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Weak amenability of $(2N)-$th dual of a Banach algebra

Mina Ettefagh[a,*], and Sima Houdfar [a]

[a] *Department of Mathematics, Tabriz Branch, Islamic Azad University, Tabriz, Iran*

Abstract. In this paper by using some conditions, we show that the weak amenability of (2n)-th dual of a Banach algebra A for some $n \geqslant 1$ implies the weak amenability of A.

Keywords: Banach algebra, Arens porducts, Arens regularity, Derivation, weak amenability.

1. Introduction and Preliminaries

Let X be a normed space and $X^{'}$ be the topological dual space of X; the value of $f \in X^{'}$ at $x \in X$ is denoted by $\langle f, x\rangle$. We set $(X^{'})^{'} = X^{''}$ and so on, and we regard X as a subspace of $X^{''}$ by natural mapping $\iota : X \to X^{''}(x \longmapsto \widehat{x})$ where $\langle \widehat{x}, f\rangle = \langle f, x\rangle(f \in X^{'})$. We denot the $n-$th dual of X by $X^{(n)}$. The weak topology on X is denoted by $w = \sigma(X, X^{'})$ and weak*-topology on $X^{'}$ is dented by $w^* = \sigma(X^{'}, X)$.
New let X, Y and Z be normed spaces and let $f : X \times Y \to Z$ be a continuous bilinear map. Arens in [1] offers two extensions f^{***} and f^{t***t} of f from $X^{''} \times Y^{''}$ to $Z^{''}$ as following:

$$(1)\ \begin{cases} f^* : Z^{'} \times X \longrightarrow Y^{'} \\ \left\langle f^*(z^{'}, x), y\right\rangle = \left\langle z^{'}, f(x, y)\right\rangle \quad (x \in X, y \in Y, z^{'} \in Z^{'}). \end{cases}$$

$$(2)\ \begin{cases} f^{**} : Y^{''} \times Z^{'} \longrightarrow X^{'} \\ \left\langle f^{**}(y^{''}, z^{'}), x\right\rangle = \left\langle y^{''}, f^*(z^{'}, x)\right\rangle \quad (x \in X, z^{'} \in Z^{'}, y^{''} \in Y^{''}). \end{cases}$$

$$(3)\ \begin{cases} f^{***} : X^{''} \times Y^{''} \longrightarrow Z^{''} \\ \left\langle f^{***}(x^{''}, y^{''}), z^{'}\right\rangle = \left\langle x^{''}, f^{**}(y^{''}, z^{'})\right\rangle \quad (z^{'} \in Z^{'}, x^{''} \in X^{''}, y^{''} \in Y^{''}). \end{cases}$$

The mapping f^{***} is the unique extension of f such that $x^{''} \longmapsto f^{***}(x^{''}, y^{''})$ from $X^{''}$ into $Z^{''}$ is $w^* - w^*-$continuous for every $y^{''} \in Y^{''}$.
Let now $f^t : Y \times X \to Z$ be the transpose of f defined by $f^t(y, x) = f(x, y)$ for $x \in X$ and $y \in Y$. We can extend f^t as above to f^{t***} and then we have the mapping $f^{t***t} : X^{''} \times Y^{''} \longrightarrow Z^{''}$. If $f^{***} = f^{t***t}$ then f is called *Arens regular*. The mapping $y^{''} \longmapsto f^{t***t}(x^{''}, y^{''})$ from $Y^{''}$ into $Z^{''}$ is $w^* - w^*-$continuous for

*Corresponding author. Email: minaettefagh@gmail.com

every $x^{''} \in X^{''}$. Arens regularity of f is equivalent to the following

$$\lim_i \lim_j \left\langle z^{'}, f(x_i, y_i) \right\rangle = \lim_j \lim_i \left\langle z^{'}, f(x_i, y_i) \right\rangle,$$

whenever both limits exist for all bounded nets (x_i) and (y_j) in X and Y, respectively and for evrey $z^{'} \in Z^{'}$.
Throughout this paper A is a Banach algebra. This algebra is called *Arens regular* if its multiplication as a bilinear map $\pi : A \times A \to A(\pi(a, b) = ab)$ is Arens regular. We shall frequently use Goldstine's theorem: for each $a^{''} \in A^{''}$, there is a net (a_i) in A such that $a^{''} = w^* - \lim_i \widehat{a}_i$. Now let $a^{''} = w^* - \lim_i \widehat{a_i}$ and $b^{''} = w^* - \lim_j \widehat{b_j}$ be elements of $A^{''}$. The first and second *Arens products* on $A^{''}$ are denoted by symbols $\Box$ and $\Diamond$ respectively and defined by

$$a^{''}\Box b^{''} = \pi^{***}(a^{''}, b^{''}) \quad , \quad a^{''}\Diamond b^{''} = \pi^{t***t}(a^{''}, b^{''}).$$

It is easy to show that

$$a^{''}\Box b^{''} = w^* - \lim_i w^* - \lim_j \widehat{a_i b_j} \quad , \quad a^{''}\Diamond b^{''} = w^* - \lim_j w^* - \lim_i \widehat{a_i b_j}.$$

On the other hand we can define above Arens products in stages as following. Let $a, b \in A$, $f \in A^{'}$ and $F, G \in A^{''}$.

