

# TEACHERS' AWARENESS AND UNDERSTANDING OF STUDENTS' CONTENT KNOWLEDGE OF GEOMETRIC SHAPES

**Balli Lelinge, Christina Svensson**

Malmo University, Sweden

E-mail: balli.lelinge@mau.se, christina.svensson@mau.se

## Abstract

*Few research studies have been conducted in a primary school in early mathematical education about the teaching of geometry. This research aims to contribute with knowledge of how teachers' awareness and understanding of necessary conditions to enhance students' abilities to discern two- and three-dimensional shapes develop. In this research, qualitative methods were used to analyse data from a lesson study in grade 4 in the subject of mathematics. Data were primarily collected through audio-recorded conversations with teachers before and after the lesson, and the results of students' pre- and post-test. The results of this research showed increased awareness of using collaboration opportunities to apply professional classroom instructions and activities to enhance students' knowledge of two- and three-dimensional shapes. This research elucidates how the practice-based professional development approach emphasised the teachers' teaching targets for understanding students' content knowledge of geometric shapes. Additionally, the result highlighted teachers' awareness and understanding of the challenges students face in learning about three-dimensional shapes from two-dimensional representations. Future research should develop a more iterative and revised research lesson design to develop more powerful content knowledge and classroom activity in this topic area.*

**Keywords:** *early mathematics education, geometric shapes, practice-based professional development, lesson study*

## Introduction

This research identified teachers' awareness and understanding of students' spatial ability in early mathematical education. From a Swedish educational context, there is a great interest in students' mathematical achievements (OECD, 2016). It is assumed that practice-based collaboration between researchers and teachers promotes teachers' teaching methods in relation to the subject of mathematics. Skott et al. (2018) research highlighted the increased importance of the subject matter knowledge in teacher education. That is, however, not enough if the teacher does not understand how to teach in the subject's meaning and links the topic to, for example, everyday life (Ball & Cohen, 1999). Klim-Klimeszwska and Nazaruk (2020) research revealed that kindergarten teachers (150 interviews) did not work systematically with early mathematical education with geometric concepts. The leading critic is linked to the use of the student's potential regarding "geometric intuition, in an effective way" (p. 345).

A mere theoretical understanding of the subject matter does not contribute to conceptual teaching understanding (Ball & Cohen, 1999). Conceptual teaching understanding is based on an understanding of the need to offer the students several representations, such as art constructions, or comparisons with other subjects and contexts, which increase the probability of more in-depth understanding of the whole (Ball & Cohen, 1999, p. 11; Smith, 2001). However, there

are few research studies about teaching in geometry that focus on developing students' spatial ability in primary school. Therefore, this research addressed the importance of contributing to more research in this topic area.

The geometry content of mathematics contributes to developing students' spatial and logical reasoning (French, 2004). Spatial ability is vital in developing understanding of and experiencing the visual world correctly (Gardner, 1993) and is divided into three areas: spatial relations, visualisation, and orientation (Thurstone, 1950). Spatial relations consist of imagining the rotations of two- and three-dimensional objects as a whole body (Olkun, 2003). Spatial visualisation consists of the ability to imagine representing two- and three-dimensional movements in different spaces (Clements & Battista, 1992). Spatial orientation is the ability to distinguish geometric shapes from different positions (Thurstone, 1950) but also to imagine two- and three-dimensional shapes based on different rotations (Maier, 1996). It contributes to a mental spatial understanding in both vertical and horizontal directions. Van Hiele (1986) has identified five levels of mental spatial understanding development. Level 1 consists of visualisation, which means that students experience the most common geometric shapes in standard orientation. Level 2 (analysis) consists of students being able to identify forms based on their characteristics. Level 3 (abstraction) consists of students being able to identify, for example, that a square is a rectangle but not all rectangles are squares. Level 4 (deduction) consists of a developed understanding of the importance of geometric evidence. Finally, in level 5 (rigor), students both understand and think abstractly about the relationships between different geometric concepts.

This research concerned Level 1, and the students' ability to express their learning regarding two- and three-dimensional shapes (circle, square, rectangle, triangle, sphere, cube, cone and cuboid). That is, being able to develop and describe their spatial ability through visualisation and being able to name the mentioned shapes in their standard positions after the research lesson.

Teaching about geometrical content is closely related to the ability to discern visual representations in different forms. Several repetitions of representations are required to develop deeper mathematical knowledge (Duval, 2006). However, Wilson and Swanson (2001) showed that students who have difficulties in mathematics seem to have a harder time preserving visual representations in working memory compared with other students. This can be further discussed in relation to the levels identified by Van Hiele (1986), where Levels 1 and 2 consist of identifying geometric shapes based on representations. In geometry teaching with the support of representations has been found that students who develop the ability to transform (i.e., reasoning between two- and three-dimensional shapes in visual representation) can better handle tasks that are based on similarity to understand that two items can have different scales (Seago et al., 2013).

From the age of 11–12, students transition occurs from concrete operations to more formal ones (Piaget, 1972). This means that they develop an understanding of the relations between different forms, such as two- and three-dimensional shapes, without relying on concrete examples (a.a.). However, it is not easy to build formal mathematical knowledge at a young age because all students need to develop abstract thinking (Piaget, 1972). Hallowell et al. (2015), for example, highlighted in their research how primary school students showed difficulty transforming when curves were involved. Chapman (1997) argued about the importance of organising classroom interaction to provide an opportunity to develop a more formal mathematical language. However, this can only happen with teacher support in the form of new communication strategies containing mathematical concepts, thinking and reasoning (Sfard, 2007). Reasoning with a mathematical register (Halliday, 1981) is an important communication strategy for developing mathematical learning between students and an expert (e.g., the teacher) in a classroom (Sfard, 2007).

Research in mathematics education has highlighted students' misconceptions that, for example, in rotation, a square is not a square unless the base is resting horizontally (Clements & Battista, 1992; Mayberry, 1983). Furthermore, students have difficulty experiencing the properties within the shapes (Mayberry, 1983), such as a square being a form of a rectangle (Marchis, 2008).

More research is needed in the mathematical area of geometry, for clarifying how students are developing their understanding of spatial abilities in the classroom. It is necessary to educate teachers about how to understand students increased learning in two-and three-dimensional shapes. This research addresses the methodological approach practice-based professional development (PBPD) for developing teachers' conceptual teaching and understanding in this topic area.

### *Practice-based Professional Development*

Practice-based professional development (PBPD) research is related to improving school teaching (Ball & Cohen, 1999), and it focuses more on teacher inquiry development than only on teacher knowledge (Harris et al., 2012). According to Ball and Cohen (1999), it is necessary to understand that improving PBPD is about developing teaching-learning as something that is a part of teaching (p. 11). Ball and Cohen argued that this teaching-learning knowledge should not be entirely depending on institutional learning: "Professional development could be substantially improved if we could develop ways to learn and teach about practice in practice" (p. 11). According to Smith (2001), this approach provides teachers with an opportunity to develop new awareness and knowledge (p. 206) about teaching.

