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# New fixed and periodic point results on cone metric spaces

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**Abstract.** In this paper, several fixed point theorems for T-contraction of two maps on cone metric spaces under normality condition are proved. Obtained results extend and generalize well-known comparable results in the literature.

**Keywords:** Cone metric space; Fixed point; Property P; Property Q; Normal cone.

## 1. Introduction

In 1922, Banach proved his famous fixed point theorem [3]. Afterward, other people consider some various definitions of contractive mappings and proved several fixed point theorems in [4, 7, 10, 11, 13, 15] and the references contained therein. In 2007, Huang and Zhang [8] introduced cone metric space and proved some fixed point theorems. Afterward, several fixed and common fixed point results on cone metric spaces proved in [1, 14, 16, 17] and the references contained therein.

Recently, Morales and Rajes [12] introduced $T$-Kannan and $T$-Chatterjea contractive mappings in cone metric space and proved some fixed point theorems. Then, Filipović et al. [5] defined $T$-Hardy-Rogers contraction in cone metric space and proved some fixed and periodic point theorems. In this work, we prove several fixed and periodic point theorems for $T$-contraction of two maps on normal cone metric spaces. Our results extend various comparable results of Filipović et al. [5], and Morales and Rajes [12].

## 2. preliminaries

Let us start by defining some important definitions.

Definition 2.1 (*See [6, 8]*). *Let $E$ be a real Banach space and $P$ a subset of $E$. Then $P$ is called a cone if and only if*
(*a*) *$P$ is closed, non-empty and $P \neq \{0\}$;*
(*b*) *$a, b \in R, a, b \geqslant 0, x, y \in P$ imply that $ax + by \in P$;*
(*c*) *if $x \in P$ and $-x \in P$, then $x = 0$.*

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Given a cone $P \subset E$, we define a partial ordering $\leqslant$ with respect to $P$ by $x \leqslant y \Longleftrightarrow y - x \in P$.
We shall write $x < y$ if $x \leqslant y$ and $x \neq y$. Also, we write $x \ll y$ if and only if $y - x \in intP$ (where $intP$ is interior of $P$). If $intP \neq \emptyset$, the cone $P$ is called solid. The cone $P$ is named normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \leqslant x \leqslant y \Longrightarrow \|x\| \leqslant K\|y\|. \tag{1}$$

The least positive number satisfying the above is called the normal constant of $P$.

*Example 2.2* (See [14]). Let $E = C_{\mathbf{R}}[0,1]$ with the supremum norm and $P = \{f \in E : f \geqslant 0\}$. Then $P$ is a normal cone with normal constant $K = 1$.

DEFINITION 2.3 (*See [8]*). *Let $X$ be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies*
*$(d1)$ $0 \leqslant d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;*
*$(d2)$ $d(x,y) = d(y,x)$ for all $x, y \in X$;*
*$(d3)$ $d(x,z) \leqslant d(x,y) + d(y,z)$ for all $x, y, z \in X$.*
*Then, $d$ is called a cone metric on $X$ and $(X,d)$ is called a cone metric space.*

*Example 2.4* (See [8]). Let $E = R^2$, $P = \{(x,y) \in E | x, y \geqslant 0\} \subset R^2$, $X = R$ and $d : X \times X \to E$ such that $d(x,y) = (|x-y|, \alpha|x-y|)$, where $\alpha \geqslant 0$ is a constant. Then $(X,d)$ is a cone metric space.

DEFINITION 2.5 (*See [5]*). *Let $(X,d)$ be a cone metric space, $\{x_n\}$ a sequence in $X$ and $x \in X$. Then*
*$(i)$ $\{x_n\}$ converges to $x$ if for every $c \in E$ with $0 \ll c$ there exist $n_0 \in \mathbf{N}$ such that $d(x_n, x) \ll c$ for all $n > n_0$.*
*$(ii)$ $\{x_n\}$ is called a Cauchy sequence if for every $c \in E$ with $0 \ll c$ there exist $n_0 \in \mathbf{N}$ such that $d(x_n, x_m) \ll c$ for all $m, n > n_0$.*

Also, a cone metric space $X$ is said to be complete if every Cauchy sequence in $X$ is convergent in $X$. In the sequel, we always suppose that $E$ is a real Banach space, $P$ is a normal cone in $E$, and $\leqslant$ is partial ordering with respect to $P$.

