POWER CHAINS IN A DIVISOR GRAPH

Satyanarayana Bhavanari¹, Srinivasa Devanaboina², Mallikarjuna Bhavanari³ & Abul Basar⁴

¹Department of Mathematics, Acharya Nagarjuna University, Andhra Pradesh, India
²Department of FEd, NRI Institute of Technology, Andhra Pradesh, India
³Department of Mechanical Engineering, Institute of Energy Engineering, National Central University, Jhongli, Taoyuan, Taiwan
⁴Department of Natural and Applied Science, Glocal University, Uttar Pradesh, India

Received: 07 Apr 2019  Accepted: 16 Apr 2019  Published: 27 Apr 2019

ABSTRACT

The divisor graph of an associative ring R (denoted as DG(R)) was introduced by Satyanarayana, Srinivasulu [9]. In this paper, we introduce a simple concept “Power Chain in a Divisor Graph”. We prove that if \( 0 = a \in R \) is nilpotent, then the power chain starting with \( a \) is of finite length. If \( DG(R) \) (the divisor graph of \( R \)) contains a power chain starting with \( a \in R \) which is of infinite length, then \( 0 = a = 1 \), \( a \) is non–idempotent and non–nilpotent element. We announce some basic results. Finally, we deduce that if \( R \) be an integral domain and \( a \in R \), then \( 0 = a = 1 \) if and only if the power chain starting with \( a \) (in \( DG(R) \)) is of infinite length.

KEYWORDS: Associative Ring, Divisor Graph of a Ring, Complete Graph

Mathematics Subject Classification: 05C07, 05C20, 05C76, 05C99, 13E15

1. INTRODUCTION

Beck [2] related a commutative ring \( R \) to a graph by using the elements of \( R \) as vertices and two vertices \( x, y \) are adjacent if and only if \( xy = 0 \). Anderson and Livingston [1] proposed a modified method of associating a commutative ring to a graph by introducing the concept of a zero-divisor graph of a commutative ring. Satyanarayana Bhavanari, Syam Prasad K and Nagaraju D [26] introduced “Prime Graph” of a ring and later studied by several authors. These concepts are different bridges connecting the two theories: Ring Theory & Graph Theory.

Now we introduce a concept called “Power Chains in a divisor graph” of a ring. This idea motivates us to prove the following results: (i) \( DG(\mathbb{Z}_p) \) contains a chain of length \( p-1 \). (ii) If \( p \)-prime, then \( DG(\mathbb{Z}_p) \) contain a max chain of length \( p-1 \).

Now we review some definitions and results for the sake of completeness.

1.1 Definitions

Let \( G = (V(G), E(G)) \) be a graph where \( V(G) \) is the set of vertices of \( G \) and \( E(G) \) the set of edges of \( G \). An edge between two vertices \( x, y \in V(G) \) is denoted by \( xy \).

Impact Factor(JCC): 3.7985 - This article can be downloaded from www.impactjournals.us
A graph \( G(V, E) \) is said to be a star graph if there exists a fixed vertex \( v \) such that \( E = \{vu / u \in V \text{ and } u \neq v \} \). A star graph is said to be an \( n \)-star graph if the number of vertices of the graph is \( n \).

(Satyanarayana, Srinivasulu D & Mallikarjuna [14]): Let \( G \) be a graph. The star number of \( G \) is defined as \( \max \{ n \mid \text{there exists an } n \text{-star graph which is a subgraph of } G \text{ and } n \text{ is an integer with } n \geq 1 \} \). We denote this star number of \( G \) by \( s_n(G) \).

(Satyanarayana Bhavanari and Syam Prasad K [25]): A complete graph is a simple graph in which each pair of distinct vertices are joined by an edge. The complete graph on 'n' vertices is denoted by \( K_n \).

(Satyanarayana Bhavanari, Srinivasulu Devanaboina, AbulBasar & Mallikarjuna Bhavanari [9]): Let \( R \) be an associative ring and \( x, y \in R \). We say that \( x \) divides \( y \) (if there exists \( z \in R \) such that \( xz = y \) or \( zx = y \)). A graph \( G = (V, E) \) is said to be the divisor graph of \( R \) (denoted by \( DG(R) \)) if \( V = R \) and \( E = \{xy/xz = y \text{ or } zx = y \text{ for some } z \in R \text{ and } x = y \} \).

### Power Chains in a Divisor Graph

#### 2.1. Definition

A chain

![Figure 1](image)

is said to be a power chain starting with \( a \) if \( x_1 = a \) and \( x_n = a^{\frac{n-1}{2}} \), and \( x_{n-1} = x_n \) for all \( n \geq 1 \).

#### 2.2. Note:

If \( a \in R \) is an idempotent then \( a = a^2 \) and so there is no edge in \( DG(R) \) between \( a \) and \( a^3 \).

#### 2.3. Examples:

If \( R = \mathbb{Z}_2 = \{0, 1\} \) the ring of integers modulo 2, then \( V(DG(R)) = \{0, 1\} \). \( E(DG(R)) = \{11\} \). Now \( DG(R) \) is given in Figure 2.

