)$ by graph in [7] and [8], where both groups have connected prime graphs.

Next we introduce useful notation. Let $p$ be a prime number. The set of all non-abelian finite simple groups $G$ such that $p \in \pi(G) \subseteq \{2, 3, 5, \ldots, p\}$ is denoted by $\mathfrak{S}_p$. It is clear that the set of all non-abelian finite simple groups is the disjoint union of the finite sets $\mathfrak{S}_p$ for all primes $p$. The sets $\mathfrak{S}_p$, where $p$ is a prime less than 1000 is given in [21].

## 2. Preliminary results

Let $G$ be a finite group with disconnected prime graph. The structure of $G$ is given in [19] which is stated as a lemma here. Let $G$ be a finite group with disconnected prime graph. Then $G$ satisfies one of the following conditions:
$s(G) = 2$, $G = KC$ is a Frobenius group with kernel $K$ and complement $C$, and the two connected components of $G$ are $\Gamma(K)$ and $\Gamma(C)$. Moreover $K$ is nilpotent, and here $\Gamma(K)$ is a complete graph.
$s(G) = 2$ and $G$ is a 2-Frobeuius group, i.e. , $G = ABC$ where $A, AB \trianglelefteq G$, $B \trianglelefteq BC$, and $AB$, $BC$ are Frobenius groups.
There exists a non-abelian simple group $P$ such that $P \leqslant \overline{G} = G/N \leqslant Aut(P)$ for some nilpotent normal $\pi_1(G)$-subgroup $N$ of $G$ and $\overline{G}/P$ is a $\pi_1(G)$-group. Moreover, $\Gamma(P)$ is disconnected and $s(P) \geqslant s(G)$. If a group $G$ satisfies condition(c) of the above lemma we may write $P = B/N$, $B \leqslant G$, and $\overline{G}/P = G/B = A$, hence in terms of group extensions $G = N \cdot P \cdot A$, where $N$ is a nilpotent normal $\pi_1(G)$-subgroup of $G$ and $A$ is a $\pi_1(G)$-group.

The above structure lemma was extended to groups with connected prime graphs satisfying certain conditions [17]. Denote by $t(G)$ the maximal number of primes in $\pi(G)$ pairwise nonadjacent in $\Gamma(G)$ and $t(2, G)$ the maximal number of primes in $\pi(G)$ nonadjacent to 2. Let $G$ be a finite group satisfying the following conditions:
There exist three pairwise distinct primes in $\pi(G)$ nonadjacent in $\Gamma(G)$, i.e. , $t(G) \geqslant 3$.
There exists an odd prime in $\pi(G)$ nonadjacent in $\Gamma(G)$ to 2, i.e. ,$t(2, G) \geqslant 2$. Then, there is a finite non-abelian simple group $S$ such that $S \leqslant \overline{G} = G/K \leqslant Aut(S)$ for the maximal normal solvable subgroup $K$ of $G$. Furthermore $t(S) \geqslant t(G) - 1$ and one of the following statements holds:

(1) $S \cong A_7$ or $L_2(q)$ for some odd $q$, and $t(S) = t(2, G) = 3$.

(2) For every prime $p \in \pi(G)$ nonadjacent to 2 in $\Gamma(G)$ a Sylow $p$-subgroups of $G$ is isomorphic to a Sylow $p$-subgroup of $S$. In particular $t(2, S) \geqslant t(2, G)$.

In the following we list some properties of the Frobenius group where some of its proof can be found in [15]. Let $G$ be a Frobenius group with kernel $K$ and complement $H$, then:
$K$ is nilpotent and $|H| \mid (|K| - 1)$.
The connected components of $G$ are $\Gamma(K)$ and $\Gamma(H)$.
$|\mu(K)| = 1$ and $\Gamma(K)$ is a complete graph.
If $|H|$ is even, then $K$ is abelian.
Every subgroup of $H$ of order $pq$, $p$ and $q$ not necessary distinct primes, is cyclic. In particular if $H$ is abelian, then it would be cyclic.
If $H$ is non-solvable, then there is a normal subgroup $H_0$ of $H$ such that $[H : H_0] \leqslant 2$ and $H_0 \cong SL_2(5) \times Z$, where every Sylow subgroup of $Z$ is cyclic and $|Z|$ is prime to 2, 3 and 5. A Frobenius group with cyclic kernel of order $m$ and cyclic complement of order $n$ is denoted by $m : n$.