(1) Define $f.a$ in $A^{'}$ by $\langle f.a, b\rangle = \langle f, ab\rangle$,
and $a.f$ in $A^{'}$ by $\langle a.f, b\rangle = \langle f, ba\rangle$.
(2) Define $F.f$ in $A^{'}$ by $\langle F.f, a\rangle = \langle F, f.a\rangle$,
and $f.F$ in $A^{'}$ by $\langle f.F, a\rangle = \langle F, a.f\rangle$.
(3) Define $F\Box G$ in $A^{''}$ by $\langle F\Box G, f\rangle = \langle F, G.f\rangle$,
and $F\Diamond G$ in $A^{''}$ by $\langle F\Diamond G, f\rangle = \langle G, f.F\rangle$.

Then $(A^{''}, \Box)$ and $(A^{''}, \Diamond)$ are Banach algebras, see [1, 5] for further details.
Now let E be a Banach $A-$bimodule, then $E^{'}$ is a Banach $A-$bimodule under actions

$$\langle a.f, x\rangle = \langle f, xa\rangle, \langle f.a, x\rangle = \langle f, ax\rangle \quad (a \in A, x \in E, f \in E^{'}), \tag{1}$$

and $E^{''}$ is a Banach $A^{''}-$bimodule under actions

$$F.\Lambda = w^* - \lim_i w^* - \lim_j \widehat{a_i x_j} \quad , \quad \Lambda.F = w^* - \lim_j w^* - \lim_i \widehat{x_j a_i} \tag{2}$$

where $F = w^* - \lim_i \hat{a}_i$ and $\Lambda = w^* - \lim_j \hat{x}_j$ such that $(a_i) \subset A$ and $(x_j) \subset E$ are bounded nets.
For a Banach A-bimodule E, the continuous linear map $D : A \to E$ is called *derivation* if $D(ab) = a.D(b) + D(a).b, (a, b \in A)$. For $x \in E$ the derivation $\delta_x : A \to E$ by $\delta_x(a) = a.x - x.a$ is called *inner derivation.* The Banach algebra A is called *amenable* if every derivation $D : A \to E'$ is inner, for each Banach A-bimodule E, [7]. If every derivation $D : A \to A^{'}$ is inner, A is called *weakly amenable*, see also [2, 4] for details.

THEOREM 1.1 *Let A be a Banach algebra and E be a Banach A-bimodule and $D : A \to E$ is a continuous derivation, then $D'' : A'' \to E''$ is a continuous derivation[5, Theorem 2.7.17].*

Remark 1 A''-bimodule structures on E'' in above theorem are as in formula (2).

In [8] it was shown that if A is complete Arens regular and every derivation $D : A \rightarrow A'$ be weakly compact, then weak amenability of $A^{(2n)}$ for some $(n \geqslant 1)$ implies weak amenability of A. In this paper we always use the first Arens product $\Box$ on Banach algebra $A^{(2n)}(n \geqslant 1)$. In section 2 we shall frequently use formulas (1) and (2) and we investigate following actions

- $\triangleright$ two A''-module actions on $A^{(3)} = (A')''$ and $A^{(3)} = (A'')'$,
- $\triangleright$ two $A^{(4)}$-module actions on $A^{(5)} = ((A')'')''$ and $A^{(5)} = ((A'')'')'$,
- $\triangleright$ two $A^{(6)}-$module actions on $A^{(7)} = (((A')'')'')''$ and on $A^{(7)} = (((A'')'')'')'$,

and we will extend our results to two different $A^{(2n)}-$module actions on $A^{(2n+1)}$ by induction. In each case we find conditions to make these two different actions equal. In a similar work in [6] two different $A''-$module actions on $A^{(3)} = (A')''$ and $A^{(3)} = (A'')'$ have been studied. Finally in section 3 we investigate the innerness of second, fourth... and $(2n)-$th dual of a derivation $D : A \rightarrow A'$. By using some conditions we will show that weak amenability of $A^{(2n)}$ for some $(n \geqslant 1)$ implies weak amenability of A.

2. $A^{(2n)}-$module actions on $A^{(2n+1)}$

We shall frequentey use formulas (1) and (2) to construct two different $A^{(2n)}-$module actions on $A^{(2n+1)} = (((A')'')\cdots)''$ and $A^{(2n+1)} = (((A'')\cdots)'')'$.