![Figure 2](image)

If \( R = \mathbb{Z}_3 = \{0, 1, 2\} \) the ring of integers modulo 3, \( V(DG(R)) = \{0, 1, 2\} \) and \( E(DG(R)) = \{01, 02, 12\} \). Now there is only one power chain in \( DG(R) \) and it is given in Figure 3.

![Figure 3](image)
If $R = \mathbb{Z}_4 = \{0, 1, 2, 3\}$ the ring of integers modulo 4, $V(DG(R)) = \{0, 1, 2, 3\}$ and $E(DG(R)) = \{01, 02, 03, 12, 13, 23\}$. Now there exist two power chains in $DG(R)$ and are given in Figure 4.

![Figure 4](image)

If $R = \mathbb{Z}_5$, then $R = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ the ring of integers modulo 5, $V(DG(R)) = \{0, 1, 2, 3, 4\}$ and $E(DG(R)) = \{01, 02, 03, 12, 13, 23, 34\}$. Now power chains in $DG(R)$ is given in Figure 5.

![Figure 5](image)

If $R = \mathbb{Z}_6$, then $R = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ the ring of integers modulo 6, $V(DG(R)) = \{0, 1, 2, 3, 4, 5\}$ and $E(DG(R)) = \{01, 02, 03, 04, 05, 12, 13, 14, 15, 23, 24, 34\}$. Now Power chains in $DG(R)$ is given in Figure 6.

![Figure 6](image)

If $R = \mathbb{Z}_7$, then $R = \mathbb{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\}$ the ring of integers modulo 7, $V(DG(R)) = \{0, 1, 2, 3, 4, 5, 6\}$ and $E(DG(R)) = \{01, 02, 03, 04, 05, 06, 12, 13, 14, 15, 16, 23, 24, 25, 34, 35, 36, 45, 46, 56\}$. Now Power chains in $DG(R)$ is given in Figure 7.
2.4. Results

- DG(\( \mathbb{Z}_n \)) contains a chain of length \( \varphi(n) - 1 \)
- (ii) If \( p \)-prime, then DG(\( \mathbb{Z}_p \)) contain a max chain of length \( p - 2 \)

2.5 Lemma: If \( 0 = a \in R \) is nilpotent then the power chain starting with \( a \) is of finite length.

Proof: Suppose that \( a \in R \) is a nilpotent element. Then there exists a positive integer \( k \) such that \( a^k = 0 \). Let \( m \) be the least positive integer such that \( a^m = 0 \). Now write \( x_1 = a, \ x_2 = a, \ldots, x_m = 0 \).

Now is the power chain starting with ‘\( a \in R \)’ and its length is \( m \), a finite length.

2.6 Lemma: If DG (\( R \)) contains a power chain starting with \( a \in R \) which is of infinite length, then \( 0 = a \neq 1 \), \( a \) is non–idempotent and non – nilpotent element.

Proof: Suppose that DG(R) contains a power chain starting with a which is of infinite length. Suppose the chain is

with \( x_1 = a \) and \( x_2 = a, (x_{m-1}) = a^m \). \( x_{m-1} = x_m \) for all \( n \).

Since \( x_1 \neq x_2 \) we have that \( \neq \neq \) and so \( a \) is not idempotent.
If \( a = 0 \) then \( a^k = 0 = a^{k+1} \), a contradiction.

Suppose \( a \) is the nilpotent element. Then by above lemma, the power chain starting with \( a \) is of finite length, a contradiction.

Therefore \( a \) cannot be a nilpotent element.

2.7 Lemma: Let \( R \) be an integral domain. If \( \mathbb{0} \neq \alpha \in R \) then \( a \) cannot be a nilpotent element.

Proof: Suppose \( a \) is nilpotent, Then there exists a positive integer such that \( a^k = 0 \) without loss of generality we assume that \( n \) is the least positive integer such that \( a^n = 0 \). Now \( a \cdot (a^{n-1}) = 0 \) and \( a \neq 0 \), \( a^{n-1} \neq 0 \), a contradiction. The proof is complete.

2.8. Theorem

Let \( R \) be an integral domain and \( a \in R \). Then \( \mathbb{0} \neq a \in R \) if and only if the power chain starting with \( a \) (in \( \text{DG}(R) \)) is of infinite length.

Proof: Suppose \( a \) is non-zero element in \( R \).

Then \( a^k = 0 \) for any positive integer. (by lemma – 2.7)

Now we prove that \( a^k \neq a^{k+1} \) for all \( k \geq 1 \). Suppose \( a^k = a^{k+1} \). Then \( a^k (1 - a) = 0 \Rightarrow (1 - a) = 0 \)

Therefore the chain given here.

\[
\begin{array}{ccccccc}
& a & a^2 & a^3 & \ldots & a^k & a^{k+1} & \ldots
\end{array}
\]

Figure 10

(that is the power chain starting with \( a \)) is an infinite chain.

Now the converse follows from Lemma 2.6.
REFERENCES


9. Satyanarayana Bhavanari, Srinivasulu Devanaboina, AbulBasar & Mallikarjuna Bhavanari “Results on the Divisor Graph of \( \mathbb{Z}_p^* \)”(Communicated)