The Swedish Government announced that Sweden needs to create a problem-based development research system in which higher education institutions within all school levels must collaborate in equal partnership (Goodlad, 1991; Herro, et al., 2019) to develop learning in school. This direction and statement from the Swedish Government to increase the need for practice-based research emphasised the collaboration between teachers and researchers (Christensson, 2018). By these approaches, the gap between theory and practice is meant to be bridged by involving teachers in the research process and working directly to change and improve classroom practice (Carlgren, 2005; Swedish Research Council, 2015). According to Christensson (2018), this approach shifts the concept of problem-based development research to teaching-development research. Thus, PBPD research is characterized by a knowledge interest that primarily aims to contribute knowledge about how teaching can be developed and improved, with the aim of promoting student development and learning (cf Ball & Cohen, 1999; Gardner, 2011; Harris et al., 2012). This means that the research must be conducted in direct connection with the teaching. What characterizes these research orientations is that the research is being e.g. iterative, collaborative, process-oriented, and useful in real contexts. By involving teachers and researchers in the research processes, the PBPD research aims to narrow the gap between theory and practice (Ball & Cohen, 1999; Christensson, 2018).

This research direction was inspired by the lesson study model, where teachers' collaborative professional development (CPD) has been the starting point in a previous school development project with a content-oriented approach.

### *Lesson Study: A Collaborative Professional Development Model*

The Lesson Study *Jugyou kenkyuu* (Yoshida, 1999) model offers the teachers a collaborative professional development approach to continuously improve their teaching (Lewis, 2009). Lesson study has been developed in Japan and translated as "lesson study" (hereafter LS) by Stigler and Hiebert (1999). According to Lewis (2002, 2009) LS is a "research

lesson", where the emphasis is on teaching for understanding instead of teaching as telling (Lewis & Tsuchida, 1998). In the past century, LS has been an important path (Wood, 2019) of enhancing teacher learning (Pêna Trapero, 2013) professional development, and triggering changes in teachers' classroom practice (Vrikki et al., 2017; Warwick et al., 2019). LS gives the teachers a great potential to observe the life and processes in the classroom, particularly regarding students' learning capability, which is subsequently collaboratively analysed (Lewis (2009). The research lesson is designed in the teacher group, although it is usually put into practice by just one of the teachers. Whilst the teacher teaches about the object of learning, the others observe.

LS aim is to improve teaching and learning to evaluate and revise the lesson, and to share lesson experiences with colleagues (Doig & Groves, 2011; Lewis, 2009; Lewis & Tsuchida, 1998; Stigler & Hiebert, 1999). Although the process may vary, it often begins with defining the problem (Hiebert, et al., 2002). Most often this problem is formulated as a learning goal that teachers found to be challenging for their students (Carlgren, 2011). In the Japanese LS design, teachers work in small groups and meet every week to collaborate and plan lessons together to develop an issue that is of concern for the students' learning and understanding (Lewis, 2009, 2016). A key factor of LS is to accomplish a process of teaching-learning goals (Takahashi & McDougal, 2016). According to Fujii (2016), LS is primarily intended to produce learning that helps teachers apply and develop future lesson instruction and classroom activities. LS can have different goals: as a professional development tool, to improve teacher instruction and subject competencies; as a research methodology; and as a means to achieving both the professional development and research goals (cf Holmqvist & Mattisson, 2008; Stigler & Hiebert, 2016). Goldsmith's, et al. (2014) synthesis of 106 research articles according to mathematics teachers' learning found that few studies were focussing on this process.

LS has been implemented in this research in grade 4 in a Swedish context with the learning objects of two- and three-dimensional shapes.

### *Problem Statement and Research Focus*

PBPD research analysis describes researchers and teachers' collaborative processes that are in great need of development. Research has shown that collaborative, iterative, learning processes increase teachers' awareness of the importance of showing what and how student learning can be identified. Examples from previous research showed that subject content must be at the foreground when teachers are planning the object of learning. Yet, there seems to be very few studies that have their focus on the content of spatial learning in the ages of 10–11 years, which is the case in this present study. Therefore, this research aimed to contribute new knowledge about students' spatial learning about two- and three-dimensional shapes, and teachers' awareness of the same through the collaborative process.

Based on the research problem highlighted above, this research aim was to contribute with knowledge of how teachers' awareness and understanding of necessary conditions to enhance students' abilities to discern two- and three-dimensional shapes develop. The research was based on the following research questions:

RQ1 What arguments and assumptions guide the teachers' construction of the research lesson?

RQ2 In what way, if any, does students' knowledge change?

RQ3 What necessary conditions, based on teachers' awareness and understanding, are required to enhance students' abilities to discern the differences between two- and three-dimensional shapes?

## Research Methodology

### *General Background*

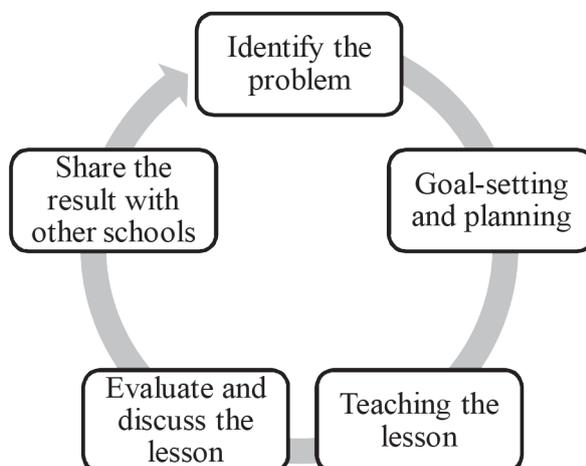
This research was part of a two-year school development project in a major city in Sweden, intended to develop different collaborative learning models. This research was conducted close to the teachers' ordinary daily work. The teachers intended to develop student knowledge, and the researchers' intentions to study the teachers' work while participating in collaborative professional development, based on the LS approach (Lewis, 2002, 2009, 2016; Stigler & Hiebert, 1999; Takahashi & McDougal, 2016). As the teachers work with developing the students' knowledge of geometrical shapes, the study explores teachers' developed awareness about how to teach students about two- and three-dimensional shapes. In accordance with the Swedish Government, the practice-based professional development research is aiming to develop teachers' professional knowledge in order to enhance their teaching skills. By linking research questions directly to identified needs for improvement and using a research methodology that actively involves the teachers, the improvement work becomes part of the research method (Christensson, 2018).

The study started with two lectures about school development based on scientific approaches, for the entire school. After the initializing lectures, the teachers were divided into smaller groups, and this research reports on the work of the researchers and three teachers who chose to conduct a lesson study to enhance students' knowledge of geometrical shapes. LS is an iterative cyclic model for analysis (see Figure 1), inspired by Stigler and Hiebert (1999) and Lewis (2002) to improve the teaching-learning goals (Takahashi & McDougal, 2016).

When the teachers select an object of learning, it is necessary to identify what is to be focused upon (Holmqvist & Mattisson, 2008). Holmqvist and Mattisson (2008) argued that the most critical factor is the choice of theoretical perspective and method to answer the research questions (p. 37). The teachers' choice of the object of learning to enhance students' abilities was related to the syllabus in the Swedish curriculum of the subject of mathematical goals: "Basic geometrical objects such as polygons, circles, spheres, cones, cylinders, pyramids, cuboids and their relationships. Basic geometrical properties of these objects. /.../ Construction of geometrical objects, both with and without digital tools" (Swedish National Agency for Education [SNAE], 2018, p. 58).