Remark 1 There are many other $A^{(2n)}-$module actions on $A^{(2n+1)}$ that we don't need to mention.

First for n= 1 we consider two A''-module actions on $A^{(3)} = (A')''$ and $A^{(3)} = (A'')'$. Let $a^{(3)} = w^* - \lim_\alpha \widehat{a'_\alpha} \in A^{(3)}$ and $a'' = w^* - \lim_\beta \widehat{a_\beta}, b'' = w^* - \lim_i \widehat{b_i}$ in which (a'_α) and $(a_\beta), (b_i)$ are bounded nets in A' and A respectively. For left A''-module action on $A^{(3)} = (A')''$ as second dual of A' we can write

$$\begin{aligned} \langle a''.a^{(3)}, b''\rangle &= \lim_\beta \lim_\alpha \langle b'', a_\beta.a'_\alpha\rangle \qquad \text{(by formula (2))} \\ &= \lim_\beta \lim_\alpha \lim_i \langle a'_\alpha, b_i.a_\beta\rangle, \end{aligned} \tag{3}$$

and for left A''-module action on $A^{(3)} = (A'')'$ as dual of A'' we can write

$$\begin{aligned} \langle a''.a^{(3)}, b''\rangle &= \langle a^{(3)}, b''\Box a''\rangle \qquad \text{(by formula (1))} \\ &= \lim_\alpha \langle b''\Box a'', a'_\alpha\rangle \\ &= \lim_\alpha \lim_i \lim_\beta \langle a'_\alpha, b_i.a_\beta\rangle. \end{aligned} \tag{4}$$

This shows that two left A''-module actions on $A^{(3)} = (A'')'$ and $A^{(3)} = (A')''$ are not equal. Similarly for right A''-module action on $A^{(3)} = (A')''$ we have

$$\begin{aligned} \langle a^{(3)}.a'', b''\rangle &= \lim_\alpha \lim_\beta \langle b'', a'_\alpha.a_\beta\rangle \qquad \text{(by formula (2))} \\ &= \lim_\alpha \lim_\beta \lim_i \langle a'_\alpha, a_\beta.b_i\rangle, \end{aligned} \tag{5}$$

and for right A''-module action coincide on $A^{(3)}=(A'')'$

$$\begin{aligned}\langle a^{(3)}.a'',b''\rangle &= \langle a^{(3)},a''\Box b''\rangle \qquad \text{(by formula (1))}\\ &=\lim_{\alpha}\langle a''\Box b'',a'_{\alpha}\rangle \\ &=\lim_{\alpha}\lim_{\beta}\lim_{i}\langle a'_{\alpha},a_{\beta}.b_i\rangle.\end{aligned} \tag{6}$$

This shows that two right A''-module actions on $A^{(3)}=(A'')'$ and $A^{(3)}=(A')''$ are equal.

Proposition 2.1 *Let A be a Banach algebra with following conditions*

(*i*) *A is Arens regular,*
(*ii*) *the map $A\times A'\to A'$ $\big((a,a')\longmapsto a.a'\big)$ is Arens regular.*

Then two A''-module actions on $A^{(3)}=(A')''$ and $A^{(3)}=(A'')'$ coincide.

Proof It is enaugh to prove that left module actions in (3) and (4) coincide. We begin with equation (3)

$$\begin{aligned}\langle a''.a^{(3)},b''\rangle &= \lim_{\beta}\lim_{\alpha}\langle b'',a_{\beta}.a'_{\alpha}\rangle\\ &=\lim_{\alpha}\lim_{\beta}\langle b'',a_{\beta}.a'_{\alpha}\rangle \qquad \text{(by (ii))}\\ &=\lim_{\alpha}\lim_{\beta}\lim_{i}\langle a_{\beta}.a'_{\alpha},b_i\rangle\\ &=\lim_{\alpha}\lim_{\beta}\lim_{i}\langle a'_{\alpha},b_i.a_{\beta}\rangle \qquad \text{(by formula (1))}\\ &=\lim_{\alpha}\lim_{i}\lim_{\beta}\langle a'_{\alpha},b_i.a_{\beta}\rangle \qquad \text{(by (i))}\end{aligned}$$

this proves the equality of (3) and (4). ■

Now for $n=2$ we consider two $A^{(4)}-$module actions on $A^{(5)}=((A')'')''$ and $A^{(5)}=((A'')'')'$. Let $a^{(5)}=w^*-\lim_{\alpha}\widehat{a^{(3)}_{\alpha}}\in A^{(5)}$ and $a^{(4)}=w^*-\lim_{\beta}\widehat{a''_{\beta}}, b^{(4)}=w^*-\lim_{i}\widehat{b''_{i}}$ in $A^{(4)}$ where $(a^{(3)}_{\alpha})$ and $(a''_{\alpha}),(b''_i)$ are bounded nets in $A^{(3)}$ and A'', respectively. For left $A^{(4)}-$module action on $A^{(5)}=((A')'')''$ we have