DEFINITION 2.6 (*See [5]*). *Let $(X,d)$ be a cone metric space, $P$ a solid cone and $S : X \to X$. Then*
*$(i)$ $S$ is said to be sequentially convergent if we have for every sequence $(x_n)$, if $S(x_n)$ is convergent, then $(x_n)$ also is convergent.*
*$(ii)$ $S$ is said to be subsequentially convergent if we have for every sequence $(x_n)$ that $S(x_n)$ is convergent, implies $(x_n)$ has a convergent subsequence.*
*$(iii)$ $S$ is said to be continuous, if $\lim_{n\to\infty} x_n = x$ implies that $\lim_{n\to\infty} S(x_n) = S(x)$, for all $(x_n)$ in $X$.*

DEFINITION 2.7 (*See [5]*). *Let $(X,d)$ be a cone metric space and $T, f : X \to X$ two mappings. A mapping $f$ is said to be a $T$-Hardy-Rogers contraction, if there exist $\alpha_i \geqslant 0$, $i = 1, \cdots, 5$ with $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 1$ such that for all $x, y \in X$,*

$$\begin{aligned} d(Tfx, Tfy) \leqslant\ & \alpha_1 d(Tx, Ty) + \alpha_2 d(Tx, Tfx) + \alpha_3 d(Ty, Tfy) + \alpha_4 d(Tx, Tfy) \\ & + \alpha_5 d(Ty, Tfx). \end{aligned} \tag{2}$$

*In previous definition, suppose that $\alpha_1 = \alpha_4 = \alpha_5 = 0$ and $\alpha_2 = \alpha_3 \neq 0$ (resp. $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and $\alpha_4 = \alpha_5 \neq 0$). Then we obtain $T$-Kannan (resp. $T$-Chatterjea) contraction from [12].*

## 3. Fixed point results

Theorem 3.1 *Suppose that $(X, d)$ be a complete cone metric space, $P$ be a normal cone with normal constant $K$, and $T : X \to X$ be a continuous and one to one mapping. Moreover, let $f$ and $g$ be two maps of $X$ satisfying*

$$d(Tfx, Tgy) \leqslant \alpha_1 d(Tx, Ty) + \alpha_2[d(Tx, Tfx) + d(Ty, Tgy)] \\ +\alpha_3[d(Tx, Tgy) + d(Ty, Tfx)], \tag{3}$$

*for all $x, y \in X$, where*

$$\alpha_i \geqslant 0 \quad for \quad i = 1, 2, 3 \qquad and \qquad \alpha_1 + 2\alpha_2 + 2\alpha_3 < 1. \tag{4}$$

*That is, $f$ and $g$ be a $T$-contraction. Then*
(1) *There exist $u_x \in X$ such that $\lim_{n\to\infty} Tfx_{2n} = \lim_{n\to\infty} Tgx_{2n+1} = u_x$.*
(2) *If $T$ is subsequentially convergent, then $\{fx_{2n}\}$ and $\{gx_{2n+1}\}$ have a convergent subsequence.*
(3) *There exist a unique $v_x \in X$ such that $fv_x = gv_x = v_x$, that is, $f$ and $g$ have a unique common fixed point.*
(4) *If $T$ is sequentially convergent, then iterate sequences $\{fx_{2n}\}$ and $\{gx_{2n+1}\}$ converge to $v_x$.*

*Proof* Suppose $x_0$ is an arbitrary point of $X$, and define $\{x_n\}$ by
$x_1 = fx_0$ , $x_2 = gx_1$ , $\cdots$ , $x_{2n+1} = fx_{2n}$ , $x_{2n+2} = gx_{2n+1}$ $for$ $n = 0, 1, 2, ....$
First, we prove that $\{Tx_n\}$ is a Cauchy sequence.

