*Journal of*
*Linear and Topological Algebra*
Vol. 01, No. 01, Summer 2012, 21- 25

# Solving the liner quadratic differential equations with constant coefficients using Taylor series with step size $h$

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**Abstract.** In this study we produced a new method for solving regular differential equations with step size $h$ and Taylor series. This method analyzes a regular differential equation with initial values and step size $h$. this types of equations include quadratic and cubic homogenous equations with constant coefficients and cubic and second- level equations.

**Keywords:** Differential equation; initial value; step length; numerical methods; Taylor series.

## 1. Introduction

In the first and second sections of this paper, the numerical solution of a initial linear differential equations of cubic and quadratic homogeneous with constant coefficients is calculated by using Taylor series with length $h$. In this part the series $\sum_{n=0}^{\infty} a_n x_i^n$ will be replaced in Taylor expansion of $y(x_i), y'(x_i)$ and $y''(x_i)$, and then the obtained series replace in the given original differential equation, and we obtain $a_0, a_1, a_2, \cdots, a_n$. In Section 3 we will solve a quadratic homogenous differential equation

$$y'' + (A_0 x + B_0)y' + (A_1 x + B_1)y = 0,$$

using the mentioned method. Details are thoroughly discusses in the books [3] and [4].
An introduction to differential and their application of Zill et. al. [2], and numerical methods for partial differential equation of Ames[2].

## 2. Method of Solution

### 2.1 *Case 1.*

We consider the following initial value problem

$$y'' + Ay' + By = 0, \qquad y(x_0) = y_0, \quad y'(x_1) = y_1. \tag{1}$$

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We assume that the solution of Equation (1) has the following form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

According to Taylor series respect to $x_0$, we have

$$y(x_i) = y(x_0 + ih) = y(x_0) + ihy'(x_0) + \frac{(ih)^2}{2!} y''(x_0) + \frac{(ih)^3}{3!} y^{(3)}(x_0) + ... \qquad (2)$$

and so

$$y'(x_i) = y'(x_0) + ihy''(x_0) + \frac{(ih)^2}{2!} y^{(3)}(x_0) + \frac{(ih)^3}{3!} y^{(4)}(x_0) + ... \qquad (3)$$

therefore we have

$$y''(x_i) = y''(x_0) + ihy^{(3)}(x_0) + \frac{(ih)^2}{2!} y^{(4)}(x_0) + \frac{(ih)^3}{3!} y^{(5)}(x_0) + ... \qquad (4)$$

If we set

$$y(x_0) = \sum_{n=0}^{\infty} a_n x_0^n,$$

then we have

$$y^{(k)}(x_0) = \sum_{n=k}^{\infty} n(n-1)...(n-k+1) a_n x_0^{n-k}, \qquad k = 1, 2, 3, ... \qquad (5)$$

Now, by substituting the Equation (5) in Equation (2) we have

$$\begin{aligned} y(x_i) = & \sum_{n=0}^{\infty} a_n x_0^n + ih \sum_{n=1}^{\infty} n a_n x_0^{n-1} + \frac{(ih)^2}{2!} \sum_{n=2}^{\infty} n(n-1) a_n x_0^{n-2} \\ & + \frac{(ih)^3}{3!} \sum_{n=3}^{\infty} (n-1)(n-2) a_n x_0^{n-3} \qquad (6) \\ & + \frac{(ih)^4}{4!} \sum_{n=4}^{\infty} (n-1)(n-2)(n-3) a_n x_0^{n-4} + ... \end{aligned}$$

or

$$\begin{aligned} y(x_i) = & \sum_{n=0}^{\infty} x_0^n \Big( a_n + ih(n+1) a_{n+1} + \frac{(ih)^2}{2!} (n+1)(n+2) a_{n+2} \\ & + \frac{(ih)^3}{3!} (n+1)(n+2)(n+3) a_{n+3} + ... \Big). \qquad (7) \end{aligned}$$