**Figure 1**

*Iterative cyclic model*



*Sample*

The research was conducted in a primary school in Sweden, grade 4, from January 2019 to April 2019 (Table 1). The planning meeting was held at the school where the teachers worked. Each session lasted about 2 hours.

**Table 1**  
*Participants and context of the working process*

School level	Participating teachers	Period	Lesson total	Sharing of knowledge
Grade 4. One group (pre-test <i>n</i> :14 post-test <i>n</i> :15).	Three primary school level teachers.	January–April 2019.	One lesson cycle.	Colleagues at their own school.

In total three teachers and two researchers participated in this research, which consisted of a lesson study cycle in grade 4. Teacher 1 (T1) is a certified teacher in art (pictures) in grades 1–9 and in religion grades 4–9. T1 has been a certified teacher for 3 years and recently began work at the current school. Neither Teachers 2 or 3 (T2, T3) are certified teachers. T2 has been working as a music teacher in grades 1–5 for 6 years, and for 1 year at the current school. T3 teaches religion and history in fourth grade and has been working at the current school for 1 year (Table 2). This is his first year as a teacher.

**Table 2**  
*Study participants*

Participant	Age	Sex	Degree	Work experience in years (current school)
Teacher (T1)	28	Male	Yes (1–9)	3 (<1)
Teacher (T2)	31	Male	No (1–4)	6 (1)
Teacher (T3)	27	Male	No (4–6)	1 (1)

The number in brackets is the number of years that the participants had been employed at the current school at the time of the research project, Spring 2019.

Fourteen students participated in this research lesson and took the pre-test. However, one student did not take the pre-test. However, the material that the researchers later received for an inter-reliability assessment of the individual responses included results from 15 students (see Tables 5 and 7). Thus, the pre-test analysis is determined from the researchers having counted 15 tests as completed. According to the teachers several students had special needs and adaptations.

*Research Instrument and Procedures*

Teachers' planning and acting during the LS is described as a background, and after this description the research procedures will be described. The focus during the teachers' work was primarily about developing their students' abilities to discern the dimensional differences of the

quadrant, triangle, circle, rectangle, cube, cone, sphere and cuboid. The teachers designed their research lesson based on the differences and similarities between two- and three-dimensional shapes (Table 3).

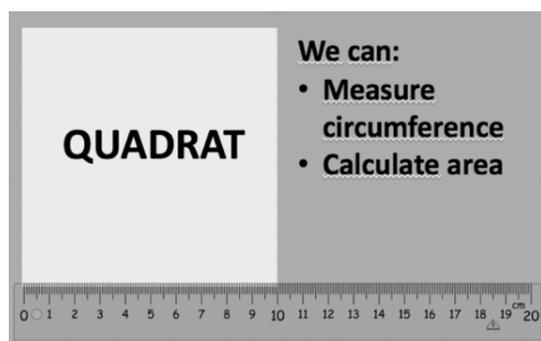
**Table 3**

*The approach of the research lesson*

Research lesson design	Part 1: PowerPoint introduction (lecture 15 minutes) contains the following:	Part 2: Lesson activity contains the following:
	2D shapes (visualisation by measurement quadrant, Figure 2) diachronic.	3D shapes in clay (visualisation, Figure 5) diachronic.
	3D shapes (counting the side surfaces of the cube (visualisation and rotation) diachronic.	2D and 3D shapes with paper and ruler, construction of quadrants and cubes (spatial visualisation, rotation) synchronic.
	2D and 3D shapes, art-construction (spatial visualisation, Figures 3 and 4) synchronic.	Short summary of the lesson.
	Differences and similarities between cubes and cuboids (visualisation) diachronic.	
	Differences between cones and cuboids and cubes (visualisation both in art and with artefacts such as toy blocks, dice and cones) diachronic.	
	Differences between cones and spheres (visualisation both in art and with artefacts such as cones and globes diachronic).	

**Figure 2**

*2D measurement of a quadrant (visualisation)*

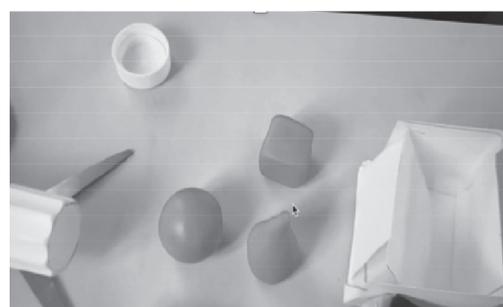
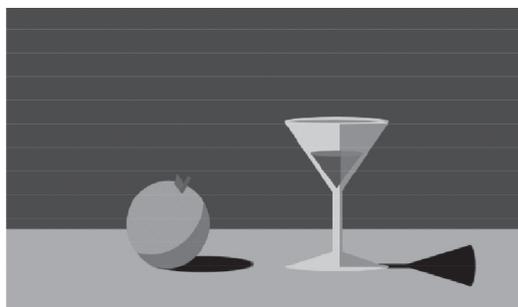


**Figure 3**

*2D and 3D shapes (spatial visualisation)*



**Figure 4** 2D and 3D shapes (spatial visualisation) 2D and 3D shapes in clay (visualisation)



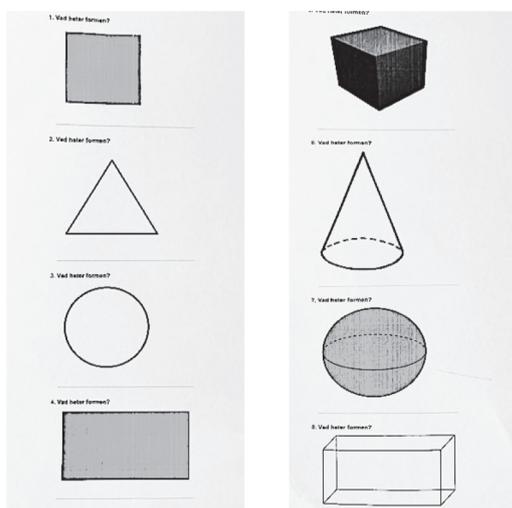
A pre- and post-test was constructed and used before and after the research lesson. The teachers' evaluation of the LS was discussed within the research group and presented for other teachers at the school. The researchers followed the teachers' learning process and scaffolded their lesson design and were also a part of the teacher's analysis of the students' learning outcomes. The researchers' analysis of the project also included a comparison between teachers' and researchers' assessments of the students' test.

#### *Data Collection*

To achieve the aim and answer the research questions, the data collection was consisted of different types of data: Pre- and post-test results, students' experience of pre- and post-test difficulties, audio-recorded teacher conversations during the design of the lesson, audio-recorded teacher conversations after the lesson, and audio recordings of teachers sharing their results with colleagues after the lesson (Table 4).