$$\begin{aligned} d(Tx_{2n+1}, Tx_{2n+2}) &= d(Tfx_{2n}, Tgx_{2n+1}) \\ &\leqslant \alpha_1 d(Tx_{2n}, Tx_{2n+1}) \\ &\quad +\alpha_2[d(Tx_{2n}, Tfx_{2n}) + d(Tx_{2n+1}, Tgx_{2n+1})] \\ &\quad +\alpha_3[d(Tx_{2n}, Tgx_{2n+1}) + d(Tx_{2n+1}, Tfx_{2n})] \\ &= \alpha_1 d(Tx_{2n}, Tx_{2n+1}) \\ &\quad +\alpha_2[d(Tx_{2n}, Tx_{2n+1}) + d(Tx_{2n+1}, Tx_{2n+2})] \\ &\quad +\alpha_3[d(Tx_{2n}, Tx_{2n+2}) + d(Tx_{2n+1}, Tx_{2n+1})] \\ &\leqslant (\alpha_1 + \alpha_2 + \alpha_3) d(Tx_{2n}, Tx_{2n+1}) \\ &\quad +(\alpha_2 + \alpha_3) d(Tx_{2n+1}, Tx_{2n+2}), \end{aligned}$$

which implies that

$$d(Tx_{2n+1}, Tx_{2n+2}) \leqslant \gamma d(Tx_{2n}, Tx_{2n+1}),$$

where $\gamma = \frac{\alpha_1+\alpha_2+\alpha_3}{1-\alpha_2-\alpha_3} < 1$.
Similarly, we get

$$d(Tx_{2n+3}, Tx_{2n+2}) \leqslant \gamma d(Tx_{2n+2}, Tx_{2n+1}),$$

where $\gamma = \frac{\alpha_1+\alpha_2+\alpha_3}{1-\alpha_2-\alpha_3} < 1$.

Thus, for all $n$

$$d(Tx_n, Tx_{n+1}) \leqslant \gamma d(Tx_{n-1}, Tx_n) \leqslant \gamma^2 d(Tx_{n-2}, Tx_{n-1}) \\ \leqslant \cdots \leqslant \gamma^n d(Tx_0, Tx_1). \tag{5}$$

Now, for any $m > n$

$$\begin{aligned} d(Tx_n, Tx_m) &\leqslant d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_{n+2}) + \cdots + d(Tx_{m-1}, Tx_m) \\ &\leqslant (\gamma^n + \gamma^{n+1} + \cdots + \gamma^{m-1}) d(Tx_0, Tx_1) \\ &\leqslant \frac{\gamma^n}{1-\gamma} d(Tx_0, Tx_1). \end{aligned}$$

From (1), we have

$$\|d(Tx_n, Tx_m)\| \leqslant K \frac{\gamma^n}{1-\gamma} \|d(Tx_0, Tx_1)\|.$$

It follows that $\{Tx_n\}$ is a Cauchy sequence by Definition 2.5.$(ii)$. Since cone metric space $X$ is complete, there exist $u_x \in X$ such that $Tx_n \to u_x$ as $n \to \infty$. Thus,

$$\lim_{n\to\infty} Tfx_{2n} = u_x, \qquad \lim_{n\to\infty} Tgx_{2n+1} = u_x. \tag{6}$$

Now, if $T$ is subsequentially convergent, $\{fx_{2n}\}$ (resp. $\{gx_{2n+1}\}$) has a convergent subsequence. Thus, there exist $v_{x_1} \in X$ and $\{fx_{2n_i}\}$ (resp. $v_{x_2} \in X$ and $\{gx_{2n_i+1}\}$) such that

$$\lim_{n\to\infty} fx_{2n_i} = v_{x_1}, \qquad \lim_{n\to\infty} gx_{2n_i+1} = v_{x_2}. \tag{7}$$

Because of continuity $T$ and by (7), we have

$$\lim_{n\to\infty} Tfx_{2n_i} = Tv_{x_1}, \qquad \lim_{n\to\infty} Tgx_{2n_i+1} = Tv_{x_2}. \tag{8}$$

Now, by (6) and (8) and because of injectivity of T, there exist $w_x \in X$ (set $v_x = v_{x_1} = v_{x_2}$) such that $Tv_x = u_x$.
On the other hand, by $(d_3)$ and (3), we have

$$\begin{aligned} d(Tv_x, Tgv_x) &\leqslant d(Tv_x, Tgx_{2n_i+1}) + d(Tgx_{2n_i+1}, Tfx_{2n_i}) + d(Tfx_{2n_i}, Tgv_x) \\ &\leqslant d(Tv_x, Tx_{2n_i+2}) + d(Tx_{2n_i+2}, Tx_{2n_i+1}) + \alpha_1 d(Tx_{2n_i}, Tv_x) \\ &\quad +\alpha_2 [d(Tx_{2n_i}, Tx_{2n_i+1}) + d(Tv_x, Tgv_x)] \\ &\quad +\alpha_3 [d(Tx_{2n_i}, Tgv_x) + d(Tv_x, Tx_{2n_i+1})] \\ &\leqslant d(Tv_x, Tx_{2n_i+2}) + d(Tx_{2n_i+2}, Tx_{2n_i+1}) + (\alpha_1 + \alpha_3) d(Tx_{2n_i}, Tv_x) \\ &\quad +\alpha_2 d(Tx_{2n_i}, Tx_{2n_i+1}) + \alpha_3 d(Tv_x, Tx_{2n_i+1}) \\ &\quad +(\alpha_2 + \alpha_3) d(Tv_x, Tgv_x). \end{aligned}$$