$$\begin{aligned}\langle a^{(4)}.a^{(5)},b^{(4)}\rangle &= \lim_{\beta}\lim_{\alpha}\langle b^{(4)},a''_{\beta}.a^{(3)}_{\alpha}\rangle \qquad \text{(by formula (2))}\\ &=\lim_{\beta}\lim_{\alpha}\lim_{i}\langle a''_{\beta}.a^{(3)}_{\alpha},b''_i\rangle\end{aligned} \tag{7}$$

and for left $A^{(4)}-$module action on $A^{(5)}=((A'')'')'$ we have

$$\begin{aligned}\langle a^{(4)}.a^{(5)},b^{(4)}\rangle &= \langle a^{(5)},b^{(4)}\Box a^{(4)}\rangle \qquad \text{(by formula (1))}\\ &=\lim_{\alpha}\langle b^{(4)}\Box a^{(4)},a^{(3)}_{\alpha}\rangle\\ &=\lim_{\alpha}\lim_{i}\lim_{\beta}\langle a^{(3)}_{\alpha},b''_i\Box a''_{\beta}\rangle.\end{aligned} \tag{8}$$

For right $A^{(4)}-$module action on $A^{(5)}=((A')'')''$ we have

$$\begin{aligned}\langle a^{(5)}.a^{(4)},b^{(4)}\rangle &= \lim_{\alpha}\lim_{\beta}\langle b^{(4)},a^{(3)}_{\alpha}.a''_{\beta}\rangle \qquad \text{(by formula (2))}\\ &=\lim_{\alpha}\lim_{\beta}\lim_{i}\langle a^{(3)}_{\alpha}.a''_{\beta},b''_i\rangle,\end{aligned} \tag{9}$$

and for right $A^{(4)}$-module action on $A^{(5)} = ((A^{''})^{''})^{'}$

$$\begin{aligned}\langle a^{(5)}.a^{(4)}, b^{(4)}\rangle &= \langle a^{(5)}, a^{(4)}\Box b^{(4)}\rangle && \text{(by formula (1))}\\ &= \lim_{\alpha}\langle a^{(4)}\Box b^{(4)}, a^{(3)}_{\alpha}\rangle \\ &= \lim_{\alpha}\lim_{\beta}\lim_{i}\langle a^{(3)}_{\alpha}, a^{''}_{\beta}\Box b^{''}_{i}\rangle.\end{aligned} \tag{10}$$

We need the equality of two $A^{''}$-module actions on $A^{(3)} = (A^{''})^{'}$ and $A^{(3)} = (A^{'})^{''}$ to prove the equality of above $A^{(4)}$-module actions on $A^{(5)}$, so we need the following lemma whose proof is streightforward.

Lemma 2.2 *Let A be a Banach algebra with following conditions*

(*i*) *$A^{''}$ is Arens regular,*
(*ii*) *the map $A^{''}\times A^{'''}\to A^{'''}$ $\big((a^{''}, a^{(3)})\longmapsto a^{''}.a^{(3)}\big)$ is Arens regular.*

Then the conditions of Proposition 2.1 hold.

Proposition 2.3 *Let A be a Banach algebra with conditions*

(*i*) *$A^{''}$ is Arens regular,*
(*ii*) *the map $A^{''}\times A^{'''}\to A^{'''}$ $\big((a^{''}, a^{(3)})\longmapsto a^{''}.a^{(3)}\big)$ is Arens regular.*

Then two $A^{(4)}$-module actions on $A^{(5)} = ((A^{'})^{''})^{''}$ and $A^{(5)} = ((A^{''})^{''})^{'}$ coincide.

Proof By Lemma 2.2 the conditions of Proposition 2.1 hold, so two A''-module actions on $A^{(3)} = (A^{'})^{''}$ and $A^{(3)} = (A^{''})^{'}$ are equal. We begin with equality (7)

$$\begin{aligned}\langle a^{(4)}.a^{(5)}, b^{(4)}\rangle &= \lim_{\beta}\lim_{\alpha}\langle b^{(4)}, a^{''}_{\beta}.a^{(3)}_{\alpha}\rangle \\ &= \lim_{\alpha}\lim_{\beta}\langle b^{(4)}, a^{''}_{\beta}.a^{(3)}_{\alpha}\rangle && \text{(by (ii) of Lemma 2.2)}\\ &= \lim_{\alpha}\lim_{\beta}\lim_{i}\langle a^{''}_{\beta}.a^{(3)}_{\alpha}, b^{''}_{i}\rangle \\ &= \lim_{\alpha}\lim_{\beta}\lim_{i}\langle a^{(3)}_{\alpha}, b^{''}_{i}\Box a^{''}_{\beta}\rangle && \text{(by Proposition 2.1)}\\ &= \lim_{\alpha}\lim_{i}\lim_{\beta}\langle a^{(3)}_{\alpha}, b^{''}_{i}\Box a^{''}_{\beta}\rangle, && \text{(by (i) of Lemma 2.2)}\end{aligned}$$