**Table 4**  
*Data collection*

Overall data collection	Unit of analysis	Date	Research Lesson
Audio-recorded data that was transcribed verbatim.	Pre-test (Figure 6)	2019/01/30	2019/02/13
	Audio-recorded teacher conversations during design <i>before</i> the research lesson.	2019/02/13	Time of lesson: 45–50 minutes.
	Post-test (Figure 6) Audio-recorded teacher conversations <i>after</i> the lesson.	2019/03/06	
	Audio recordings of teachers sharing their experiences of the lesson with colleagues.	2019/04/10	

**Figure 6***Pre- and post-test three dimensional shapes**Data Analysis*

The empirical material consisted of teachers' introduction by the PowerPoint presentation on the selected learning object (see Table 3). The verbatim transcripts, mainly those from conversations done before the research lesson, were analysed to identify what arguments and assumptions guided the teachers during the planning of the research lesson about mathematical spatial ability (RQ1). To answer RQ2, pre- and post-test results were collected and analysed to identify changes in students' knowledge and awareness of dimensional differences of two- and three-dimensional shapes. To answer RQ3, the verbatim transcripts, mainly those from conversations after the research lesson and when the teachers shared their experiences from the same research lesson, were analysed to identify the teachers' understanding of which necessary conditions are required to enhance students' abilities to discern the differences between two- and three-dimensional shapes.

The analysis was firstly based on the assessment of students' changed knowledge about the learning object based on the pre- and post-tests. Secondly, the pre- and post-test results have guided the researchers to capture the teacher's experience about necessary conditions in relation to changes, if any, in the students' understanding of the two- and three-dimensional shapes. The experiences about the teachers' necessary conditions and assumptions regarding any changes in the students' understanding have been analysed based on the previous research about students' developed spatial ability. The audio-recordings were transcribed and analysed separately and discussed by both researchers to increase the validity and credibility of the selected excerpts.

*Ethical Considerations*

In today's society, research occupies a prominent position, and the expectations are high. According to the Swedish Research Council guideline for *Good Research Practice* (2017), the researchers have a specific responsibility regarding validity and trust to the participants. It is vital to strive to conduct research on a high-quality level, regardless of whether the results are positive or negative. The researchers are obligated to follow the Good Research Practice guidelines and be free from influences and manipulation or act in personal interests. In this context, good research is systematic, shared with the participating teachers, students, parents

and colleagues. There are no conflicts of interest, methods and reliability. The research process and ethical requirements rest on society's general ethical norms and values. The participating teachers provided written consent to participate in this study and to be audio-recorded. The students and their parents provided both verbal and written informed consent about the study and that any personal information would remain anonymous.

## Research Results

### *Arguments and Assumptions that Guided the Teachers' Construction of the Research Lesson*

The analysis of the arguments and assumptions that guided the research lesson design showed how the teachers discussed their experiences before the lesson. In their discussions T2 mentioned that some students had "a red flag in front of their eyes immediately when mathematics is mentioned". T2 summarised (Excerpt 1) his ideas about the lesson plan and developed teaching methods by discussing less about mathematics and tests.

#### **Excerpt 1:**

Teacher 2: We should talk less about math, and less talk about the pre- and post-test /.../.

Later, the teachers planned the first part of the lesson (see Table 3) from a synchronic visualisation perspective of the object in daily life. The excerpt below shows their notions of visualising mathematical properties through everyday objects and bringing with them some three-dimensional artefacts.

#### **Excerpt 2:**

Teacher 1: /.../ I intend to bring in a tennis ball, I intend to bring in dice, I intend to bring in a cone. Yes, I intend to bring in items.

Teacher 2: Yes. You could spin a globe and ask, "What can we call this?" Lots of such things can be used with tactile and visual stimulation. They develop their knowledge of basic geometric shapes [with the students next to them].

The teachers also mentioned and discussed how they could use paper instead of shapes. "You might be able to talk about the properties of the paper" (T2). The teachers discussed their experiences regarding the conditions necessary for this class. For example, T3 stated that the lesson should not be longer than 45–50 minutes. The teachers had also identified that this is students' need to work in smaller groups, "It is more comfortable for students with special needs and adaptations to have the same structure that they [have] always had, working together as a group, and sitting around a table" (T1). Furthermore, the teachers discussed how this design supports the goal of having the students talk openly to each other about how and what they are doing. T1 has a concrete idea of how to stimulate the students' ability to discern the differences between the three-dimensional shapes:

#### **Excerpt 3:**

Teacher 1: They [shall] roll [around] a ball and talk about what the difference is about rolling a ball around the table versus trying to get a cone to the other side of the table. [It is that] it will spin around in circles. The ball spins straight. "Why so?" So, it becomes a bit more that they can describe the differences between the different forms.

The teachers' arguments and assumptions guided them to use everyday objects together with a PowerPoint presentation to elucidate and offer a tactile feeling of the similarities and differences by synchronically visualising the different three-dimensional shapes. The second assumption that guided the teachers' construction of the lesson consists organisational factors such as the short lesson, using less talk about maths, sitting around a table, dividing the class into two groups, and providing a structure that allows the students to learn from each other.

*Identifying Changes in Students' Awareness of  
Two- and Three-dimensional Shapes*

In the teachers' design of pre- and post-test, they added a final question (Tables 6 and 8) about the students' experiences of the test, whether it was difficult or easy. The results indicated that students' understanding of the object of learning, geometrical shapes increased between the pre- and post-test (Tables 5 and 7). The results have been analysed due to the percentage of correct answers before and after each lesson. The teachers assessed the students' answer of the term "fyrkant" (Swedish) square for quadrant as correct. In the Swedish language, this is problematic because "fyrkant", square, any combination of four sides and edges between the side surfaces, so it can be used to describe both two- and three-dimensional shapes.

**Table 5**  
*Researchers' and teachers' pre-test assessments*

Pre-test (n=14 students)			
Geometric shapes	Percent (%)		Qualitative analysis of student-answers
	Researchers' analysis	Teachers' analysis	
Quadrant	71	93	10 of 14 had correct answer. Four wrote "square".
Triangle	86	86	12 of 14 had correct answer. Two wrote "rectangle".
Circle	100	100	14 of 14 had correct answer.
Rectangle	64	64	9 of 14 had correct answer. One wrote "quadrant". Four did not answer.
Cube	57	57	8 of 14 had correct answer. Two wrote "rectangle". Four did not answer.
Cone	14	14	2 of 14 had correct answer. Five wrong answer: Two wrote "hat", one wrote "castle", another wrote "tube", one wrote "triangle". Seven did not answer.
Sphere	7	7	1 of 14 had correct answer. Seven wrong answer: One answered "2D". Four answered "ball". One answered "basketball", and one answered "circle". Four did not answer.
Cuboids	0	0	No one has answered correctly. Six had a wrong answer: one answered "square", one answered "three-dimension square", one answered "box", one answered "prism", and two answered "rectangle". Eight did not answer.

*Students' Experiences of the Pre-test*

The teachers added a final question (Table 6): "Was it easy or difficult?" Here the students were able to leave a comment. Fourteen students completed the pre-test.

**Table 6**  
*Students' pre-test experiences*

Correct answers	Easy	Medium	Difficult	No answer	Total (n=8)
8					0
7		1			1
6	1				1
5		2	1		3
4		1	1		2
3	1	2		2	5
2		1			1
1			1		1
0					0

None of the students achieved 8 of 8 correct answers on the pre-test. One student had only 1 of 8 correct answers, and this student also answered that the pre-test was "difficult". One of five students who had three correct answers wrote that it was an "easy" test; this indicated that the student's self-assessment of his/her prior knowledge did not correspond with reality. One student had seven correct answers and commented that he/she felt that the pre-test was of "medium" difficulty.

