A Semi-Analytic Method for Solving Partial Differential Equations with Non-Local Boundary Conditions

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Abstract
This paper deals with the non-local boundary and initial value problems for two-dimensional two-sided partial differential equation model by using the semi analytic method. Tested examples and the obtained results demonstrate efficiency of the proposed method.

Keywords
two-dimensional, two-sided, partial differential equation, non-local boundary condition problem

Introduction
Adomian decomposition method can solve large classes of linear and nonlinear differential equations and it is much simpler in computation and quicker in convergence than any other method available in the open literature [1,2].

There are many literatures developed concerning Adomian decomposition method [3-6] and the related modification to investigate various scientific model [7-10]. E. Babolian et al. introduced the restart method to solve the equation \( f(x) = 0 \) [11], and the integral equations [12]. H. Jafari et.al used a correction of decomposition method for ordinary and nonlinear systems of equations and show that the correction accelerates the convergence [13, 14]

In this paper, we present computationally efficient numerical method for solving the partial differential equation with boundary integral conditions:

\[
D_t \Psi(x, y, t) - D_{xx} \Psi(x, y, t) - D_{yy} \Psi(x, y, t) - D_{x+y} \Psi(x, y, t) + \Psi(x, y, t) = k(x, y, t)
\]

with the initial condition

\[
\Psi(x, y, 0) = f(x, y), 0 \leq x \leq T, 0 \leq y \leq T
\]

and the non-local boundary conditions

\[
\int_0^T \int_0^T \Psi(x, y, t) dxdy = g_1(t), 0 < t \leq T
\]

\[
\int_0^T \int_0^T P(x, y) \Psi(x, y, t) dxdy = g_2(t), 0 < t \leq T
\]

Where \( f, g_1, g_2, \Psi \) and \( h \) are known functions. \( T \) is given constant. In the present work, we apply the modified Adomian’s decomposition method for solving eq.(1) and compare the results with exact solution. The paper is organized as follows: In section 2 the two-dimensional partial differential equations with boundary integral conditions and its solution by modified decomposition method is presented. In section 3 an example is solved numerically using the modified decomposition method. Finally, we present conclusion about solution of the two-dimensional partial differential equation.
Numerical Method
In this section, we present modified decomposition method for solving two-dimensional two-sided partial differential equations with boundary integral given in eq.(1). In this method we assume that:

\[ \Psi(x, y, t) = \sum_{n=0}^{\infty} \Psi_n(x, y, t) \]

eq.(1) can be rewritten:

\[ L_t \Psi(x, y, t) = L_{xx} \Psi(x, y, t) + L_{xy} \Psi(x, y, t) + L_{yy} \Psi(x, y, t) - \Psi(x, y, t) + k(x, y, t) \]  (2)

Where

\[ L_t(\cdot) = \frac{\partial}{\partial t}, \quad L_{xx} = \frac{\partial^2}{\partial x^2}, \quad \text{and} \quad L_{xy} = \frac{\partial^2}{\partial x \partial y} \]

The inverse

\[ L^{-1} = \int_{0}^{t} (\cdot) dt \]  (3)

Take \( L^{-1} \) on both sides of eq (2) we have

\[ L^{-1}(L_t \Psi((x, y, t))) = L^{-1}(L_{xx} (\Psi(x, y, t))) + L^{-1}(L_{xy} (\Psi(x, y, t))) + L^{-1}(L_{yy} (\Psi(x, y, t))) - L^{-1}(\Psi(x, y, t)) + L^{-1}(k(x, y, t)) \]

Then, we can write,

\[ \Psi(x, y, t) = \Psi(x, y, 0) + L_t^{-1}(L_{xx} (\sum_{n=0}^{\infty} \Psi_n)) + L_t^{-1}(L_{xy} (\sum_{n=0}^{\infty} \Psi_n)) + L_t^{-1}(L_{yy} (\sum_{n=0}^{\infty} \Psi_n)) - L_t^{-1}(\Psi(x, y, t)) + L_t^{-1}(k(x, y, t)) \]  (4)