this proves the equality of (7) and (8). For equality of right-module actions, we continue equality 9

$$\begin{aligned}\langle a^{(5)}.a^{(4)}, b^{(4)}\rangle &= \lim_{\alpha}\lim_{\beta}\lim_{i}\langle a^{(3)}_{\alpha}.a^{''}_{\beta}, b^{''}_{i}\rangle \\ &= \lim_{\alpha}\lim_{\beta}\lim_{i}\langle a^{(3)}_{\alpha}, a^{''}_{\beta}\Box b^{''}_{i}\rangle && \text{(by Proposition 2.1)}\end{aligned}$$

and this proves the equality of (9) and (10). ■

Now suppose that $n = 3$, we consider two $A^{(6)}-$module actions on $A^{(7)} = (((A^{'})^{''})^{''})^{''}$ and $A^{(7)} = (((A^{''})^{''})^{''})^{'}$. Let $a^{(7)} = w^{*} - \lim_{\alpha}\widehat{a^{(5)}_{\alpha}} \in A^{(7)}$ and $a^{(6)} = w^{*} - \lim_{\beta}\widehat{a^{(4)}_{\beta}}, b^{(6)} = w^{*} - \lim_{i}\widehat{b^{(4)}_{i}} \in A^{(6)}$ where $(a^{(5)}_{\alpha})$ and $(a^{(4)}_{\beta}), (b^{(4)}_{i})$ are bounded nets in $A^{(5)}$ and $A^{(4)}$, respectively. For left $A^{(6)}-$module action on $A^{(7)} = (((A^{'})^{''})^{''})^{''}$ we can write

$$\langle a^{(6)}.a^{(7)}, b^{(6)}\rangle = \lim_{\beta}\lim_{\alpha}\lim_{i}\langle a^{(4)}_{\beta}.a^{(5)}_{\alpha}, b^{(4)}_{i}\rangle, \tag{11}$$

and for left $A^{(6)}-$module action on $A^{(7)} = (((A'')'')'')'$ we can write

$$\langle a^{(6)}.a^{(7)}, b^{(6)}\rangle = \lim_{\alpha}\lim_{i}\lim_{\beta}\langle a^{(5)}_{\alpha}, b^{(4)}_{i}\Box a^{(4)}_{\beta}\rangle. \tag{12}$$

For right $A^{(6)}-$module action on $A^{(7)} = (((A')'')'')''$ we can write

$$\langle a^{(7)}.a^{(6)}, b^{(6)}\rangle = \lim_{\alpha}\lim_{\beta}\lim_{i}\langle a^{(5)}_{\alpha}.a^{(4)}_{\beta}, b^{(4)}_{i}\rangle, \tag{13}$$

and for right $A^{(6)}-$module action on $A^{(7)} = (((A'')'')'')'$ we can write

$$\langle a^{(7)}.a^{(6)}, b^{(6)}\rangle = \lim_{\alpha}\lim_{\beta}\lim_{i}\langle a^{(5)}_{\alpha}, a^{(4)}_{\beta}\Box b^{(4)}_{i}\rangle. \tag{14}$$

We need the equality of two A''-module actions on $A^{(5)} = ((A'')'')'$ and $A^{(5)} = ((A')'')''$ to prove the equality of above $A^{(6)}$-module actions on $A^{(7)}$, so we need the following Lemma that is similar to Lemma 2.2.

Lemma 2.4 *Let A be a Banach algebra with following conditions*

(i) $A^{(4)}$ is Arens regular,
(ii) the map $A^{(4)} \times A^{(5)} \to A^{(5)}$ $\big((a^{(4)}, a^{(5)}) \longmapsto a^{(4)}.a^{(5)}\big)$ is Arens regular.

Then the conditions of Proposition 2.3 hold.

Proposition 2.5 *Let A be a Banach algebra with conditions*

(i) $A^{(4)}$ is Arens regular,
(ii) the map $A^{(4)} \times A^{(5)} \to A^{(5)}$ $\big((a^{(4)}, a^{(5)}) \longmapsto a^{(4)}.a^{(5)}\big)$ is Arens regular.

Then two $A^{(6)}-$module actions on $A^{(7)} = (((A'')'')'')'$ and $A^{(7)} = (((A')'')'')''$ coincide.