*Result of Students' Responses to the Post-test*

In the post-test, no students had fewer than five correct answers. In the pre-test (Table 5), all students achieved correct answers for only one shape: the circle. Conversely, in the post-test, all students achieved correct answers for all the two-dimensional shapes and one three-dimensional shape: the cube (Table 7). The ability to identify two-dimensional shapes increased most for the rectangle (36%) and the ability to identify three-dimensional shapes increased most at cone (66%).

**Table 7**

*Post-test research analysis n=15, teachers' analysis n=14. Result and inter-reliability and equivalences of the post-test*

Geometric shapes	Post-test				Result and Analysis
	Researchers' analysis Percent	Change in percent pre- and post-test	Teachers' analysis Percent	Change in percent pre- and post-test	
Quadrant	100	29	100	7	15 of 15 had correct answer.
Triangle	100	14	100	14	15 of 15 had correct answer
Circle	100	0	100	0	15 of 15 had correct answer
Rectangle	100	36	100	0	15 of 15 had correct answer
Cube	100	43	100	43	15 of 15 had correct answer
Cone	80	66	71	57	12 of 15 had correct answer. Two students answered wrong and one student did not answer.
Sphere	67	60	57	50	10 of 15 had correct answer. Five did not answer.
Cuboid	53	53	50	50	8 of 15 had correct answer. One wrote "cuboids-sphere". Six did not answer.

#### *Students' Experiences of the Post-test*

In both the pre-test and post-test (Table 8), the teachers added a closing question: "Was it easy or difficult?", where the students were able to leave a comment.

**Table 8**

*Students' post-test experiences*

Correct answers	Easy	Medium	Difficult	No answer	Total (n=8)
8	6				6
7	1	1			2
6	1	2	2		5
5	2				2
4					0
3					0
2					0
1					0
0					0

In the post-test, six students' (compared with none in the pre-test) achieved 8 of 8 correct answers. Also, more students in the post-test stated that they had more easily distinguished the geometric shapes: 10 in the post-test verses 2 in the pre-test. These indicated that the student's self-assessment of their prior knowledge corresponded with reality. The result also showed that there are changes in knowledge before and after the research lesson. First, the students' understanding is made apparent in the differences between the pre- and post-test results regarding both two- and three-dimensional shapes. In the post-test, all students identified the two-dimensional shapes, as well as the cube. There was also an increase in the students' ability to identify the remaining three three-dimensional shapes. The second change regarding the student's knowledge was related to their understanding of their self-assessed experience which was in line with reality.

*Necessary Conditions for Students to Understand the Differences Between  
Two- and Three-dimensional Shapes*

The teachers discussed the students' knowledge from the post-test as it was related to the lesson. In the audio-recorded material it is clear that T1 was not completely satisfied with his introduction of the object of learning: "I do not think much about the increase itself, but more about my own clarity /.../." The teacher's statement indicates that he has a greater interest in his introduction and clarity regarding the content than whether the students' post-test results will increase. The teacher (T1) is reflecting on his own awareness of how the teaching actions will affect the students' understanding of the content.

After the lesson, the teachers discussed how they perceived the opportunities for the students to discern the geometric shapes.

**Excerpt 4:**

Teacher 1: It became easier for them when I showed them a cone, instead of just talking about cones in the presentation; they suddenly knew what a cone was. Then they knew, and then they could name it. So that was good. And I used, for example, toy blocks /.../, which I also gave out to the students when we talked about the cube.

Researcher: So, you not only showed the things you had with you [in the lesson], but you also let the students touch [experience] them?

Teacher 1: Yes, and we also talked about whether we can find more things in the room that are cubes, we looked specifically at the cuboids. Then we discussed boxes and furniture and other stuff that was in the room.

Before the lesson, the teachers discussed the importance of having three-dimensional artefacts to use during the lesson, and to synchronically visualise the mathematical shapes as they relate to everyday objects. Later, they discussed the opportunities for distinguishing the spatial properties and spatial relations of the quadrant and the cube by measuring the area and volume.

**Excerpt 5:**

Teacher 1: But if I said, "square we can check the area of", then after a while, someone may be able to explain volume, but the two students know it and get a lot more space to talk, which allows them to learn from each other /.../.

Researcher: Could you recognise the students' knowledge in any way?

Teacher 1: /.../ I could see them collaborating, even if they work individually, they collaborate. For example, somebody could say, "I need a large rectangle, can you measure that for me?" Or, "Has anyone seen my cube?" So, they use the words.

The excerpt above shows that the teachers are still discussing collaboration opportunities to increase students' knowledge. There is also a connection to the organisational design where the teachers discuss students' collaboration about the object of learning. However, still, the teachers emphasised instruction of peer learning as a knowledge-generating activity. The students are given opportunities through the teachers' lesson design by visualising daily artefacts synchronic with the picture of the three-dimensional shapes through the PowerPoint instructions to interact and thereby develop and extend their previous knowledge of the object of learning.

The activity allowed the students to create geometric shapes using clay and paper (after the PowerPoint introduction) and encouraged them to make the three-dimensional shapes visibly and recognisably (see Figure 5).

#### **Excerpt 6:**

Teacher 1: I think it was because they got to see these shapes [in the teacher's PowerPoint presentation], that they could communicate the differences between the shapes, the properties of the shapes and then also that the shapes were set against each other, and also linked to their two-dimensional shapes. So, it was linked to something they recognised before and compared with something that was different, and that they had to work with the shapes with their hands. So maybe they got an understanding of what the shapes look like.

Researcher: Do you mean with the support of the clay?

Teacher 1: Yes, they got to know it. It was tactile, it was visual.

Teacher 2: Together with tactile and visual stimulation, they developed their knowledge of basic geometric shapes [with those sitting next to them].

The teachers continued to reflect on why the students were able to perceive the cube's properties. One reflection is related to the communications between the peers when they discussed the structure of the cube: "Because of the sides, that side is bigger than that side, so it's a rectangle" (students voice through the teachers' reflections). But the latter can also be reflected in the object of learning.

Another condition necessary to acquire knowledge about three-dimensional shapes is, according to the teachers, when the students are guiding each other to create round shapes.

#### **Excerpt 7:**

Teacher 1: I make the circle, then [someone says there] it is three-dimensional, and then maybe someone says "globe". And then someone else could say, "But you shouldn't do like that. It is better to roll the meatball or the chocolate ball" and then they started to roll and show with their hands.

The teachers indicated that the students had gained knowledge that a circle and a sphere have the same circular properties. The teachers end their conversation by highlighting how they rolled a ball and slid the cube on the table between the students.

In the next excerpt the students' developed conceptual understanding through art constructions according to the teachers.

### Excerpt 8:

Teacher 1: We have talked about that, we have looked at that when we talked about differences between two- and three-dimensional shapes, really, what differentiates them. /.../ Then we talked about cubes and squares a lot, and they drew and talked about how to draw, for example, how many directions you can draw lines in three-dimensional shapes, etc. So, there was a little more focus on the dimensions.

In the excerpt the teachers are sharing their experiences of the research lesson. The teachers highlighted how the students talked about geometrical shapes in other subjects as well.