Now, by (4) and (5) we have

$$\begin{aligned}
d(Tv_x, Tgv_x) &\leqslant \frac{1}{1-\alpha_2-\alpha_3} d(Tv_x, Tx_{2n_i+2}) + \frac{1}{1-\alpha_2-\alpha_3} d(Tx_{2n_i+2}, Tx_{2n_i+1}) \\
&\quad + \frac{\alpha_1+\alpha_3}{1-\alpha_2-\alpha_3} d(Tx_{2n_i}, Tv_x) + \frac{\alpha_2}{1-\alpha_2-\alpha_3} d(Tx_{2n_i}, Tx_{2n_i+1}) \\
&\quad + \frac{\alpha_3}{1-\alpha_2-\alpha_3} d(Tv_x, Tx_{2n_i+1}) \\
&= A_1 d(Tv_x, Tx_{2n_i+2}) + A_2\gamma^{2n_i} + A_3 d(Tx_{2n_i}, Tv_x) \\
&\quad + A_4 d(Tv_x, Tx_{2n_i+1}),
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= \frac{1}{1-\alpha_2-\alpha_3} \quad , \quad A_2 = \frac{\alpha_2+\gamma}{1-\alpha_2-\alpha_3} d(Tx_0, Tx_1) \\
A_3 &= \frac{\alpha_1+\alpha_3}{1-\alpha_2-\alpha_3} \quad , \quad A_4 = \frac{\alpha_3}{1-\alpha_2-\alpha_3}.
\end{aligned}$$

From (1), we have

$$\begin{aligned}
\|d(Tv_x, Tgv_x)\| &\leqslant A_1 K \|d(Tv_x, Tx_{2n_i+2})\| + A_2 K \gamma^{2n_i} \|d(Tx_0, Tx_1)\| \\
&\quad + A_3 K \|d(Tx_{2n_i}, Tv_x)\| + A_4 K \|d(Tv_x, Tx_{2n_i+1})\|
\end{aligned}$$

Now the right hand side of the above inequality approaches zero as $i \to \infty$. The convergence above give us that $\|d(Tv_x, Tgv_x)\| = 0$. Hence $d(Tv_x, Tgv_x) = 0$, that is, $Tv_x = Tgv_x$. Since $T$ is one to one, then $gv_x = v_x$. Now, we shall show that $fv_x = v_x$.

$$\begin{aligned}
d(Tfv_x, Tv_x) &= d(Tfv_x, Tgv_x) \\
&\leqslant \alpha_1 d(Tv_x, Tv_x) + \alpha_2[d(Tv_x, Tfv_x) + d(Tv_x, Tgv_x)] \\
&\quad + \alpha_3[d(Tv_x, Tgv_x) + d(Tv_x, Tfv_x)] \\
&= (\alpha_2 + \alpha_3) d(Tv_x, Tfv_x).
\end{aligned}$$

which, using the definition of partial ordering on $E$ and properties of cone $P$, gives $d(Tfv_x, Tv_x) = 0$. Hence, $Tfv_x = Tv_x$. Since $T$ is one to one, then $fv_x = v_x$. Thus, $fv_x = gv_x = v_x$, that is, $v_x$ is a common fixed point of $f$ and $g$. Now, we shall show that $v_x$ is a unique common fixed point. Suppose that $v'_x$ be another common fixed point of $f$ and $g$, then

$$\begin{aligned}
d(Tv_x, Tv'_x) &= d(Tfv_x, Tgv'_x) \\
&\leqslant \alpha_1 d(Tv_x, Tv'_x) + \alpha_2[d(Tv_x, Tfv_x) + d(Tv'_x, Tgv'_x)] \\
&\quad + \alpha_3[d(Tv_x, Tgv'_x) + d(Tv'_x, Tfv_x)] \\
&= (\alpha_1 + 2\alpha_3) d(Tv_x, Tv'_x).
\end{aligned}$$

By the same arguments as above, we conclude that $d(Tv_x, Tv'_x) = 0$, which implies the equality $Tv_x = Tv'_x$. Since $T$ is one to one, then $v_x = v'_x$. Thus $f$ and $g$ have a unique common fixed point.