Proof By Lemma 2.4 the conditions of Proposition 2.3 hold, so two $A^{(4)}$-module actions on $A^{(5)} = ((A')'')''$ and $A^{(5)} = ((A'')'')'$ are equal. We begin with equality (11)

$$\begin{aligned}
\langle a^{(6)}.a^{(7)}, b^{(6)}\rangle &= \lim_{\beta}\lim_{\alpha}\langle b^{(6)}, a^{(4)}_{\beta}.a^{(5)}_{\alpha}\rangle \\
&= \lim_{\alpha}\lim_{\beta}\langle b^{(6)}, a^{(4)}_{\beta}.a^{(5)}_{\alpha}\rangle && \text{(by (ii) of Lemma 2.4)} \\
&= \lim_{\alpha}\lim_{\beta}\lim_{i}\langle a^{(4)}_{\beta}.a^{(5)}_{\alpha}, b^{(4)}_{i}\rangle \\
&= \lim_{\alpha}\lim_{\beta}\lim_{i}\langle a^{(5)}_{\alpha}, b^{(4)}_{i}\Box a^{(4)}_{\beta}\rangle && \text{(by Proposition 2.3)} \\
&= \lim_{\alpha}\lim_{i}\lim_{\beta}\langle a^{(5)}_{\alpha}, b^{(4)}_{i}\Box a^{(4)}_{\beta}\rangle, && \text{(by (i) of Lemma 2.4)}
\end{aligned}$$

this prove the equality of (11) and (12). For equality of right-module actions, we continue equality (13)

$$\begin{aligned}
\langle a^{(7)}.a^{(6)}, b^{(6)}\rangle &= \lim_{\alpha}\lim_{\beta}\lim_{i}\langle a^{(5)}_{\alpha}.a^{(4)}_{\beta}, b^{(4)}_{i}\rangle \\
&= \lim_{\alpha}\lim_{\beta}\lim_{i}\langle a^{(5)}_{\alpha}, a^{(4)}_{\beta}\Box b^{(4)}_{i}\rangle && \text{(by Proposition 2.3)}
\end{aligned}$$

and this proves the equality of (13) and (14). ■

Now by induction process we have the following extended result.

Proposition 2.6 *Let A be a Banach algebra with following conditions for some $n \geqslant 1$*

(*i*) A^{2n-2} *is Arens regular,*
(*ii*) *the map* $A^{(2n-2)} \times A^{(2n-1)} \to A^{(2n-1)}$ $((a,f) \longmapsto a.f)$ *is Arens regular.*

Then two $A^{(2n)}$*−module actions on* $A^{(2n+1)} = ((((A^{''})^{''})\cdots)^{''})^{'}$ *and* $A^{(2n+1)} = ((((A^{'})^{''})\cdots)^{''})^{''}$ *coincide.*

3. Main results

In this section we consider the transposes $D^{''}, D^{(4)}, \cdots, D^{(2n)}$ of a continuous derivation $D : A \to A^{'}$. We know by Theorem 1.1 that the following maps will be continuous derivations

$$\begin{aligned}
D^{''} &: A^{''} \longrightarrow A^{(3)} = (A^{'})^{''}\\
D^{(4)} &: A^{(4)} = (A^{''})^{''} \longrightarrow A^{(5)} = ((A^{'})^{''})^{''}\\
D^{(6)} &: A^{(6)} = ((A^{''})^{''})^{''} \longrightarrow A^{(7)} = (((A^{'})^{''})^{''})^{''}\\
&\vdots\\
D^{(2n)} &: A^{(2n)} = (((A^{''})^{''})\cdots)^{''} \longrightarrow A^{(2n+1)} = ((((A^{'})^{''})^{''})\cdots)^{''}.
\end{aligned}$$

Proposition 3.1 *Let A be a Banach algebra with hypothesis of Proposition 2.1. If the second transpose $D^{''}$ of continuous derivation $D : A \to A^{'}$ is inner, then D is inner.*

Proof Let $D : A \longrightarrow A^{'}$ be a dervation, then by Theorem 1.1 and Proposition 2.1, $D^{''} : A^{''} \longrightarrow A^{(3)} = (A^{'})^{''} = (A^{''})^{'}$ is also a derivation. Since $D^{''}$ is inner, there exists $a^{''} \in A^{''}$ such that $D^{''}(a^{''}) = a^{''}.a^{(3)} - a^{(3)}.a^{''}, (a^{(3)} \in A^{(3)})$. Let $a^{'} = \iota^*(a^{(3)})$, where $\iota : A \longrightarrow A^{''}$ is the natural map. Then for each $a, b \in A$ we can write

$$\begin{aligned}
\langle D(a), b\rangle &= \langle D^{''}(\widehat{a}), \widehat{b}\rangle\\
&= \langle \widehat{a}.a^{(3)} - a^{(3)}.\widehat{a}, \widehat{b}\rangle\\
&= \langle a^{(3)}, \widehat{b}\square\widehat{a} - \widehat{a}\square\widehat{b}\rangle \quad \text{(by Proposition 2.1)}\\
&= \langle a^{(3)}, \widehat{b.a - a.b}\rangle\\
&= \langle a^{(3)}, \iota(b.a - a.b)\rangle\\
&= \langle \iota^*(a^{(3)}), b.a - a.b\rangle\\
&= \langle a^{'}, b.a - a.b\rangle\\
&= \langle a.a^{'} - a^{'}.a, b\rangle,
\end{aligned}$$

hence $D(a) = a.a^{'} - a^{'}.a$ and so D is inner. ∎

Proposition 3.2 *Let A be a Banach algebra with hypothesis of Proposition 2.3. If the fourth transpose $D^{(4)}$ of continuous derivation $D : A \to A^{'}$ is inner, then D is inner.*