### Excerpt 9:

Teacher 1: We build things now [at the art lesson] that are two and three-dimensional, and then you hear people working together, they usually do. I could see them collaborating, even if they work individually, they collaborate. For example, somebody could say, "I need a large rectangle, can you measure that for me?" Or, "Has anyone seen my cube?" So, they use the words.

By offering daily artefacts simultaneously with pictures in the PowerPoint, the students are giving opportunities to transforming mathematical representations in other contexts, i.e. in the subject of art.

In the following excerpt, the teacher discusses their awareness of their professional development after the lesson together with the researcher.

### Excerpt 10:

Researcher: What I hear you say /.../ is that the difference is that you find that *you* talk less and instead let the students discuss and explain more based on the material that you have bring to the classroom? And then you hear the students' use words like "depth", "volume", "area"? and that they describe and help each other?

Teacher 1: Mm. Yes exactly. /.../ I think also it can have been depending on me.

Teacher 2: It is not only about you and your learning – it is also about to be able to *see* when you can step in into the students' conversation and raise student's and give them the opportunity to discuss and share their knowledges.

Unlike Excerpt 1, when the teachers plan their lesson, they discuss using the concept of mathematics sparingly since, in this student's group, they have experienced that some students react inhibitably and with a "red flag" for the subject. Their reflections after the lesson were slightly different. There seems to be a developed thought between what the teachers discussed before the lesson and after the students' learning outcomes. When the teachers reflected together with the researcher after their lesson, they discussed their experiences. They highlighted the possibility that several mathematical concepts can be made visible in other subjects (see Excerpt 8). Similarly, in Excerpt 10, the teachers discuss the importance of paying attention to the students' learning outcomes as well as capturing their professional awareness.

## Discussion

This research identified teachers' awareness of how students in fourth grade distinguished all two-dimensional shapes and three-dimensional cube, and cone through visual representation (Van Hiele, 1986). The teacher's PowerPoint instruction showed that most of the shapes (Table 2) are Level 1 visual representations (Van Hiele, 1986). Only the cube offered a distinction from

Level 2 through different rotation directions (Excerpt 6). The latter is also synchronic with the two-dimensional shape quadrant (Excerpt 6). In Excerpt 8, the teacher reflected about “how many directions you can draw with lines in three-dimensional shapes” as cubes synchronic with two-dimensional squares. According to Van Hiele (1986), the teacher offered a conceptual understanding of the cubes on Level 5. This can also be related addressed to Hallowell et al. (2015) results about difficulties at this year of age to transform curves from two- to three-dimensional. In the post-test (Table 7) and the teachers' assumptions, it is the cube that has the most increased understanding (see Excerpts 4, 5, 6, 7 and 8). The sphere does not increase at all, even though the students' during the pre-test showed an understanding of a circle. However, the organisation in the classroom provides the students to interact and discuss all the shapes. But all the three-dimensional shapes were not highlighted by the teacher in the formal mathematical language (cf Sfard, 2007). For example, they used “globes” and “meatballs” instead of the sphere (Excerpt 7). After the lesson, the teachers reflected on the importance of incorporating everyday objects such as cones, toy blocks and globes

while showing the shapes in their PowerPoint presentation. In the result, the teachers illustrated the cuboids only by pointing to furniture and boxes in the classroom. Regarding the cuboids, students are allowed to see the similarities only based on a non-rotating visual representation: both in the PowerPoint presentation and the everyday objects. Cuboids, such as the rectangle, are distinguished only in their horizontal form (Level 1 Van Hiele, 1986). Even though the students are working to identify the similarities between the cuboid and rectangle (cf Maybarr, 1983) during the lesson, there are still 8 of 15 students who do not respond correctly during the post-test (see Table 7). The latter indicated that even if the students are offered concrete examples of the shapes (Piaget, 1972), they still do not develop their understanding and awareness of all the three-dimensional shapes equally. By just showing concrete examples does not contribute to increased understanding.

Through tactile visualisation with the support of clay and paper, teachers discussed on different techniques, such as meatball production as support for developing the spatial ability for identifying spheres. Sfard (2007) argued for the importance of making the mathematical content visible when developing collaborating classrooms. However, organising for a more collaborative, linguistic environment is not enough (Excerpt 2). The teacher must also contribute to the formal mathematical language (Sfard, 2007). It can thus be debated whether the mathematical concept of the globe has been highlighted in the various activities, such as taking in a globe (though not naming it as a sphere) and discussing “meatball” production rather than describing a “sphere”. The result also emphasized that the shape of “sphere” as a concept also has the second-worst percentage during the post-test (see Table 7).

According to the teacher's assumption by incorporating everyday objects, it increased the students' understanding of three-dimensional shapes (cf Smith, 2001) (Excerpt 4). The teachers assume that the students have developed their spatial ability to understand their surroundings (Gardner, 1993). This is also related to the abstract thinking to be able to discern the rotation of the globe in different spatial spaces (Clements & Battista, 1992). In the results, the teachers offered students opportunities for tactile and visual representations through clay and paper activities which stimulated student's imagination of properties of the shapes (Ball & Cohen, 1999; Klim-Klimaszewska & Nazaruk, 2020). For example, in Excerpts 6 and 8, teachers assume that students need to see and experience the shapes at the same time so that they can discern similarities and differences in the proportions of the shapes. This could contribute to both horizontal and vertical mental spatial understanding through visual rotation (Maier, 1996).

The teachers are not certified mathematics teachers, still there is an increased understanding by the students of several two-and three-dimensional shapes (Table 5 and 7). One of the teachers is certified in art in grades 1–9 and in the syllabus in the Swedish curriculum of the subject of art some of the goals is: “Drawing, painting, printing and three-dimensional

production [...]. Different elements that make up and create a sense of space in pictures, such as lines [...] and how these can be used when creating pictures" (SNAE, 2018, p. 28). This could be interpreted as the goals in the subject of art providing support for the teachers in this research to interpret even the mathematical goals. The students' increased understanding of two- and three-dimensional shapes could therefore be related to the similarities of these curriculum goals regarding the importance of developing students' spatial abilities (Ball & Cohen, 1999). Furthermore, the teachers' instruction by offering the students an added final question in the pre- and post-test about their experiences of the test, whether it was difficult or easy, better aligns with the students' actual post-test knowledge. It could be interpreted as the teachers' construction of the research lesson contributing to the students being offered to reason similarities and differences between two- and three-dimensional shapes (Excerpts 2, 3 and 5). This could also achieve the curriculum's following goals in the subject of art: "Words and terms for interpreting, writing and discussing the picture's design and message" (SNAE, 2018, p. 29), and through these objectives at the same time may have contributed to reasoning and thus contributed to students' conceptual understanding of the cube and cone (Ball & Cohen, 1999). Nevertheless, it is possible to think that a limitation is that the teachers in this research are not certificated mathematics teachers and therefore explain why the pre- and post-tests only test the students' spatial knowledge on Level 1 (Van Hiele, 1986). Despite this it could be discussed in relation to the fact that students mathematical two- and three-dimensional abilities increase (Clements & Battista, 1992; Olkun, 2003; Thurstone, 1950).

