Journal of
Linear and Topological Algebra
Vol. 01, No. 01, Summer 2012, 41- 44

Commutativity degree of $\mathbb{Z}_p \wr \mathbb{Z}_{p^n}$

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Abstract. For a finite group G the commutativity degree denote by $d(G)$ and defind:

$$d(G) = \frac{|\{(x,y)|x,y \in G, xy = yx\}|}{|G|^2}.$$

In [2] authors found commutativity degree for some groups,in this paper we find commutativity degree for a class of groups that have high nilpontencies.

Keywords: Presentation of groups,Finite groups,commutativity degree.

1. Introduction

For a finite group G the commutativity degree

$$d(G) = \frac{|\{(x,y)|x,y \in G, xy = yx\}|}{|G|^2}.$$

is defined and studied by several authors (see for example [2, 3, 7]).
When $d(G) \geqslant \frac{1}{2}$, it is proved by P.Lescot in 1995 that G is abelain ,or $\frac{G}{Z(G)}$ is elementary abelian with $|\acute{G}| = 2$,or G is isoclinic with S_3 and $d(G) = 1$.

Throughout this paper n is positive integer and p is odd prime number.
We consider the wreath product $G_n = \mathbb{Z}_p \wr \mathbb{Z}_{p^n}$ where ,the standard wreath product $G \wr H$ of the finite groups G and H is defined to be semidirect product of G by direct product B of $|G|$ copies of H.

In [1] it is proved that G_n has efficient presentation as follows:

$$G_n = \langle x, y | y^p = x^{p^n} = 1 \ \ , \ \ [x, x^{y^i}] = 1 \ \ , \ \ 1 \leqslant i \leqslant \frac{p-1}{2} \rangle \ .$$

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Main theorems in this paper are:

THEOREM 1.1

$$d(G_n) = \frac{p^{(p-1)n} + (p^2 - 1)}{p^{(p-1)n+2}}.$$

THEOREM 1.2

$$\lim_{n\to\infty} d(G_n) = \frac{1}{p^2}.$$

THEOREM 1.3

$$\frac{1}{p^2} < d(G_n) < \frac{1}{p}.$$

2. Proofs

We need some lemmas for proving Theorems 1.1, 1.2 and 1.3.

LEMMA 2.1 *In group G_n every element z has an unique presentations as follows:*

$$z = y^{\alpha}(x)^{\beta_0}(x^y)^{\beta_1}(x^{y^2})^{\beta_2}...(x^{y^{p-1}})^{\beta_{p-1}}$$

where $\alpha \in \{0, 1, 2, ..., p-1\}$ and $\beta_i \in \{0, 1, 2, ..., p^n - 1\}$ $\quad (0 \leqslant i \leqslant p-1)$.

Proof By presentation of G_n,it is clearly. ■

LEMMA 2.2 *Let z_1,$z_2 \in G_n$ and $z_1 = y^{\alpha_1}(x)^{\beta_0}(x^y)^{\beta_1}(x^{y^2})^{\beta_2}...(x^{y^{p-1}})^{\beta_{p-1}}$ and $z_2 = y^{\alpha_2}(x)^{\gamma_0}(x^y)^{\gamma_1}(x^{y^2})^{\gamma_2}...(x^{y^{p-1}})^{\gamma_{p-1}}$.Then $z_1z_2 = z_2z_1$ if and only if:*

$$\beta_i + \gamma_{\alpha_2+i} \equiv \beta_{\alpha_2+i} + \gamma_{\alpha_2-\alpha_1+i} \quad (mod \ \ p^n) \quad ,(i = 0, 1, 2, ..., p-1)$$

where indices are reduced module of p.

Proof We have:
$z_2z_1 =$

$$y^{\alpha_1+\alpha_2}(x^{y^{\alpha_1}})^{\gamma_0}(x^{y^{\alpha_1+1}})^{\gamma_1}...(x^{y^{\alpha_1+p-1}})^{\gamma_{p-1}}(x)^{\beta_0}(x^y)^{\beta_1}(x^{y^2})^{\beta_2}...(x^{y^{p-1}})^{\beta_{p-1}}$$

and
$z_1z_2 =$

$$y^{\alpha_1+\alpha_2}(x^{y^{\alpha_2}})^{\beta_0}(x^{y^{\alpha_2+1}})^{\beta_1}...(x^{y^{\alpha_2+p-1}})^{\beta_{p-1}}(x)^{\gamma_0}(x^y)^{\gamma_1}(x^{y^2})^{\gamma_2}...(x^{y^{p-1}})^{\gamma_{p-1}}.$$

By lemma 2.1 every element in G_n has unique presentation ,so we have:

$$\begin{cases} \beta_0 + \gamma_{\alpha_2} & \equiv & \beta_{\alpha_2} + \gamma_{\alpha_2 - \alpha_1} \quad (mod \ \ p^n) \\ \beta_1 + \gamma_{\alpha_2+1} & \equiv & \beta_{\alpha_2+1} + \gamma_{\alpha_2 - \alpha_1 + 1} \quad (mod \ \ p^n) \\ \vdots & \vdots & \\ \beta_{p-1} + \gamma_{\alpha_2+p-1} & \equiv & \beta_{\alpha_2+p-1} + \gamma_{\alpha_2 - \alpha_1 + p-1} \quad (mod \ \ p^n). \end{cases}$$