Proof Let $D : A \longrightarrow A^{'}$ be a dervation, then by Theorem 1.1 and Proposition 2.3, $D^{(4)} : ((A^{''})^{''}) \longrightarrow (((A^{'})^{''})^{''}) = (((A^{''})^{''})^{'})$ is also a derivation. Since $D^{(4)}$ is inner, there exists $a^{(4)} \in A^{(4)}$ such that $D^{(4)}(a^{(4)}) = a^{(4)}.a^{(5)} - a^{(5)}.a^{(4)}, (a^{(5)} \in A^{(5)})$. Let $a^{'} = \iota^* \circ \iota^{***}(a^{(5)})$, where $\iota : A \longrightarrow A^{''}$ is the natural map. Then for each $a, b \in A$ we can write

$$\begin{aligned}
\langle D(a), b\rangle &= \langle D''(\widehat{a}), \widehat{b}\rangle \\
&= \langle \widehat{D''(\widehat{a})}, \widehat{\widehat{b}}\rangle \\
&= \langle D^{(4)}(\widehat{\widehat{a}}), \widehat{\widehat{b}}\rangle \\
&= \langle \widehat{\widehat{a}}.a^{(5)} - a^{(5)}.\widehat{\widehat{a}}, \widehat{\widehat{b}}\rangle \\
&= \langle a^{(5)}, \widehat{\widehat{b}}\Box\widehat{\widehat{a}} - \widehat{\widehat{a}}\Box\widehat{\widehat{b}}\rangle \qquad \text{(by Proposition 2.3)} \\
&= \langle a^{(5)}, \widehat{\widehat{b}\Box\widehat{a} - \widehat{a}\Box\widehat{b}}\rangle \\
&= \langle a^{(5)}, \iota^{**}(\widehat{b}\Box\widehat{a} - \widehat{a}\Box\widehat{b})\rangle \\
&= \langle \iota^{***}(a^{(5)}), \widehat{b.a - a.b}\rangle \\
&= \langle \iota^{***}(a^{(5)}), \iota(b.a - a.b)\rangle \\
&= \langle \iota^{*} \circ \iota^{***}(a^{(5)}), b.a - a.b\rangle \\
&= \langle a', b.a - a.b\rangle \\
&= \langle a.a' - a'.a, b\rangle,
\end{aligned}$$

hence $D(a) = a.a' - a'.a$ and so D is inner. ■

Proposition 3.3 *Let A be a Banach algebra with hypothesis of Proposition 2.5. If the sixth transpose $D^{(6)}$ of continuous derivation $D : A \rightarrow A'$ is inner, then D is inner.*

Proof Let $D : A \longrightarrow A'$ be a dervation, then by Theorem 1.1 and Proposition 2.3, $D^{(6)} : A^{(6)} = ((A'')'')'' \longrightarrow (((A')'')'')'' = (((A'')''))'')' = A^{(7)}$ is also a derivation. Since $D^{(6)}$ is inner, there exists $a^{(6)} \in A^{(6)}$ such that $D^{(6)}(a^{(6)}) = a^{(6)}.a^{(7)} - a^{(7)}.a^{(6)}, (a^{(7)} \in A^{(7)})$. Let $a' = \iota^{*} \circ \iota^{***} \circ \iota^{*****}(a^{(7)})$, where $\iota : A \longrightarrow A''$ is the natural map. Then for each $a, b \in A$ we can write