Ultimately, if $T$ is sequentially convergent, then we replace $n$ for $n_i$. Thus, we have

$$\lim_{n\to\infty} fx_{2n} = v_x, \qquad \lim_{n\to\infty} gx_{2n+1} = v_x.$$

Therefore if $T$ is sequentially convergent, then iterate sequences $\{fx_{2n}\}$ and $\{gx_{2n+1}\}$ converge to $v_x$. ■

The following results is obtained from Theorem 3.1.

COROLLARY 3.2 *Let $(X, d)$ be a complete cone metric space, $P$ be a normal cone and $T : X \to X$ be a continuous and one to one mapping. Moreover, let mapping $f$ be a map of $X$ satisfying*

$$\begin{aligned} d(Tfx, Tfy) \leqslant \alpha_1 d(Tx, Ty) + \alpha_2[d(Tx, Tfx) + d(Ty, Tfy)] \\ +\alpha_3[d(Tx, Tfy) + d(Ty, Tfx)], \end{aligned} \tag{9}$$

*for all $x, y \in X$, where*

$$\alpha_i \geqslant 0 \quad for \quad i = 1, 2, 3 \qquad and \qquad \alpha_1 + 2\alpha_2 + 2\alpha_3 < 1. \tag{10}$$

*That is, $f$ be a $T$-contraction. Then,*
(1) *For each $x_0 \in X$, $\{Tf^n x_0\}$ is a cauchy sequence.*
(2) *There exist $u_{x_0} \in X$ such that $\lim_{n\to\infty} Tf^n x_0 = u_{x_0}$.*
(3) *If $T$ is subsequentially convergent, then $\{f^n x_0\}$ has a convergent subsequence.*
(4) *There exist a unique $v_{x_0} \in X$ such that $fv_{x_0} = v_{x_0}$, that is, $f$ has a unique fixed point.*
(5) *If $T$ is sequentially convergent, then for each $x_0 \in X$ the iterate sequence $\{f^n x_0\}$ converges to $v_{x_0}$.*

Recently, Fillipović et al. prove that the Corollary 3.2 for a non-normal cone.

COROLLARY 3.3 *Let $(X, d)$ be a complete cone metric space, $P$ be a solid cone and $T : X \to X$ be a continuous and one to one mapping. Moreover, let mapping $f$ be a $T$-Hardy-Rogers contraction. Then, the results of previous Corollary hold.*

*Proof* See [5]. ■

## 4. Periodic point results

Obviously, if $f$ is a map which has a fixed point $z$, then $z$ is also a fixed point of $f^n$ for each $n \in \mathbf{N}$. However the converse is not true [2]. If a map $f : X \to X$ satisfies $Fix(f) = Fix(f^n)$ for each $n \in \mathbf{N}$, where $Fix(f)$ stands for the set of fixed points of $f$ [9], then $f$ is said to have property $P$. Recall also that two mappings $f, g : X \to X$ is said to have property $Q$ if $Fix(f) \bigcap Fix(g) = Fix(f^n) \bigcap Fix(g^n)$. The following results extend some theorems of [2].

THEOREM 4.1 *Let $(X, d)$ be a cone metric space, $P$ be a normal cone and $T : X \to X$ be a one to one mapping. Moreover, let mapping $f$ be a map of $X$ satisfyiing (i) $d(fx, f^2x) \leqslant \lambda d(x, fx)$ for all $x \in X$, where $\lambda \in [0, 1)$ and or (ii) with strict inequality, $\lambda = 1$ for all $x \in X$ with $x \neq fx$. If $Fix(f) \neq \emptyset$, then $f$ has property $P$.*

*Proof* See [5]. ■

THEOREM 4.2 *Let $(X, d)$ be a complete cone metric space, and $P$ a normal cone with normal constant $K$. Suppose that mappings $f, g : X \to X$ satisfy all the conditions of Theorem 3.1. Then $f$ and $g$ have property $Q$.*

