ON F-PRIME RINGS AND THEIR F-RINGS OF QUOTIENTS

Abstract: All rings are associative in what follows. We will also assume that all rings are unital and all ring homomorphisms preserve the identity. Throughout the paper, B stands for a unitary associative ring and f is an automorphism of B. The main purpose of the work is to describe the left f-ring of quotients $Q_f(B)$ of B. Our methods is not connected with right flat epimorphic hull, but we use a fruitful construction based on a direct limit to build the f-ring of quotients. The principal results to be given are Theorems 1 and 2 below. Theorem 1 establishes that $Q_f(B)$ is embeddable in the complete left ring of quotients $Q_{\text{max}}(B)$ and the ring B is a subring of $Q_f(B)$. Theorem 2 asserts that the centres of the left and right f-ring of quotients rings coincides. The authors have intention to continue this study in their subsequent articles. Therefore, at the end of the paper, we formulate a hypothesis, for the proof of which we need the results of this work.

Key words: associative rings; f-rings of quotients; f-prime rings.

Language: English


Introduction

All rings in this paper are associative with multiplicative identity and all ring homomorphisms are assumed preserve the identity.

Definition 1. Let R be a ring and F be an injective ring endomorphism of R.

Let N be a subset of R. We say that N is an F-subset if $F^{-1}(N) = N$. Similarly, we shall say that an ideal N of R is an F-ideal if N is an F-subset (see [3], [4], [6], [14]). A ring R is said to be F-prime if the product of any two nonzero F-ideals of R is nonzero. In other words, ring R is F-prime if the product of any two F-ideals P and Q is equal to the zero ideal if and only if either $P = 0$ or $Q = 0$.

Lemma 1. Let A be a ring and f be an automorphism of A. Then the following conditions (1)–(5) are equivalent:

1. $A$ is f-prime;
2. For any two left f-ideals I and J of A, the equality $IJ = 0$ implies that either $I = 0$ or $J = 0$;
3. $a \sum_{i \in \mathbb{Z}} A f^i(b) \neq 0$ for every $a, b \in A$;
4. $r_A(I) = 0$ for every nonzero f-ideals I of A;
5. $l_A(I) = 0$ for every nonzero f-ideals I of A.

Materials and Methods

Throughout the sequel, B will denote a ring, f will stand for an automorphism of B and we will always assume that the ring B is f-prime. We denote by $\Phi(B)$ the set of all nonzero f-ideals of B.

Despite the fact that rings of rings for along time are classical objects of study, the rings of quotients have been actively studied recently (see,
for example, [1], [2]). Now we define the left f-ring of quotients of the ring B.

We denote by M the set of all the pair of the form (I, α) where I ∈ Φ(B) and α : l → B is a homomorphism of left B-modules. Since B is f-prime, we have 0 ≠ I ⊆ l ∩ J for all I, J ∈ Φ(B).

Определение отношения эквивалентности на множество M max: положим (I, α) ~ (J, β), если существует ненулевой f-идеал K ⊆ l ∩ J такой, что α(x) = β(x) для всех. Let us define the equivalence relation θ on the set M as follows: we put (I, α) ~ (J, β) if there exists a nonzero f-ideal K ⊆ l ∩ J such that α(x) = β(x) for all x ∈ K. We denote by [I, α] the equivalence class containing the pair (I, α). Let Q_f(B) = M/θ (the set of all equivalence classes).

The set Q_f(B) turn into a ring if we define on it the following operations:

\[
[I, \alpha] + [J, \beta] = [I \cap J, \alpha + \beta],
\]
\[
-[I, \alpha] = [I, -\alpha],
\]
\[
[I, \alpha] \cdot [J, \beta] = [I \cap J, \alpha \circ \beta].
\]

Definition 2. The ring Q_f(B) described above is called the left (Martindale) f-ring of quotients of B.

Remark. One can formalize the construction of Q_f(B) as the direct limit

\[
Q_f(B) = \lim_{\rightarrow}(\text{Hom}_B(g_\lambda, B_\gamma); I \in \Phi(B)).
\]

If every ideal of B is an f-ideal, then Q_f(B) coincides with the left ring of quotients in the sense of Martindale.