The results of this research addressed the importance of teachers' developing knowledge about designing tests based on the chosen object of learning (Ball & Cohen, 1999). We find further support that teachers are choosing the object of learning based on their need to develop their everyday practice in the classroom (Smith, 2001). According to Smith (2001), teachers should further develop an understanding of topics, and pedagogy—components of a teacher's knowledge base for teaching—by designing tasks that are central to teaching.

The results from this research strengthen earlier discussions that PBDP research is rooted in the issues and challenges that school professionals face and deal with in their day-to-day practice, and to give equal opportunities to develop abilities and build new knowledge (Ball & Cohen, 1999; Christensson, 2018). According to Harris et al. (2012), a teacher-driven approach promotes new skills in the classroom. This research leads to knowledge that teachers can use to improve their teaching and working methods as well as their ability awareness to make professional judgments about development and learning (Carlgren, 2005, 2011).

In this research, the learning process (Wood, 2019) has emphasised the teaching targets for understanding instead of teaching as telling, which had been the important direction for the teachers (Lewis, 2009). In two Excerpts (4 and 6), the researchers guided the teachers in their collaborative effort to improve their classroom PBDP, reflecting over the teacher learning process and analysing the merits and disadvantages of the research lesson.

The research model for the teachers' research design was inspired by the Japanese collaborative professional development model lesson study (Doig & Groves, 2011; Lewis, 2002, 2009; Lewis & Tsuchida, 1998; Stigler & Hiebert, 1999). The result showed increased awareness of using collaboration opportunities to apply classroom (Vrikki et al., 2017; Warwick et al., 2019) instructions and activities to increase students' knowledge about the object of learning (Fujii, 2016) two-and three-dimensional shapes.

## Conclusions

The results of this research highlight the students' (in primary school) increased understanding of two- and three-dimensional shapes even though the teachers in this research are not certified maths teachers. This research illuminates how the PBDP approach stressed the

teachers' teaching targets for understanding students' content knowledge of geometric shapes. The results provide evidence that PBPD research approach, containing reflective discussions with a researcher, developed teacher's awareness and understanding of necessary conditions to enhance students' abilities to discern two- and three-dimensional shapes. The analysis of the research data identified teachers' awareness and understanding, which challenges students need to face in learning about three-dimensional shapes from two-dimensional representations.

Concerning this result, further research should effort more iterative and revised research lesson design to develop more powerful content knowledge and classroom activity in this topic area. The findings also pointed out that there is a need to examine why artefacts did not increase all three-dimensional shapes. Additionally, more studies are needed to develop a PBPD framework to promote a more conceptual teacher understanding for lesson design and support a more forceful evidence-based teaching approach in geometry in primary school.

## Acknowledgements

The authors would like to thank their supervisor's professor Mona Holmquist and the lecturer Jonas Alwall for guidance and support. The authors would also want to thank the reviewers for the comments and guidance. The authors want to thank colleagues at Malmo University and the faculty of Education and Society. This research is a part of the Swedish National Research School of Special Education for Teacher Educators (SET), funded by the Swedish Research Council (grant no. 2017-06039), for which the authors are grateful.

## References

- Ball, D., & Cohen, D. (1999). Developing practice, developing practitioners: Toward a practice-based theory of professional education. In L. Darling-Hammond & G. Sykes (Eds.), *Teaching as the learning profession* (pp. 3–32). Jossey-Bass Education Series.
- Carlgren, I (2005). *Forskning och skola i samverkan – Kartläggningar av forskningsresultat med relevans för praktiskt arbete i skolväsendet* [Research and school in cooperation. Survey of research results relevant to practical work in the school system.] Vetenskapsrådet [Swedish research council.]
- Carlgren, I. (2011). Forskning ja, men i vilket syfte och om vad? Om avsaknaden och behovet av en 'klinisk' mellanrumsforskning. [Research yes, but for what purpose and for what? About the lack and need for a 'clinical' gap research] *Forskning om undervisning och lärande*. [Research on Teaching and Learning], 5(2), 64–79.
- Chapman, R. S. (1997). Language development in children and adolescents with Down syndrome. *Mental Retardation and Developmental Disabilities Research Reviews*, 3(4), 307–312. [https://doi.org/10.1002/\(SICI\)1098-2779\(1997\)3:4<307::AID-MRDD5>3.0.CO;2-K](https://doi.org/10.1002/(SICI)1098-2779(1997)3:4<307::AID-MRDD5>3.0.CO;2-K)
- Christersson, C. (2018). *Forska tillsammans: samverkan för lärande och förbättring* [Research together: Collaboration for learning and improvement]. The Government's Official Investigations, SOU 2018:19. Norstedts juridik.
- Clements, D. H., & Battista, M. T. (1992). Geometry and spatial reasoning. In D. A. Grouws (Ed.), *Handbook on mathematics teaching and learning* (pp. 420–464). Macmillan.
- Doig, B., & Groves, S. (2011). Japanese lesson study: Teacher professional development through communities of inquiry. *Mathematics Teacher Education and Development*, 13(1), 77–93.
- Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, 61, 103–131. <http://dx.doi.org/10.1007/s10649-006-0400-z>
- French, D. (2004). *Teaching and learning geometry: Issues and methods in mathematical education*. Continuum.
- Fujii, T. (2016). Designing and adapting tasks in lesson planning: A critical process of lesson study. *ZDM Mathematics Education*, 48(4), 411–423. <http://dx.doi.org/10.1007/s11858-016-0770-3>