$$\begin{aligned}
\langle D(a), b\rangle &= \langle D''(\widehat{a}), \widehat{b}\rangle \\
&= \langle \widehat{D''(\widehat{a})}, \widehat{\widehat{b}}\rangle \\
&= \langle \widehat{D^{(4)}(\widehat{\widehat{a}})}, \widehat{\widehat{\widehat{b}}}\rangle \\
&= \langle D^{(6)}(\widehat{\widehat{\widehat{a}}}), \widehat{\widehat{\widehat{b}}}\rangle \\
&= \langle \widehat{\widehat{\widehat{a}}}.a^{(7)} - a^{(7)}.\widehat{\widehat{\widehat{a}}}, \widehat{\widehat{\widehat{b}}}\rangle \\
&= \langle a^{(7)}, \widehat{\widehat{\widehat{b}}}\Box\widehat{\widehat{\widehat{a}}} - \widehat{\widehat{\widehat{a}}}\Box\widehat{\widehat{\widehat{b}}}\rangle \qquad \text{(by Proposition 2.5)} \\
&= \langle a^{(7)}, \widehat{\widehat{\widehat{b}}\Box\widehat{\widehat{a}} - \widehat{\widehat{a}}\Box\widehat{\widehat{b}}}\rangle \\
&= \langle a^{(7)}, \iota^{****}(\widehat{\widehat{b}}\Box\widehat{\widehat{a}} - \widehat{\widehat{a}}\Box\widehat{\widehat{b}})\rangle \\
&= \langle \iota^{*****}(a^{(7)}), \iota^{**}(\widehat{b}\Box\widehat{a} - \widehat{a}\Box\widehat{b})\rangle \\
&= \langle \iota^{***} \circ \iota^{*****}(a^{(7)}), \widehat{b.a - a.b}\rangle \\
&= \langle \iota^{***} \circ \iota^{*****}(a^{(7)}), \iota(b.a - a.b)\rangle \\
&= \langle \iota^{*} \circ \iota^{***} \circ \iota^{*****}(a^{(7)}), b.a - a.b\rangle \\
&= \langle a', b.a - a.b\rangle \\
&= \langle a.a' - a'.a, b\rangle,
\end{aligned}$$

hence $D(a) = a.a' - a'.a$ and so D is inner. ■

Using the similar reasoning as in the proof of previous lemmas we have the following proposition.

Proposition 3.4 *Let A be a Banach algebra with hypothesis of Proposition 2.6. If the $(2n)-th$ transpose $D^{(2n)}$ of continuous derivation $D : A \rightarrow A'$ is inner, then D is inner.*

PROPOSITION 3.5 *Let A be a Banach algebra with hypothesis of Proposition 2.1. If A'' is weakly amenable, then A is weakly amenable.*

Proof Suppose that $D : A \to A'$ be a continuous derivation. Then $D'' : A'' \longrightarrow A^{(3)} = (A')''$ is a continuous derivation by Theorem 1.1. But two $A''-$module actions on $A^{(3)} = (A')''$ and $A^{(3)} = (A'')'$ are equal by Proposition 2.1, hence $D'' : A'' \longrightarrow A^{(3)} = (A'')'$ is also a continuous derivation in which $A^{(3)} = (A'')'$ is considered as dual of A''. Since A'' is weakly amenable, then D'' is inner. Therefore D is inner by Proposition 3.1. This completes the proof. ■

Using the same reasoning as in the proofs of previous propositions we have next results, so we omit the details in proofs.

PROPOSITION 3.6 *Let A be a Banach algebra with hypothesis of Proposition 2.3. If $A^{(4)}$ is weakly amenable, then A is weakly amenable.*

Proof This is a consequence of Proposition 3.2. ■

PROPOSITION 3.7 *Let A be a Banach algebra with hypothesis of Proposition 2.5. If $A^{(6)}$ is weakly amenable, then A is weakly amenable.*

Proof This is a consequence of Proposition 3.3. ■

Finally by Propositions 2.6 and 3.4 we have the following extended result.

PROPOSITION 3.8 *Let A be a Banach algebra with hypothesis of Proposition 2.6. If $A^{(2n)}$ is weakly amenable, then A is weakly amenable.*

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References

[1] R. ARENS, The adjoint of a bilinear operation, Proc. Amer. Math. Soc. 2 (1951), 839-848.
[2] W.G. BADE, P.C. CURTIS AND H.G. DALES,, Amenability and weak amenability for Bearling and Lipschitz algebra, Proc. London Math. Soc. , 55 (1987), no. 3, 359-377.
[3] A. BODAGHI, M.ETTEFAGH, M.E. GORDJI AND A. MEDGHALCHI, Module structures on iterated duals of Banach algebras, An.st.Univ.Ovidius Constanta, 18(1) (2010) 63-80.
[4] H.G. DALES, F. GHAHRAMANI, AND N. GRONBAEK, Derivations into iterated duals of Banach algebras, Studia Math, 128 (1998), no.1, 19-54.
[5] H.G. DALES, Banach algebra and Automatic continuity, Oxford university Press, (2000).
[6] M. ETTEFAGH, The third dual of a Banach algebra, Studia. Sci. Math. Hung, 45(1) (2008) 1-11.
[7] B.E. JOHNSON, Cohomology in Banach algebras, Mem. Amer. Math. Soc, 127 (1972).
[8] A. Medghalchi and T.Yazdanpanah, Problems concerning n-weak amenability of a Banach algebra, Czecholovak Math. J, 55(130) (2005) 863-876.