We define analogously the right f-ring of quotients of B (in the sense of Martindale) as follows:

\[
Q^r_f(B) = \lim_{\rightarrow}(\text{Hom}_B(B_\gamma, g_\lambda); I \in \Phi(B)).
\]

Lemma 2. Let I ∈ Φ(B), and be a left ideal of B and α, β : l → B be a homomorphism of left B-modules. Suppose that α(x) = β(x) for all x ∈ l \cap J. Then α = β.

Proof. Take any b ∈ J. If a ∈ l then ab ∈ l \∩ J and α(ab) = β(ab). Hence, α(ab) = β(ab) = 0 for all a ∈ I. But r^α(I) = 0 by Lemma 1. It follows that α(b) = β(b) for all b ∈ J. QED.

The above f-ring of quotients Q_f(B) possesses some properties of the usual Martindale ring of quotients. In particular, we have the following theorem.

Theorem 1. The f-ring of quotients Q_f(B) is embeddable in the maximal left ring of quotients Q_{max}(B) and the ring B is a subring of the ring Q_f(B).

Proof. If I ∈ Φ(B) then r^α(I) = 0 by Lemma 1 and, hence, f is a dense right ideal of B. Therefore, to each element q = [I, α] ∈ Q_f(B) can be matched up with the element φ(q) ∈ Q_{max}(B) such that xφ(q) = α(x) for all elements x ∈ I.

One can easily verify that the mapping q → φ(q) is a correctly defined injective ring homomorphism. Furthermore, according to the construction φ(Q_f(B)) = {q ∈ Q_f(B) : there exists an ideal I ∈ Φ(B) such that Iq ⊆ B}. Each element α ∈ B defines the homomorphism of left B-modules

\[
a_\alpha : x \mapsto xa (x \in B).
\]

Since B ∈ Φ(B), the equivalence class [B, α] belongs to Q_f(B). This gives a natural embedding B ⊆ Q_f(B). QED.

In what follows we identify the rings Q_f(B) and Q_f(B).

Let q ∈ Q_f(B). We denote by J(q) the sum of all f-ideals I of B such that Iq ⊆ B. Observe that J(q) itself is also an f-ideal and J(q) : q ∈ B. Таким образом, J(q) – наибольший идеал среди f-идеалов I кольца B, обладающих свойством Iq ⊆ B.

Thus, J(q) is the largest ideal among f-ideals I of B having the property Iq ⊆ B.

If q ∈ Q_{max}(B) then we set D(q) = {α ∈ Q_{max}(B) : αq ∈ B}. As is well known in the theory of rings of quotients, the automorphism f can be uniquely extended to an automorphism of Q_{max}(B).

We will denote this extension by the same symbol f.

\[
D(f(q)) \cdot q = f(D(q))q \in f(B) = B
\]

and

\[
f^{-1}(D(f(q))) \cdot q = f^{-1}(D(f(q)))f(q) \in f^{-1}(B) = B.
\]

We get f(D(q)) = D(f(q)).

Let us consider any element q ∈ Q_f(B) and note the following: if x ∈ I(q) then both f(x) and f^{-1}(x) lie in J(q), and therefore we have that xφ(q) = f^{-1}(xφ(q)) = f^{-1}(f^{-1}(xφ(q))) = f^{-1}(f^{-1}(f(q))) ∈ B. It follows that f(q) ∈ B and f(q) = f^{-1}(f(q)) ⊆ B. Consequently, f(Q_f(B)) ⊆ Q_f(B) and f^{-1}(Q_f(B)) ⊆ Q_f(B). Thus, f can be regarded as an automorphism of Q_f(B).

Proposition 1. The f-ring of quotients Q_f(B) introduced above has the following properties.

(1). For any elements q_1, q_2, ..., q_n ∈ Q_f(B), there exists an ideal I ∈ Φ(B) such that Iq_i ⊆ B for all i = 1, 2, ..., n.

(2). If Iq_i = 0 for some ideal I ∈ Φ(B) and some element q ∈ Q_f(B), then q = 0.

(3). If qI = 0 for some ideal I ∈ Φ(B) and some element q ∈ Q_f(B), then q = 0.

(4). If I ∈ Φ(B) and γ : l → B is a homomorphism of left B-modules, then there is an element q ∈ Q_f(B) such that γ(x) = xq for all elements x ∈ I.