The following result is also used in this paper whose proof is included in [3]. Every 2-Frobenius group is solvable. [6] Let $G$ be a finite solvable group all of whose elements are of prime power order, then the order of $G$ is divisible by at most two distinct primes. [12] Let $G$ be a finite group, $K \trianglelefteq G$, and let $G/K$ be a Frobenius group with kernel $F$ and cyclic complement $C$. If $(|F|, |K|) = 1$ and $F$ dose not lie in $(K \cdot C_G(K))/K$, then $r \cdot |C| \in w(G)$ for some prime divisor $r$ of $|K|$. [18]
If there exists a primitive prime divisor $r$ of $q^n - 1$, then $L_n(q)$ has a Frobenius subgroup with kernel of order $r$ and cyclic complement of order $n$.
$L_n(q)$ contains a Frobenius subgroup with kernel of order $q^{n-1}$ and cyclic complement of order $(q^{n-1} - 1)/(n, q - 1)$. Using [1], we can find $\mu(G_2(5)) = \{20, 21, 24, 25, 30, 31\}$.Therefore, the prime graph of $G_2(5)$ is as a follows.

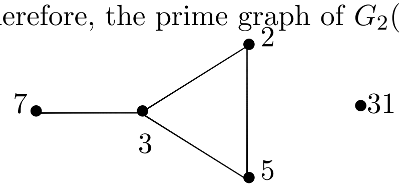


Figure 1: The prime graph of $G_2(5)$

Our main results are the following: If $G$ is a finite group such that $\Gamma(G) = \Gamma(G_2(5))$, then $G$ has a normal subgroup $N$ such that $\pi(N) \subseteq \{2, 3, 5\}$ and $G/N \cong G_2(5)$.

## 3. Proof of the theorem

We assume $G$ is a group with $\Gamma(G) = \Gamma(G_2(5))$. By Figure 1, we have $s(G) = 2$, hence, $G$ has disconnected prime graph and we can use Lemma 2.1 here:
$G$ is non-solvable.
If $G$ is solvable, then consider a $\{5, 7, 31\}$-Hall subgroup of $G$ and call it $H$. By Figure 1, $H$ dose not contain elements of order $5 \cdot 7$, $7 \cdot 31$, $5 \cdot 31$, and since it is solvable, by [5] we deduce $|t(H)| \leqslant 2$, a contradiction.
$G$ is neither a Frobenius nor a 2-Frobenius group.
By (a) and Lemma 2.4, $G$ is not a 2-Frobenius group. If $G$ is a Frobenius group, then by lemma 2.1, $G = KC$ with Frobenius kernel $K$ and Frobenius complement $C$ with connected components $\Gamma(K)$ and $\Gamma(C)$. Obviously $\Gamma(K)$ is a graph with vertex $\{31\}$ and $\Gamma(C)$ with vertex set $\{2, 3, 5, 7\}$. Since $G$ is non-solvable, by Lemma 2.3(a) $C$ must be non-solvable. Therefore, by Lemma 2.3(f) $C$ has a

subgroup isomorphic to $H_0$ and $[C : H_0] \leqslant 2$, where $H_0 \cong SL_2(5) \times Z$ with $Z$ cyclic of order prime to $2, 3, 5$. But $\mu(SL_2(5)) = \{4, 6, 10\}$ from which we can observe that $H_0$ has no element of order 15. This implies that $C$ has no element of order 15, contradicting Figure 1.

(a) and (b) imply that case (c) of Lemma 2.1 holds for $G$. Hence, there is a non-abelian simple group $P$ such that $P \leqslant \overline{G} = G/N \leqslant Aut(P)$ where $N$ is a nilpotent normal $\pi_1(G)$-subgroup of $G$ and $\overline{G}/P$ is a $\pi_1(G)$-group and $s(P) \geqslant 2$. We have $\pi_1(G) = \{2, 3, 5, 7\}$ and $\pi(G) = \{2, 3, 5, 7, 31\}$. Therefore, $P$ is a simple group with $\pi(P) \subseteq \{2, 3, 5, 7, 31\}$, i.e. , $P \in \mathfrak{S}_p$ where $p$ is a prime number satisfying $p \leqslant 31, p \neq 11, 13, 17, 19, 23, 29$. Using [21] we list the possibilities for $P$ in Table I.

Table I: Simple groups in $\mathfrak{S}_p$, $p \leqslant 31, p \neq 11, 13, 17, 19, 23, 29$.

| P | $\|P\|$ | $\|out(P)\|$ | P | $\|P\|$ | $\|out(P)\|$ |
|---|---|---|---|---|---|
| $A_5$ | $2^2 \cdot 3 \cdot 5$ | 2 | $L_2(8)$ | $2^3 \cdot 3^2 \cdot 7$ | 3 |
| $A_6$ | $2^3 \cdot 3^2 \cdot 5$ | 4 | $U_3(3)$ | $2^5 \cdot 3^3 \cdot 7$ | 2 |
| $S_4(3)$ | $2^6 \cdot 3^4 \cdot 5$ | 2 | $A_7$ | $2^3 \cdot 3^2 \cdot 5 \cdot 7$ | 2 |
| $L_2(7)$ | $2^3 \cdot 3 \cdot 7$ | 2 | $L_2(49)$ | $2^4 \cdot 3 \cdot 5^2 \cdot 7^2$ | 4 |