Then we have:

$$\beta_i + \gamma_{\alpha_2+i} \quad \equiv \quad \beta_{\alpha_2+i} + \gamma_{\alpha_2-\alpha_1+i} \quad (mod \ \ p^n) \quad , (i = 0, 1, 2, ..., p-1).$$

■

Remark: On set $G_n \times G_n$,we consider:

$$\zeta(G_n) = \{(z_1, z_2) | z_1, z_2 \in G_n, z_1 z_2 = z_2 z_1\}.$$

LEMMA 2.3

$$|\zeta(G_n)| = p^{(p+1)n}(p^{(p-1)n} + p^2 - 1).$$

Proof Let $z \in G_n$ and $z = y^{\alpha}(x)^{\beta_0}(x^y)^{\beta_1}(x^{y^2})^{\beta_2}...(x^{y^{p-1}})^{\beta_{p-1}}$.
We consider $\psi(z) = \alpha$. Now let

$$\zeta_{\alpha_1,\alpha_2}(G_n) = \{(z_1, z_2) | z_1, z_2 \in G_n, z_1 z_2 = z_2 z_1, \psi(z_1) = \alpha_1, \psi(z_2) = \alpha_2\}.$$

So we have:

$$\bigcup_{\alpha_1=0}^{p-1} \bigcup_{\alpha_2=0}^{p-1} \zeta_{\alpha_1,\alpha_2}(G_n) = \zeta(G_n).$$

More over:

$$|\zeta(G_n)| = \sum_{\alpha_1=0}^{p-1} \sum_{\alpha_2=0}^{p-1} |\zeta_{\alpha_1,\alpha_2}(G_n)|.$$

Now we have two cases.
Case I: $\alpha_1 = 0$,$\alpha_2 = 0$
let $z_1 = x^{\beta_0}(x^y)^{\beta_1}(x^{y^2})^{\beta_2}...(x^{y^{p-1}})^{\beta_{p-1}}$ and $z_2 = x^{\gamma_0}(x^y)^{\gamma_1}(x^{y^2})^{\gamma_2}...(x^{y^{p-1}})^{\gamma_{p-1}}$ where β_i,$\gamma_j \in \{0, 1, ..., p^n - 1\}$ and $0 \leqslant i, j \leqslant p - 1$.
Since $z_1 z_2 = z_2 z_1$ then:

$$|\zeta_{0,0}(G_n)| = \underbrace{p^n \times p^n \times \cdots \times p^n}_{2p} = p^{2pn}.$$

Case II: $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$,
let $z_1 = y^{\alpha_1}(x)^{\beta_0}(x^y)^{\beta_1}(x^{y^2})^{\beta_2}...(x^{y^{p-1}})^{\beta_{p-1}}$ and $z_2 = y^{\alpha_2}(x)^{\gamma_0}(x^y)^{\gamma_1}(x^{y^2})^{\gamma_2}...(x^{y^{p-1}})^{\gamma_{p-1}}$.If $z_1 z_2 = z_2 z_1$ by lemma 2.2 we have:

$$\beta_i + \gamma_{\alpha_2+i} \quad \equiv \quad \beta_{\alpha_2+i} + \gamma_{\alpha_2-\alpha_1+i} \quad (mod \ \ p^n) \quad , (i = 0, 1, 2, ..., p-1) \qquad (*)$$

where indices are reduced module of p.
Now we can choose $\beta_0, \beta_1, ..., \beta_{p-1}, \gamma_0$ and find$\gamma_1, \gamma_2, ..., \gamma_{p-1}$ uniquely by $(*)$, then

$$|\zeta_{\alpha_1,\alpha_2}(G_n)| = \underbrace{p^n \times p^n \times ... \times p^n}_{p+1} = p^{n(p+1)}.$$

Finally we have

$$|\zeta(G_n)| = \sum_{\alpha_1=0}^{p-1} \sum_{\alpha_2=0}^{p-1} |\zeta_{\alpha_1,\alpha_2}(G_n)| = p^{2np} + (p^2-1)p^{n(p+1)} = p^{(p+1)n}(p^{(p-1)n} + p^2 - 1).$$

■

Proof theorems 1.1,1.2 and 1.3:
For 1.1 since $d(G_n) = \frac{|\zeta(G_n)|}{|G_n|^2}$ so by lemma 2.3 we find $d(G_n) = \frac{p^{(p-1)n} + (p^2-1)}{p^{(p-1)n+2}}$.
For 1.2 and 1.3 we have $d(G_n) = \frac{1}{p^2} + \frac{p^2-1}{p^{(p-1)n+2}}$, so

$$\lim_{n\to\infty} d(G_n) = \frac{1}{p^2}$$

and $d(G_n) > \frac{1}{p^2}$. $d(G_n) < \frac{1}{p}$ is simple . □

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