If we substitute Equation (5) in Equation (3), we have

$$y'(x_i) = \sum_{n=0}^{\infty} x_0^n \Big((n+1)a_{n+1} + ih(n+1)(n+2)a_{n+2} + \frac{(ih)^2}{2!}(n+1)(n+2)(n+3)a_{n+3}$$
$$+\frac{(ih)^3}{3!}(n+1)(n+2)(n+3)(n+4)a_{n+4} + ...\Big). \quad (8)$$

Finally, by substituting the Equation (5) in Equation (4) we have

$$y''(x_i) = \sum_{n=0}^{\infty} x_0^n \Big((n+1)(n+2)a_{n+2} + ih(n+1)(n+2)(n+3)a_{n+3}$$
$$+\frac{(ih)^2}{2!}(n+1)(n+2)(n+3)(n+4)a_{n+4}$$
$$+\frac{(ih)^3}{3!}(n+1)(n+2)(n+3)(n+4)(n+5)a_{n+5} + ...\Big). \quad (9)$$

By substituting Equations (7)-(9) in Equation (1), we have

$$\sum_{n=0}^{\infty}(x_0)^n \{((n+1)a_{n+1}(B(ih)+1)$$
$$+\sum_{k=2}^{m}\left(a_{n+k}\left(B\frac{(ih)^k}{(k)!} + \frac{A(ih)^{k-1}}{(k-1)!} + \frac{(ih)^{k-2}}{(k-2)!}\Pi_{i=1}^{k}(n+i)\right)\right)\} = 0, \quad (10)$$

so

$$a_{n+k} = \frac{-(n+1)a_{n+1}(B(ih)+1) + \sum_{k=2}^{m-1}(a_{n+k}(B\frac{(ih)^k}{(k)!} + \frac{A(ih)^{k-1}}{(k-1)!} + \frac{(ih)^{k-2}}{(k-2)!}\alpha_{n,m})}{\left(B\frac{(ih)^m}{(m)!} + \frac{A(ih)^{m-1}}{(m-1)!} + \frac{(ih)^{m-2}}{(m-2)!}\right)\alpha_{n,m}}. \quad (11)$$

where $\alpha_{n,k} = \Pi_{i=1}^{k}(n+i))$.

**Example 1.** Consider the following initial value problem

$$y''(x) + 2y'(x) + y(x) = 0, \qquad y(0) = 0, y'(0) = 1,$$

according to the above algorithm for $y(0.1)$ and $h = 0.1$, if we set $m = 2$, then

$$a_0 = 0, a_1 = 1, a_{n+2} = \frac{-(a_{n+1}(n+1)(Bh+A))}{(B\frac{(ih)^2}{2!} + A(ih) + 1)\Pi_{i=1}^{2}(n+i)}$$

so $a_2 = -0.8714$. Also if we set $m = 3$, then $a_3 = 0.00067$ and $y(0.1) = 0.0913$. We know that the exact solution is $y(0.1) = 0.0905$ and absolute error is $8 \times 10^{-4}$.

### 2.2 *Case 2.*

In this case we consider the following problem

$$y''' + Ay'' + By' + Cy = 0, \quad y(x_0) = y_0, y'(x_1) = y_1, y''(x_2) = y_2, A, B, C \in R. \quad (12)$$

According the Case 1. we have

$$\sum_{n=0}^{\infty}(x_0)^n\left\{((n+1)a_{n+1}(B(ih)+1)\ +\right.$$

$$\left.\sum_{k=3}^{m}\left(a_{n+k}\left(B\frac{(ih)^{k-1}}{(k-1)!}+\frac{A(ih)^{k-2}}{(k-2)!}+\frac{(ih)^{k-3}}{(k-3)!}\right)\prod_{i=1}^{k}(n+i)\right)\right\}=0,$$

and then

$$a_{n+k}=\frac{-((Ca_n+B(n+1)a_{n+1}+\sum_{k=3}^{m-1}a_{n+k}(B\frac{(ih)^{k-1}}{(k-1)!}+\frac{A(ih)^{k-2}}{(k-2)!}+\frac{(ih)^{k-3}}{(k-3)!})\alpha_{n,k}}{\left(B\frac{(ih)^{m-1}}{(m-1)!}+\frac{A(ih)^{m-2}}{(m-2)!}+\frac{(ih)^{m-3}}{(m-3)!}\right)\alpha_{n,m}} \tag{13}$$