- Gardner, R. B., & Kamran, N. (1993). Characteristics and the geometry of hyperbolic equations in the plane. *Journal of Differential Equations*, 104(1), 60-116. <https://doi.org/10.1006/jdeq.1993.1064>
- Gardner, D.C. (2011). Characteristic collaborative processes in school-university partnerships. *Planning & Changing*, 42(1/2), 63–86.
- Goldsmith, L. T., Doerr, H. M., & Lewis, C. C. (2014). Mathematics teachers' learning: A conceptual framework and synthesis of research. *Journal of Mathematics Teacher Education*, 17(1), 5–36. <http://dx.doi.org/10.1007/s10857-013-9245-4>
- Goodlad, J.I. (1991). School–university partnerships: Fundamental concepts. *School of Education Review*, 3(2), 36–42.
- Halliday, M. A. K. (1981). Linguistic function and literary style: An inquiry into the language of William Golding's *The Inheritors*. In D. C. Freeman (Ed.), *Essays in modern stylistics* (pp. 325-360). Routledge.
- Hallowell, D. A., Okamoto, Y., Romo, L. F., & La Joy, J. R. (2015). First-graders' spatial-mathematical reasoning about plane and solid shapes and their representations. *The International Journal on Mathematics Education*, 47(3), 363-375. <http://dx.doi.org/10.1007/s11858-015-0664-9>
- Harris, K. R., Lane, K. L., Graham, S., Driscoll, S. A., Sandmel, K., Brindle, M., & Schatschneider, C. (2012). Practice-based professional development for self-regulated strategies development in writing: A randomized controlled study. *Journal of Teacher Education*, 63(2), 103–119. <http://dx.doi.org/10.1177/0022487111429005>
- Herro, D, Hirsch, S. E., & Quigley, C. (2019). A faculty-in-residence programme: Enacting practice-based professional development in a STEAM-focused middle school. *Professional Development in Education, December 2019*, 1–17. <https://doi.org/10.1080/19415257.2019.1702579>
- Hiebert, J., Gallimore, R., & Stigler, J.W. (2002). A knowledge base for the teaching profession: What would it look like and how can we get one? *Educational Researcher*, 31(5), 3–15. <https://doi.org/10.3102/0013189X031005003>
- Holmqvist, M., & Mattisson, J. (2008). Variation theory: A tool to analyse and develop learning at school. *Problems of Education in the 21st Century*, 7(7), 31–38. <http://oaji.net/articles/2014/457-1392234817.pdf>
- Klim-Klimaszewska, A., & Nazaruk, S. (2017). The scope of implementation of geometric concepts in selected kindergartens in Poland. *Problems of Education in the 21st Century*, 75(4), 345–353. <http://www.scientiasocialis.lt/pec/node/1072>
- Lewis, C. (2002). *A handbook of teacher-led instructional change*. Research for Better Schools.
- Lewis, C. (2009). What is the nature of knowledge development in lesson study? *Educational Action Research*, 17(1), 95-110. <https://doi.org/10.1080/09650790802667477>
- Lewis, C. (2016). Learning to lead, leading to learn: How facilitators learn to lead lesson study. *ZDM: The International Journal on Mathematics Education*, 48(4), 527–540. <http://dx.doi.org/10.1007/s11858-015-0753-9>
- Lewis, C., & Tsuchida, I. (1998). A lesson is like a swiftly flowing river: How research lessons improve Japanese education. *American Educator*, 22(4), 12–17, 50–52.
- Maier, P. H. (1996). Spatial geometry and spatial ability – how to make solid geometry solid? In E. Cohors-Fresenborg, K. Reiss, G. Toener, & H-G. Weigand (Eds.), *Selected Papers from the Annual Conference of Didactics of Mathematics 1996* (pp. 69–81). Osnabrueck.
- Marchiş, I. (2008). Geometry in primary school mathematics. *Educația*, 21(6), 131–139.
- Mayberry, J. (1983). The van Hiele levels of geometric thought in undergraduate preservice teachers. *Journal for Research in Mathematics Education*, 14(1), 58–69.
- OECD (2016). *PISA 2015 Results (Volume I): Excellence and equity in education*. OECD.
- Olkun, S. (2003). Making connections: Improving spatial abilities with engineering drawing activities. *International Journal of Mathematics Teaching and Learning*, 3(1), 1–10.
- Pêna Trapero, N. (2013). Lesson study and practical thinking: A case study in Spain. *International Journal for Lesson and Learning Studies*, 2(2), 2013, 115–136. <https://doi.org/10.1108/20468251311323379>
- Piaget, J. (1972). Intellectual evolution from adolescence to adulthood. *Human Development*, 15(1), 1–12, 72.
- Seago, N., Jacobs, J., Driscoll, M., Nikula, J., Matassa, M., & Callahan, P. (2013). Developing teachers' knowledge of a transformations-based approach to geometric similarity. *Mathematics Teacher Educator*, 2(1), 74–85. <https://www.jstor.org/stable/10.5951/mathteaceduc.2.1.0074>

- Sfard, A. (2007). When the rules of discourse change, but nobody tells you: Making sense of mathematics learning from a commognitive standpoint. *Journal of the Learning Sciences*, 16(4), 565-613. <http://dx.doi.org/10.1080/10508400701525253>
- Skott, J., Mosvold, R., & Sakonidis, C. (2018). Classroom practice and teachers' knowledge, beliefs and identity. In T. Dreyfus, M. Artigue, D. Potari, S. Prediger, & K. Ruthven (Eds.), *Developing research in mathematics education: Twenty years of communication, cooperation, and collaboration in Europe* (pp. 162–180). Routledge.
- Smith, M. S. (2001). *Practice-based professional development for teachers of mathematics*. National Council of Teachers of Mathematics, U.S.
- Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. Free Press.
- Stigler, J. W., & Hiebert, J. (2016). Lesson study, improvement, and the importing of cultural routines. *ZDM: The International Journal on Mathematics Education*, 48(4), 581–587. <http://dx.doi.org/10.1007/s11858-016-0787-7>
- Swedish National Agency for Education. (2018). *Curriculum for the compulsory school, preschool class and school-age educare*. 2011 Revised 2018. Stockholm: Norstedts Juridik kundservice.
- Swedish Research Council (2017). *God forskningssed*. [Good Research Practice]. Swedish Research Council.
- Takahashi, A., & McDougal, T. (2016). Collaborative lesson research: Maximizing the impact of lesson study. *ZDM: The International Journal on Mathematics Education*, 48(4), 513–526. <http://dx.doi.org/10.1007/s11858-015-0752-x>
- Thurstone, L. L. (1950). Some primary abilities in visual thinking. *Proceedings of the American Philosophical Society*, 94(6), 517–521. American Philosophical Society.
- Van Hiele, P. M. (1986). *Structure and insight. a theory of mathematics education*. Academic Press.
- Vrikki, M., Warwick, P., Vermunt, J. D., Mercer, N., & Van Halem, N. (2017). Teacher learning in the context of Lesson Study: A video-based analysis of teacher discussions. *Teaching and Teacher Education*, 61(2017), 211–224. <https://doi.org/10.1016/j.tate.2016.10.014>
- Warwick, P., Vrikki, M., Færøyvik Karlsen, A. M., Dudley, P., & Vermunt J. D. (2019). The role of pupil voice as a trigger for teacher learning in Lesson Study professional groups. *Cambridge Journal of Education*, 49(4), 435–455. <https://doi.org/10.1080/0305764X.2018.1556606>
- Wilson, K. M., & Swanson, H. L. (2001). Are mathematics disabilities due to a domain-general or a domain-specific working memory deficit? *Journal of Learning Disabilities*, 34(3), 237–248. <https://doi.org/10.1177/002221940103400304>
- Wood, K. (2019). The path of teachers' learning through lesson and learning studies. *International Journal for Lesson & Learning Studies*, 9(2), 93–99. <https://doi.org/10.1108/IJLLS-12-2019-0083>
- Yoshida, M. (1999). *Lesson study: A case study of a Japanese approach to improving instruction through school-based teacher development*. Ph.D. Dissertation, The University of Chicago, Illinois, Chicago.

Received: May 05, 2020

Accepted: September 20, 2020

Cite as: Lelinge, B., & Svensson, C. (2020). Teachers' awareness and understanding of students' content knowledge of geometric shapes. *Problems of Education in the 21<sup>st</sup> Century*, 78(5), 777-798. <https://doi.org/10.33225/pec/20.78.777>

**Balli Lelinge**  
(Corresponding author)

PhD Student, University Lecturer, Malmö University, 20506 Malmö, Sweden.  
E-mail: [balli.lelinge@mau.se](mailto:balli.lelinge@mau.se)  
Website: <https://mau.se>  
ORCID: <https://orcid.org/0000-0003-2435-0913>

**Christina Svensson**

PhD Student, University Lecturer, Malmö University, 20506 Malmö, Sweden.  
E-mail: [christina.svensson@mau.se](mailto:christina.svensson@mau.se)  
Website: <https://mau.se>  
ORCID: <https://orcid.org/0000-0003-1815-887X>