| P | $\|P\|$ | $\|out(P)\|$ | P | $\|P\|$ | $\|out(P)\|$ |
|---|---|---|---|---|---|
| $U_3(5)$ | $2^4 \cdot 3^2 \cdot 5^3 \cdot 7$ | 6 | $S_6(2)$ | $2^9 \cdot 3^4 \cdot 5 \cdot 7$ | 1 |
| $L_3(4)$ | $2^6 \cdot 3^2 \cdot 5 \cdot 7$ | 12 | $O_8^+(2)$ | $2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$ | 6 |
| $A_8$ | $2^6 \cdot 3^2 \cdot 5 \cdot 7$ | 2 | $L_2(31)$ | $2^5 \cdot 3 \cdot 5 \cdot 31$ | 2 |
| $A_9$ | $2^6 \cdot 3^4 \cdot 5 \cdot 7$ | 2 | $L_3(5)$ | $2^5 \cdot 3 \cdot 5^3 \cdot 31$ | 2 |
| $J_2$ | $2^7 \cdot 3^3 \cdot 5^2 \cdot 7$ | 2 | $L_2(5^3)$ | $2^2 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 31$ | 6 |
| $A_{10}$ | $2^7 \cdot 3^4 \cdot 5^2 \cdot 7$ | 2 | $G_2(5)$ | $2^6 \cdot 3^3 \cdot 5^6 \cdot 7 \cdot 31$ | 1 |
| $U_4(3)$ | $2^7 \cdot 3^6 \cdot 5 \cdot 7$ | 8 | $L_5(2)$ | $2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$ | 2 |
| $S_4(7)$ | $2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$ | 2 | $L_6(2)$ | $2^{15} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 31$ | 2 |

$\{31\} \subseteq \pi(P)$

By Table I, $|Out(P)|$ is a number of the form $2^\alpha \cdot 3^\beta$, therefore, if $G/N = P \cdot S$ where $S \leqslant Out(P)$, then $|P|_p = |G/N|_p/|S|_p$ for all $p \in \pi(G)$, where $n_p$ denotes the $p$-part of the integer $n \in N$. Hence, $|N|_p = \frac{|G|_p}{|P|_p \cdot |S|_p}$, from which the claim follows because $\pi(N) \subseteq \{2, 3, 5, 7\}$.

Therefore only the following possibilities arise for $P$: $L_2(31)$, $L_5(2)$, $L_6(2)$, $L_3(5)$, $L_2(5^3)$ and $G_2(5)$.

$P \cong G_2(5)$

By [4], we have $\mu(L_5(2)) = \{8, 12, 14, 15, 21, 31\}$ and $\mu(L_6(2)) = \{8, 12, 28, 30, 31, 63\}$. Therefore, if $P \cong L_5(2)$ or $L_6(2)$, then, we have $2 \sim 7$ in $\Gamma(G)$, is a contradiction.

By [10], we have $\mu(L_2(5^3)) = \{5, 62, 63\}$. Therefore, if $P \cong L_2(5^3)$, then, we have $2 \sim 31$ in $\Gamma(G)$, a contradiction.

By [1], we have $\mu(L_2(31)) = \{15, 16, 31\}$. Therefore, if $P \cong L_2(31)$, then, $7 \in \pi(N)$. By Lemma 2.7, $P$ has a Frobenius subgroup $31 : 15$, then, by Lemma 2.6, $G$ has an element of order $5 \cdot 7$, a contradiction.

By [1], we have $\mu(L_3(5)) = \{20, 24, 31\}$. Therefore, if $P \cong L_3(5)$, then, $7 \in \pi(N)$. By Lemma 2.7, $P$ has a Frobenius subgroup $25 : 24$, then, by Lemma 2.6, $G$ has an element of order $2 \cdot 7$, a contradiction. Therefore $P \cong G_2(5)$.

$G/N \cong G_2(5)$
So far we proved that $P \leqslant G/N \leqslant Aut(P)$ where $P \cong G_2(5)$. But $Aut(G_2(5)) = G_2(5)$, therefore, $G/N \cong G_2(5)$.
$\pi(N) \subseteq \{2, 3, 5\}$
We Know that $N$ is a nilpotent normal $\{2, 3, 5, 7\}$-subgroup of $G$. Regarding Figure 1 we obtain:
If $2, 5 \mid |N|$, then $\pi(N) \subseteq \{2, 3, 5\}$
If $3 \mid |N|$, then $\pi(N) \subseteq \{2, 3, 5, 7\}$
If $7 \mid |N|$, then $\pi(N) \subseteq \{3, 7\}$

Now we observe that the group $G_2(5)$ contains Frobenius subgroup $31 : 5$. We may assume $N$ is elementary abelian $p$-group for $p \in \{2, 3, 5, 7\}$. Now if $7 \mid |N|$, then by Lemma 2.6, $G$ has an element of order $5 \cdot 7$, a contradiction. Therefore, $\pi(N) \subseteq \{2, 3, 5\}$.

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