The modified decomposition method was introduced by Wazwaz [6]. This method is based on the assumption that the function \( \gamma(x, y) \) can be divided into two parts, namely \( \gamma_1(x, y) \) and \( \gamma_2(x, y) \). Under this assumption we set

\[ \gamma(x, y) = \gamma_1(x, y) + \gamma_2(x, y) \]

Then the modification

\[ u_0 = \gamma_1 \]

\[ \Psi_1 = \gamma_2 + L_t^{-1}(L_{xx} \Psi_0) + L_t^{-1}(L_{xy} \Psi_0) + L_t^{-1}(L_{yy} \Psi_0) - L_t^{-1}(\Psi_0) \]

\[ \Psi_{n+1} = L_t^{-1}(L_{xx} (\sum_{n=0}^{\infty} \Psi_n)) + L_t^{-1}(L_{xy} (\sum_{n=0}^{\infty} \Psi_n)) + L_t^{-1}(L_{yy} (\sum_{n=0}^{\infty} \Psi_n)) - L_t^{-1}(\sum_{n=0}^{\infty} \Psi_n), \quad n \geq 1 \]

Numerical Example
Consider two-dimensional two-sided partial differential equation with non-local boundary condition for the equation (1):

\[ D_t \Psi - D_{+xx} \Psi - D_{-xx} \Psi - D_{+yy} \Psi - D_{-yy} \Psi + \Psi = 2t + t^2 + x + y \]

subject to the initial condition

\[ \Psi(x, y, 0) = x^2 + y^2 \quad \text{for} \quad x, y \in (0,1), \quad 0 \leq t \leq T \]

and the non-local boundary conditions
\[ \int_0^1 \int_0^1 (x^2 + y^2 + t^2) \, dx \, dy = t^2 + \frac{2}{3}, \quad 0 \leq t \leq T \]
\[ \int_0^1 \int_0^1 x^2 y^2 (x^2 + y^2 + t^2) \, dx \, dy = \frac{t^2}{9} + \frac{2}{15}, \quad 0 \leq t \leq T \]

We apply the above proposed method; we obtain:
\[
\Psi_0(x, y, t) = x^2 + y^2 + t^2
\]
\[
\Psi_1(x, y, t) = 0
\]
\[
\Psi_2(x, y, t) = 0
\]
\[
\Psi_3(x, y, t) = 0
\]

Then the series form is given by:
\[
\Psi(x, y, t) = \Psi_0(x, y, t) + \Psi_1(x, y, t) + \Psi_2(x, y, t) + \Psi_3(x, y, t)
\]
\[
= x^2 + y^2 + t^2
\]

This is the exact solution:
\[
\Psi(x, y, t) = x^2 + y^2 + t^2
\]

Table 1 shows the analytical solutions for partial differential equation with boundary integral condition obtained for different values and comparison between exact solution and analytical solution.

| x = y t | Exact Solution | Modified Adomian Decomposition Method | \(| \Psi_{MADM} - \Psi_{MADM} | |
|---------|----------------|---------------------------------------|----------------|
| 0 2 4.0 | 4.0            |                                      | 0.0000         |
| 0.1 2 4.02| 4.02          |                                      | 0.0000         |
| 0.2 2 4.08| 4.08          |                                      | 0.0000         |
| 0.3 2 4.18| 4.18          |                                      | 0.0000         |
| 0.4 2 4.32| 4.32          |                                      | 0.0000         |
| 0.5 2 4.50| 4.50          |                                      | 0.0000         |
| 0.6 2 4.72| 4.72          |                                      | 0.0000         |
| 0.7 2 4.98| 4.98          |                                      | 0.0000         |
| 0.8 2 5.28| 5.28          |                                      | 0.0000         |
| 0.9 2 5.62| 5.62          |                                      | 0.0000         |
| 1 2 6.0 | 6.0           |                                      | 0.0000         |

Conclusion

In this paper, we have applied the modified decomposition method for the solution of the two-dimensional two-sided the partial differential equation with ono-local boundary condition. This algorithm is simple and easy to implement. The obtained results confirmed a good accuracy of the method.

References