**Example 2.** Consider the following initial value problem

$$y'''(x)-6y''+11y'(x)-6y(x)=0, \qquad y(0)=0, y'(0)=1, y''(0)=0.$$

According to the above algorithm, we have

$$y(x)=x+x^2-29.01099x^3+918.9361x^4,$$

so $y(0.1)=0.1728$, and the absolute error is 0.0566.

## 3. Case 3.

Finally, we consider the following problem

$$y''(x)+(A_0x+B_0)y'(x)+(A_1x+B_1)y(x)=0, \qquad y(x_0)=y_0, \quad y'(x_1)=y_1 \tag{14}$$

According the above algorithm in Case 1., we have

$$a_{n+k}=\frac{-(a_nA_1+a_{n+1}(B_1+A_1(ih)(n+1)+A_0(ih)(n+1))+E}{\left(\left(\left(B_0\frac{(ih)^{m-2}}{(m-2)!}+B_1\frac{(ih)^{m-1}}{(m-1)!}\right)\prod_{i=2}^{k}(n+i\right)\right)+F} \tag{15}$$

where

$$E=\sum_{k=3}^{m}a_{n+k}\left(\left(B_0\frac{(ih)^{k-2}}{(k-2)!}+B_1\frac{(ih)^{k-1}}{(k-1)!}\right)\prod_{i=2}^{k}(n+i)\right)$$
$$+\left(A_0\frac{(ih)^{k-1}}{(k-1)!}+B_1\frac{(ih)^{k}}{(k)!}\prod_{i=1}^{k}(n+i)+\frac{(ih)^{k-3}}{(k-3)!}\right)\prod_{i=2}^{k}(n+i),$$

and

$$F=A_0\frac{(ih)^{m-1}}{(m-1)!}+A_1\frac{(ih)^{m}}{(m)!}\prod_{i=1}^{m}(n+i)+\frac{(ih)^{m-3}}{(m-3)!}\prod_{i=2}^{k}(n+i).$$

## 4. Conclusions

The results obtained from the method introduced in this paper, can be used to numerical solution of $n$th order differential equation with constant coefficient and with initial value and with step size $h$ by series $\sum a_n x^n$, and thus for obtaining the answer of homogenous linear differential equation of $n$th order, $a_n(x)y^{(n)} + a_{n-1}(x)y^{n-1} + \cdots + a_1(x)y' + a_0 y = 0$ can get with initial values and with step length.

## References


[1] F.S. Acton, *Numerical methods that work*,Harpor and Row, New york, 1970.
[2] W.F. Ames,*Numerical methods for partial dofferential equations*, Barnes and Noble, New york, 1969.
[8] G. Birlhoff, G. ROTA, *orfonary differntial equations*, Blaisdell, wal tmn. moll, 1969.
[4] S.L. Campbell, *An introduction to differntail and thair applications*, 2nd edition, wadswor th publication co, 1990.
[5] W. Derrick, S. Grassman, *elementary diffential equations with applications*, 2nd edition, Addison-wasely, 1981.
[6] L.M. Kelly, *Elementary differontial equation*, 6th. MCGeaw hill,New york, 1965.
[7] J.M. Ortega,*Numerical analysis - Asecond course*, Academic press, New york, 1972.
[8] G.F. Simmans, *Differentioal equatons with application*, MCGAW-Hill, 1972.
[9] D.G. Zill, *Differential equations with bondary walae problems*, PWS KENT publishing co, 1989.