*Proof* From Theorem 3.1, $f$ and $g$ have a unique common fixed point in $X$. Suppose that $z \in Fix(f^n) \bigcap Fix(g^n)$, thus we have

$$\begin{aligned} d(Tz, Tgz) &= d(Tf(f^{n-1}z), Tg(g^n z)) \\ &\leqslant \alpha_1 d(Tf^{n-1}z, Tg^n z) + \alpha_2[d(Tf^{n-1}z, Tf^n z) + d(Tg^n z, Tg^{n+1}z)] \\ &\quad +\alpha_3[d(Tf^{n-1}z, Tg^{n+1}z) + d(Tg^n z, Tf^n z)] \\ &= \alpha_1 d(Tf^{n-1}z, Tz) + \alpha_2[d(Tf^{n-1}z, Tz) + d(Tz, Tgz)] \\ &\quad +\alpha_3 d(Tf^{n-1}z, Tgz) \\ &\leqslant (\alpha_1 + \alpha_2 + \alpha_3)d(Tf^{n-1}z, Tz) + (\alpha_2 + \alpha_3)d(Tz, Tgz), \end{aligned}$$

which implies that

$$d(Tz, Tgz) \leqslant \gamma d(Tf^{n-1}z, Tz),$$

where $\gamma = \frac{\alpha_1+\alpha_2+\alpha_3}{1-\alpha_2-\alpha_3} < 1$ (by relation (4)). Now, we have

$$d(Tz, Tgz) = d(Tf^n z, Tg^{n+1}z) \leqslant \gamma d(Tf^{n-1}z, Tz) \leqslant \cdots \leqslant \gamma^n d(Tfz, Tz).$$

From (1), we have

$$\|d(Tz, Tgz)\| \leqslant \gamma^n K\|d(Tfz, Tz)\|.$$

Now the right hand side of the above inequality approaches zero as $n \to \infty$. Hence, $\|d(Tz, Tgz)\| = 0$. It follows that $d(Tz, Tgz) = 0$, that is, $Tgz = Tz$. Since $T$ is one to one, then $gz = z$. Also, Theorem 3.1 implies that $fz = z$ and $z \in Fix(f) \bigcap Fix(g)$. ■

THEOREM 4.3 *Let (X, d) be a complete cone metric space, and P a solid cone. Suppose that mapping $f : X \to X$ satisfies all the conditions of Corollary 3.2. Then $f$ has property P.*

*Proof* From Corollary 3.2, $f$ has a unique common fixed point in $X$. Suppose that $z \in Fix(f^n)$, we have

$$\begin{aligned} d(Tz, Tfz) &= d(Tf(f^{n-1}z), Tf(f^n z)) \\ &\leqslant \alpha_1 d(Tf^{n-1}z, Tf^n z) + \alpha_2[d(Tf^{n-1}z, Tf^n z) + d(Tf^n z, Tf^{n+1}z)] \\ &\quad +\alpha_3[d(Tf^{n-1}z, Tf^{n+1}z) + d(Tf^n z, Tf^n z)] \\ &\leqslant (\alpha_1 + \alpha_2 + \alpha_3)d(Tf^{n-1}z, Tz) + (\alpha_2 + \alpha_3)d(Tz, Tfz), \end{aligned}$$

which implies that
$d(Tz, Tfz) \leqslant \gamma d(Tf^{n-1}z, Tz)$ where $\gamma = \frac{\alpha_1+\alpha_2+\alpha_3}{1-\alpha_2-\alpha_3} < 1$, (by relation (10)). Hence,

$$d(Tz, Tfz) = d(Tf^n z, Tf^{n+1}z) \leqslant \gamma d(Tf^{n-1}z, Tz) \leqslant \cdots \leqslant \gamma^n d(Tfz, Tz).$$

Therefore, we have $d(Tz, Tfz) \leqslant \gamma^n d(Tfz, Tz) \leqslant \gamma d(Tfz, Tz)$. By the same arguments as Theorem 4.2, we conclude that $d(Tfz, Tz) = 0$, that is, $Tfz = Tz$. Since $T$ is one to one, then $fz = z$ and proof is complete. ■

COROLLARY 4.4 *Let $(X, d)$ be a complete cone metric space, and $P$ be a solid cone. Suppose that mapping $f : X \to X$ satisfies all the conditions of Corollary 3.3. Then $f$ has property P.*

*Proof* See [5]. ■

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