(5). Q_f(B) is an f-prime ring.
Proof. (1). As the required ideal, we can take the ideal
\[ I = J(q_1) \cap J(q_2) \cap \ldots \cap J(q_n). \]
It works, because B is an f-prime ring and
\[ 0 \neq f(q_1)f(q_2) \cdots f(q_n) \subseteq J(q_1) \cap J(q_2) \cap \ldots \cap J(q_n). \]

(2). This follows from Lemma 2 and the definition of the f-ring of quotients.

(3). Observe that \( P = \sum a_i J(a_i) f_i(q_i) \) is the left f-ideal of B and \( P \cap \sum a_i J(a_i) f_i(q_i) = 0. \) Since is an f-prime ring, then \( P = 0. \) Therefore, \( f(q_i)q = 0 \) and \( q = 0. \)

(4). This assertion follows immediately from Definition 2.

(5). Let a and b be two non-zero elements of \( Q_f(B). \) Then it follows from (2) that \( \sum a_i J(a_i) f_i(a) \) and \( \sum a_i J(b_i) f_i(b) \) are non-zero left f-ideals. Since B is an f-prime ring, we get that \( \sum a_i J(a_i) f_i(a) \cdot \sum a_i J(b_i) f_i(b) = 0. \) Hence, it follows that a ⋅ \( \sum a_i J(b_i) f_i(b) \) = 0 and, by Lemma 1, \( Q_f(B) \) is an f-subring. QED.

We denote by \( C(B) \) and \( C^*(B) \) the center of the left maximal and right maximal rings of quotients of B, correspondingly. One more piece of notation: \( C_f(B) = \{ c \in C(B) : f(b) = b \} \) and \( C_f^*(B) = \{ c \in C^*(B) : f(b) = b \} \).

Theorem 2.

(1). If \( q \in C_f(B) \), then the set \( D(q) = \{ b \in B : b \in B \} \) is an f-ideal.

(2). \( C_f(B) \) is the center of \( Q_f(B) \).

(3). If \( q = [1, a] \in Q_f(B) \), then \( q \in C_f^*(B) \) if and only if \( a \) is a homomorphism of bimodules over the ring \( B \).

(4). The rings \( C_f(B) \) and \( C_f^*(B) \) are isomorphic.

Proof. (1). Let \( a \in B \). Then \( qa \in B \) if and only if \( f(qa) = f(a) q \in B \). Therefore, \( D(q) \in \Phi(B) \).

Assertion (2) follows from assertion (1) since \( C_f(B) \) is the centralizer of the set \( B \) in \( Q_{max}(B) \).

(3). If \( a \) is a homomorphism of bimodules over the ring \( B \), then, as easily seen, \( q \in C(B) \) and therefore \( q \in C_f(B) \). The converse follows from assertion (1).

(4). Using assertion (3), one can prove that both rings \( C_f(B) \) and \( C_f^*(B) \) are isomorphic to the ring \( \{ c \in \text{Hom}(B, B) : f(c) = c \} \).

QED.

Theorem 3. Let B be an f-prime ring and \( q \neq 0 \in Q_f(B) \). If \( f(q) \) = \( q \) for all elements \( a \in B \), then \( q \) is an invertible element of the ring \( Q_f(B) \) and \( f \) is an inner automorphism of the ring \( Q_f(B) \) defined by the element \( q \):

\[ f(p) = q^{-1}pq \quad (\forall p \in Q_f(B)). \]
Conclusion
At the end of the present work, let us make the following hypothesis, which will be verified in the subsequent author’s works:

Let $R$ be a ring and $F$ be an injective endomorphism of $R$. Suppose that the skew polynomial ring $R[x, F]$ is semiprime. Let $(A, f)$ be the Cohn-Jordan extension of the pair $(R, f)$. This is extension described in [3], [4], and [5]. We denote by $Q$ the maximal left ring of quotients of $A$ and denote by $D_0$ the orthogonal completion of the center of the ring of oblique Laurent polynomials $Q(x, f)$ in the maximal left ring of quotients of $Q(x, f)$.

Then the extended centroid of the ring $R[x, F]$ is isomorphic to the complete left classical ring of quotients $D_0^{-1}D_0$.

References: