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AN EXPLORATION OF HIGH SCHOOL LEARNERS' UNDERSTANDING OF GEOMETRIC CONCEPTS

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Abstract

There is much concern in South Africa about the poor performance of learners in mathematics, particularly in geometry. The aim of this research was to explore the understanding of basic geometry concepts by grade 10 and grade 11 learners in terms of the van Hiele's levels of geometry thinking. The participants of the research were 147 learners from three high schools in a rural area in the south of KwaZulu Natal, South Africa. The results showed that the learners had difficulties with problems involving definitions of geometric terms, interrelations of properties and shapes, class inclusion and changing semiotic representations. It was also found that most of the learners were operating at the visual and the analysis levels of the van Hiele levels of geometric thinking. It is recommended that teachers should provide learners with tasks that require movements between semiotic representations, and to also focus attention on improving learners' skills in proving aspects of mathematical relations.
Keywords: *geometry, high school, van Hiele theory, class inclusion, mathematical proof, necessary and sufficient conditions.*

Introduction

Mathematics learning outcomes in South Africa are very low which has led many researchers to express concerns about the poor performance in mathematics, especially in geometry (Lee & Ginsburg, 2009; Mthembu, 2007; Singh, 2006). According to Patkin and Lavenberg (2012), Geometry is seen as the most complicated strand of the mathematics curriculum and learners mistakenly assume that the subject is irrelevant to their daily lives. In South Africa, education authorities themselves were uncertain about the importance of geometry. In 2006, the curriculum was revised so that geometry was no longer compulsory for learners in Grades 10 -12 (DoE, 2006). A few years later, in 2011, geometry was made compulsory again for these grades (DoBE, 2011). When geometry was made optional, many learners chose not to study the section and hence did not gain access to the particular type of geometric reasoning encountered in geometry. When the geometry strand was brought back into the core mathematics curriculum, teachers did not feel as confident about the strand since it had not been taught for such a long time. These issues illustrate the need to find out more about the particular areas of geometry that pose challenges to the learners.

Research Focus

The focus of this research was high school mathematics learners' understanding of geometry in terms of van Hiele's levels of geometry thinking. The learners were in Grades 10 and 11 which is part of the Further Education and Training (FET) band of schooling. The research sought to answer the following questions:

1. What can be deduced about the van Hiele levels of geometric thought that the learners are working within?

2. What are some factors which impede the development of their geometric understanding?

Literature Review

Geometry is a strand of mathematics that involves analysing the properties of shapes and exploring the relationships between and within figures (Bassarear, 2012). Geometry has different applications in science and technology in industries such as construction, design, architecture and engineering, amongst others (Knight, 2006). These real life applications of geometry give the teacher opportunities to make the subject mathematics more relevant to the learners (Chambers, 2008). Usiskin (2002) proposed two reasons why geometry is important to teach; it provides opportunities for mathematics modelling and also allows people to visualise concepts which may be related to other areas of mathematics. It has links with culture, history, art and design and it is the interaction with these vital human activities that provide opportunities to make geometry lessons interesting and stimulating (Chambers, 2008).

An integral part of the study of geometry is the use of proof when establishing properties and relations within and amongst figures. A proof involves the creation of a narrative which starts from some known fact and proceeds in a step by step manner where each step is deduced from the result of the previous one until the unknown fact is justified. Each of the sequential statements, comprising the proof, should be supported by valid reasons (Serra, 1997). De Villiers (2004) viewed a proof as a formal written argument of the complete thinking procedures that are used to reach a valid conclusion; the steps of these procedures are supported by theorems, postulates or definitions verifying the validity of each step and explaining why these steps are achievable. The inclusion of the study of formal proof as part of the study of geometry provides a means for the development of deductive reasoning skills (Mudaly & de Villiers, 2004). The use of proof in geometry also provides learners with the experience of a formal axiomatic system for the first time (Mudaly & De Villiers, 2004).

The study of geometric shapes requires a mix of visual and analytical strategies and juggling between these two perspectives may result in misconceptions because learners' perceptions may be different to that of their teacher. Misconceptions are conceptual or reasoning difficulties that hinder learners' from constructing a concept in a sound or mathematically endorsed manner. A misconception often arises when a rule is applied incorrectly, or when a learner over-generalises a rule, under- generalises a rule or presents an alternative conception of the situation (Hansen, Drews, Dudgeon, Lawton, & Surtees, 2017). Swan (2001) views the development of misconceptions as a natural part of conceptual development. As students learn more about a concept, the students may develop a misconception which may naturally be overcome as their concept image extends to consider other settings in which the concept is encountered. Sometimes however misconceptions may persist and interfere with later learning. Students' backgrounds, the context within which learning takes place as well as the teaching styles may contribute to the formation of misconceptions. One of the common reasons for misconceptions is because "students have difficulties in understanding the instructional strategies adopted by the teacher" Luneta (2015, p.2).

The van Hiele Model of Geometric Thought

This research was underpinned by the van Hiele model of geometric thought that explains how children develop spatial geometry concepts (Crowley, 1987). The van Hiele model (1999,

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1986) proposes five levels of geometric thinking, which students progress through as part of their development of geometric reasoning. These are the visual, analysis, informal deduction, formal deduction and the rigour levels respectively. There are particular terms and phrases used to detail the differences in the reasoning that learners use in each of the levels. According to the theory, learners move step by step from the first level (visual), through each of others, when constructing different concepts. The role of the teacher is crucial because it is the teacher who decides what experience is suitable at each level, and for which learner that experience is suitable.

Learners who are at a visual level have a very simple concept of space. They see geometric shapes or figures as a complete whole. They recognize geometric figures by their appearance not by their properties. Learners are able to identify the given shape because they associate the shape with what they know. Learners see these figures as a whole, without being able to analyse their properties (Burger & Shaughnessy, 1986)

Those learners who are able to analyse shapes in terms of their parts and properties, have progressed to the analysis level, although they may not be able to make connections between different shapes (Mason, 2010). A learner at the analysis level should be able to recognize that a square is a figure which has 4 equal sides and 4 equal angles. The diagonals of a square are equal and perpendicular bisectors of each other. However, the learners placed at this level may have an incomplete understanding of how properties of shapes are related to each other.

Learners who are at the informal deduction level can analyse the properties of the figures and understand relationships between the properties of a figure and relationships between figures. Learners are able to follow all the logical arguments using the properties of the figures, but they may not be able to create a new proof from scratch. At this stage they are able to reason about the properties of class inclusion.

The learners who are able to understand and use the ideas of formal geometry show that they have progressed to the formal deduction level. They understand how important deduction is, and can use it to build up a geometric theory based upon axioms and proofs, in the same way as Euclid did. Learners now learn to do formal proofs. They now understand the role played by terminology, definitions, axioms and theorems in Euclidean geometry.

Learners who are at the rigour level can work within a variety of axiomatic systems, non-Euclidean geometries and different systems can be compared, thus geometry is seen as abstract.

Research Methodology

Background

This research was qualitative in nature with the aim of finding out the van Hiele level of geometric thinking of the learners and also identifying factors which impeded the development of the learners' geometric understanding. The research was conducted with learners from three schools which were all located in rural area in Southern KwaZulu-Natal. The research was conducted towards the end of the year when the learners had already completed the topics in the geometry curriculum.

Sample

The participants of the study were made up of 147 Grade 10 and Grade 11 learners from three schools. The sample was that of convenience because of the proximity of the schools to the first author. The details of the participants are presented in Table 1 below.

Table1. Distribution of learners according to grades and schools

	Number of Learners		
School	Grade 10	Grade 11	Total
School A	31 learners	31 learners	62 learners
School B	32 learners	23 learners	55 learners
School C	11 learners	19 learners	30 learners
Total	74 learners	73 learners	147 learners

The learners were informed about the details of the study at a meeting. Each learner signed an informed consent form granting us permission to use their responses in the research. All protocols, in line with the ethical clearance procedures prescribed by the University of KwaZulu-Natal, were observed.

Instruments and Procedures

The research instruments consisted of a questionnaire with 15 multiple choice questions adapted from Usiskin (1982), a worksheet with six open-ended questions and an interview schedule. The multiple choice questions were targeted at the various levels of the van Hiele's model, while the worksheet probed the understanding of the learners of questions based on parallel lines, triangles and quadrilaterals. Semi-structured interviews were carried out on a purposive sample of 18 learners, who were selected on the basis of how they responded in the questionnaire and worksheet. During the interviews the participants were given the opportunity to express their perceptions and understanding of the geometry concepts of parallel lines, triangles and quadrilaterals. They were furthermore probed about their written responses to the items.

Data Analysis

The learners' written responses were first assessed and the average percentage of correct responses for questions targeted at each of the van Hiele levels was calculated. This was done by adding the percentage of correct responses per item at that level and dividing the sum by the number of items at that particular level.

Average Percentage of Correct Responses =

The learners were then linked to the different van Hiele levels based on their responses to particular items. This was dependent on the number of questions they were able to answer correctly at that particular level. In terms of identifying the challenges faced by the learners, the written responses were analysed in detail. Common patterns were identified across the scripts. During the interviews, learners' responses were used to provide a deeper understanding of the emerging patterns and these were then developed into themes. Codes were used for the learners to preserve their anonymity, for example LSA1 refers to learner number 1 from school A, LSB3 for learner number 3 from school B.

Results of the Research

Learners' Performance in terms of the van Hiele Levels

The questions from the questionnaire were grouped according to the van Hiele levels of geometry thinking that the questions required, and the average percentage of correct responses were then determined and recorded as in the table below.

Table2. Learners' performance at each Van Hiele Level for Questionnaire A.

Van Hiele level of geometric thought	Items	Short description	Average percentage of correct response per Level
Level 1:Visual Level	1 and 2	Identifying shapes by their appearance	100
Level Two: Analysis Level	3, 4, 6, 7 and 9	Recognising shapes by their properties	42
Level Three: Informal Deduction Level	5, 8, 10, 11, 12, 13 and 14	Analyse properties of figures and understand the relations between the properties	30
Level Four: Formal Deduction level	15	Develop a series of statements and start to understand the importance of deduction and vital role of axioms, theorems and proofs.	4
Level : Rigour	No items were set at this level	Reason formally about mathematical systems and understand geometric figures which are abstract	N/A

There was a decline in the number of correct responses at given Van Hiele levels moving from the most basic level, the visual level to the formal deduction level. 42% of the learners' responses were correct for items at the analysis level. 30% of the learners' responses were correct for items set at the informal deduction level whereas 4% of the learners' responses were correct for items set at the formal deduction level. No items were set at the rigour level which requires learners to reason formally about different mathematical systems, which is beyond the work that is done at school level. It is important to note that as the item level increases, the success rate decreases.

Learners were placed on the different van Hiele levels depending on the items they were able to answer correctly at that particular level. For a learner to be placed at a higher van Hiele level he/she should first meet all the requirements for the lower levels. However out of the 147 learners there were about five cases where a learner met the requirements for a higher level having failed to meet the requirements for the lower levels, but these cases were too few to be significant and these cases were then attributed to either guessing or copying. So these learners

were placed at a lower level where they meet the requirements. Only item 15 was set at the formal deduction level. Learners were said to be operating within the formal deduction level if they were able to meet all the requirements for levels 1, 2 and 3 and also to get item 15 correct. Getting item 15 correct, without meeting the requirements for levels 1, 2 and 3, was not enough for a learner to be placed at the formal deduction level as the learners could have copied. There were only two learners who were classified at this level with respect to their responses in Questionnaire A, and both these learners were interestingly, Grade 10 learners.

Seven items were targeted at the informal deduction level. Learners were said to be operating at the informal deduction level if they were able to get more than three items correct at this level, but they should also have met the requirements for levels 1 and 2, but not being able to meet the requirements for level 4. There were two reported cases where 2 learners did not meet the requirements for level 2 placement but they met the minimum requirements for the placement into level 3. These two learners were placed into level 1, as these 2 cases were too few to be significant. There are a wide range of skills associated with particular van Hiele levels. For example, with the informal deduction level, a learner may be beginning to understand and being able to work with problems, while another learner may be at an advanced stage and thus will be able to work with more complicated problems at that particular van Hiele level.

Five items were targeted at the analysis level of the van Hiele levels of geometry thinking. A learner was placed at the analysis level (level 2) if he/she was able to meet the requirements for level 1, was able to get more than two items correct at level 2, and was not able to meet the requirements for placement into the informal deduction level. All the learners who failed to meet the requirements for placement into levels 2, 3 and 4 were then placed into the visual level. The total number of learners at each van Hiele level was then summarised and is presented in the table below.

Table3. Summary of the van Hiele levels of the FET learners.

Van Hiele Level	Number (percentage) of learners operating within a particular Level
Visual Level [Level 1]	23 (16)
Analysis Level [Level 2]	77 (52)
Informal Deduction Level [Level 3]	45 (31)
Formal Deduction [Level 4]	2 (1)
Rigour Level [Level 5]	0

Table 3 shows that 16% of the learners involved in the study did not progress beyond the visual level of the van Hiele levels of geometry thinking. Even though the van Hiele levels are not age dependent, one would expect learners at the FET level to be operating at level 3 and 4, as they have been exposed to many opportunities of working with geometric figures and thus are expected to show advanced knowledge of geometry. These learners have a very simple concept of space and have not moved beyond the stage of identifying shapes by their appearances only.

Slightly more than half of the FET learners involved in the research (52%), were operating within the analysis level of the van Hiele's of geometry thinking. These learners' responses to the items showed understanding of the properties of geometric figures and they could classify

properties of some different shapes but they could not make any connections between shapes and their properties. The learners at the analysis level were able to investigate, understand, deduce and make generalisations from the properties of figures.

Thirty one per cent (31%) of the learners involved in the study were operating within the informal deduction level according to the Questionnaire results. These learners showed some evidence of being able to analyse and understand the relationships between properties of figures. It is at this level that the learners can start putting the properties of the figures in the correct order and be in a position to follow logical arguments. Only 1% of the learners showed signs of engagement within the formal deduction level.

Challenges Experienced by Learners.

Students were also given six questions in a worksheet to respond to, and 18 learners were interviewed on their responses to certain tasks to find out more about some of factors that impeded their success in in solving geometry tasks. Some challenges that were identified were misconceptions; concept of class inclusion; dealing with necessary and sufficient conditions; and, changing between semiotic registers.

There were many misconceptions that were revealed in the responses to the questionnaire. For example, from the response to Item 6 of the questionnaire, 18 learners believe that parallel lines are the lines which never lie in the same plane and never meet. It is evident that some FET learners still struggle to understand properties of parallel lines.

There were further misconceptions related to parallel lines that were revealed in the responses to Question 1 from the worksheet appearing in Figure 2, with the response of Learner LSC3.
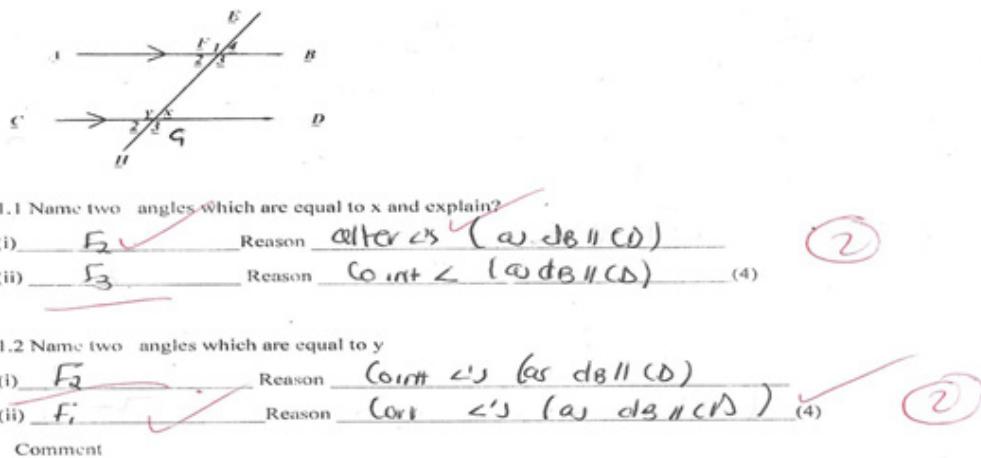
Figure1. Response by Learner LSC3.

The learner wrote F_3 , and wrote that they are co-interior angles between parallel lines. He correctly identified the pair as forming co-interior angles but had a misconception about the relationship between co-interior angles formed between two parallel lines. Co-interior angles between parallel lines are supplementary (they add up to 180º). The learner recognised the co-interior angles but was not so clear about the relationship between co-interior angles between parallel lines. The learner's lack of understanding of co-interior angles was revealed in the interview below:

Researcher: In both questions 1.1 and 1.2, you gave one correct answer and one wrong answer. In the wrong responses you gave the same reason of co-interior angles. Can you explain to me why you chose this and if possible the meaning of co-interior angles."

LSC3: To be honest I just thought since alternating angles between parallel lines are equal then co-interior angles will also be the same. I am still able to recognise that angle F3 and x are co-interior but I can't remember what will be the relationship between them. When we were taught geometry, the terms were never explained to us, we were just told the angles were equal because they are alternating or because they are vertically opposite and that was that.

The learners' response indicates that when he was introduced to the concept, he was not given much time to consolidate these relationships. It is also evident that they did not get a chance to investigate and discover the properties but were just told these facts by the teacher.

Another common misconception identified was that if two angles lie on the same straight line, then they add up to 180^0 even if they are not adjacent to each other. One learner's work illustrating this misconception is presented below.
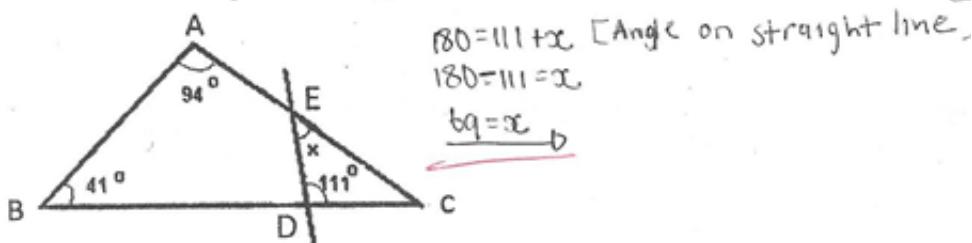
Figure 2. Learner LSC1 response to open ended question 2.

The learners with this type of misconception believed that angle and angle lie on a straight line and the sum of angles on straight line gives 180 degrees. Out of the 77 learners who did not get this question correct, nearly 50 of them had the same misconception as learner LSC1. An interview with learner LSC1 regarding question 2 resulted in the following observations.

Researcher: When you added and, you said they must give 180 degrees. Can you please explain the reason why you responded in that way?

Learner LSC1: When we were learning properties of geometric shapes, I still remember one property which says the sum of angles on a straight line add up to 180º. If we check angles and, they are both lying on the straight line ED, so if we add them they must give us 180º

The learner applied the well-known fact to angles which are situated on the same line, but had a misconception about the meaning of "angles on a straight line". The result is only true if the angles are adjacent to each other and they lie on the same line. Adjacent angles share a common ray.

The analysis also revealed that learners struggled with items involving class inclusion problems. Class inclusion is a property of geometric figures whereby one set, or class of figures is included in the set of another larger set. For example, a rectangle is a special type of parallelogram, because it is a parallelogram which has the additional property of having angles all equal to 90°. Hence the set of rectangles is a subset of the set of parallelograms. Two items

from the questionnaire (12 and 13) probed learners' understanding of the interrelationships between the properties of figures, specifically the issue of class inclusion. Question 12 asked about the interrelationship between squares and rectangles, while Item 13 focused on the interrelations between parallelograms and rectangles. For Item 12, only 25 learners (17%) were able to tell that all properties of rectangles are properties of all squares. 25 learners did not attempt to answer the item, whereas 69 learners believed that all properties of squares are properties of parallelograms. This was a misconception as there are some parallelograms which are not squares. A rectangle is a parallelogram but it is not a square. For Item 13, learners were expected to identify one property that all rectangles have, that some parallelograms do not have (which is, that diagonals are equal). Only 40 learners responded correctly (27%). It was a concern to note that 54 learners believed that in all rectangles opposite sides are parallel, whereas some parallelograms do not have parallel sides. This shows that these learners struggle with the notion of class inclusion.

A related issue to class inclusion is that of distinguishing between necessary and sufficient conditions. Question 5 from the worksheet required the learner to prove that the quadrilateral was a square elicited by many responses where students just showed that one property of squares was met by the given figure. The question appears in Figure 3.

Below is a quadrilateral BRAT. Use the quadrilateral to answer the following questions.
5.1 Write down the coordinates of R.
5.2. Is BRAT a square? Why or why not?
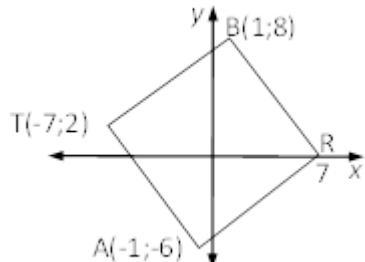
Figure 3. Question 5.

The results for Question 5 showed that only two learners were able to prove that the shape was a square. Most of the learners' responses showed that learners were unable to distinguish between the necessary and sufficient conditions for a quadrilateral to be a square. So 99% of the learners were unable to establish sufficient conditions, with many proving that the quadrilateral satisfied the property of four equal sides. However, a quadrilateral with all sides equal can be a rhombus, but not a square, where angles may not be 90°. This then implies that the property that all sides are equal is a necessary but it is not a sufficient condition for a shape to be a square. An example of such an argument was given by learner LSC11 who only proved that the sides are equal and then concluded that the shape is a square.
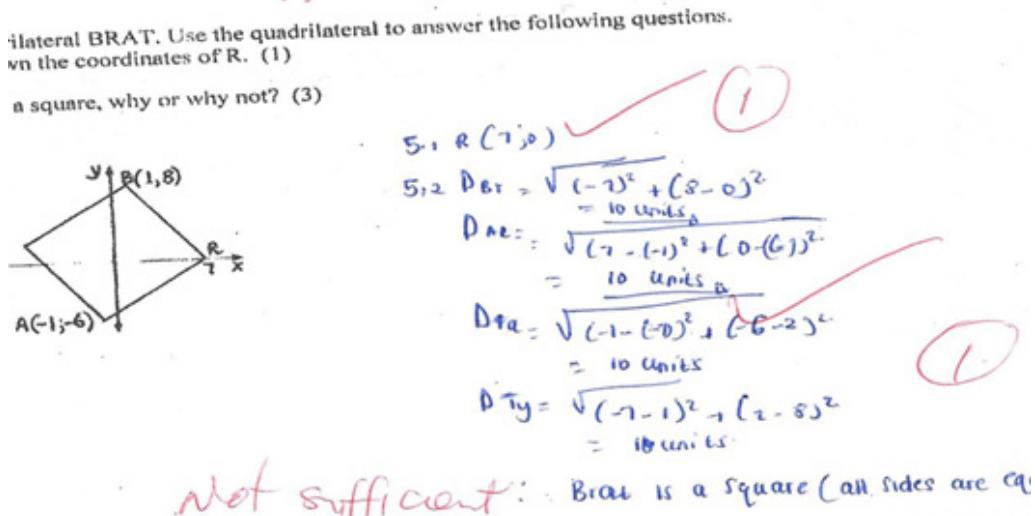
Figure 4. Learner LSC11's response to question 5.

The response by learner LSC11 showed that the learner understood the properties of a square in isolation as he failed to make connections between the properties so as to prove that the shape BRAT is a square. He believed that if the sides are all equal, then it means that the shape is a square. The extract below was taken from the interview conducted with Learner LSC11.

> *Researcher:* Can you describe to me what you understand by a square and some of its properties.
> *Learner LSC11***:** When I was growing up I knew that a square has equal sides, so I did not think about other properties, the first thing that came to my mind was to show that all sides are equal and that's what I did.
> *Researcher***:** Can you look at these two rhombuses that I am drawing on this graph and tell me what you think, whether you still agree with your definition or needs some refining.
> The researcher then drew a square and a rhombus whose angles are not 90 degrees.
> *Learner LSC11:* Mmmm (long pause). It seems like the one with angles 90° is the square, the other rhombus cannot be a square because the angles are not right angles even though the sides are equal.

The interview responses showed that he initially believed that if a figure had equal sides, then that figure was a square. However, after being challenged by the researcher, he was able to see that the property of equal sides was not sufficient to qualify a figure to be a square.

The results from the questionnaire showed that some learners struggled with the items given in the natural language, in the form of explanations and definitions of certain geometric concepts. From the interviews conducted with the learners, it became clear that the learners struggled with those items because it focused on the properties of geometric figures but the diagrams of the geometric figures were not provided. This suggested that the learners found it difficult to move between the natural language representation and the diagrammatic representations. Some of the interview extracts provided some insight about why they struggled with the items.

> *LSC2: I* know parallelograms but I failed to link them to the question, so I had to guess the answer from the given options.
> *LSB1:* We have done parallelograms before but the way the question was asked was challenging to me. I even drew my own parallelogram but I failed to create the triangles which they are talking

about in the item, then there was this term equiangular which I don't even know its meaning.
LSA1: At first I struggled to answer the question, but then I decided to draw the parallelogram and after naming the sides and angles it became easy for me to answer it.

Learner LSC2 explained that he could not link the question to the properties of a parallelogram, showing that his concept image (a mental picture or image) was not connected to the concept definition (a specific definition of a shape or its properties) hence he struggled to move from one representation to the other. Learner LSB1 explained that he could not identify the triangles that the question referred to, showing that he had difficulties in translating the verbal (written) representation into the diagrammatic representation, while LSA1 found the question easier because he was able to move between the two representations and the diagrammatic form helped him find the answer. These learners' responses indicated that visualisation of the figures is important when trying to identify the relationships within a figure. The issue of visualisation was emphasised by learner LSC11 in the interview (following Figure 4) when the responses to question 5 from the worksheet were discussed. The learner was being probed about whether all quadrilaterals with equal sides were squares and was then presented with two figures with equal sides: one square with equal angles and the other one a rhombus with only opposite equal angles. The learner quickly explained why the one with unequal angles was not a square, something he did not see when the diagrams were not provided.

The learners' responses to items requiring formal and logical reasoning suggest that learners struggle with using formal deductive reasoning and creating proofs. Only seven learners answered item 15 (based on logical reasoning and deductions) Questionnaire correctly, while only five learners scored 2 or more marks out of the possible 8 marks in the open-ended questions in the worksheet from the 147 learners who took part in the study. Many learners did not even attempt to answer question 6 of the worksheet which required a proof that a given figure was a parallelogram.

Discussion

Levels of Geometric Thinking

The research revealed that many of the learners were still operating at the visual level, even though they had spent at least ten years working with geometric figures in the time that they were at school. These learners have not moved beyond the recognition of shapes and mentioning of properties and showed no evidence of knowing how the properties are connected. The research showed that more than 1/3 of the group were limited to reasoning skills at the Visual level of van Hiele's model, while less than 40% were reasoning within the informal deduction levels. This is a concern because van Hiele theory emphasises that if learners are at a lower level and the teaching is targeted for learners whose reasoning is at a higher level, then the learner at the lower level is not likely to progress further. This is because the language and discourse associated with higher levels is different from that at the lower levels. De Villiers (2004) argued that teachers' presentation of material ought to be within a certain level that is close to where the learners are at, so that the learner will understand what is being taught and progression to the next level will be facilitated.

These findings concur with the findings of the studies by Siyepu (2005), and Atebe & Schafer (2011), whose studies indicated that the majority of the learners were found to be operating at the pre-recognition level and that a very small number of the students had progressed to the second van Hiele level. Mateya (2008) found similar results in a study that was conducted with Grade 12 students. Of the 50 students who participated in the study, 19 (38%) were at the pre-cognition level, 11 (22%) were at the first van Hiele level, 13 (26%) had

progressed to the second van Hiele level, while only 4 (8%) were at the third van Hiele level 3. Similarly Usiskin (1982), found that many secondary school learners are on the visual or analysis levels of the van Hiele levels.

The results showed that most of the learners were operating at the visual and analysis levels. This finding implies that most of the learners' levels of geometry reasoning are lower than that required by the mathematics curriculum. The curriculum is quite specific that learners should be able to engage in deductive reasoning and to construct simple proofs (DoBE, 2011). In this research, it was clear that most learners could not cope with questions, which involved proofs or needed more than two steps. The majority of learners found it difficult to use deductive reasoning to prove that angles were equal, that triangles were congruent and that a shape was a parallelogram.

The analysis of the results also revealed that sometimes learners made progress towards developing informal deduction skills but it was limited in scope. For example, a learner was able to show that the shape in question 4 of the worksheet was a parallelogram but could not prove that the shape in question 5 was a square. Yet both questions required thinking at the same van Hiele level. This shows that reasoning at a van Hiele level is not static but constantly developing; hence a learner can show some competence at a level but still struggle with other aspects at the same level.

Patkin & Lavenberg (2012) made suggestions for using examples of tried and tested activities designed to promote and develop geometric thinking. The pedagogical and didactic functions of these activities are to offer interesting and unusual mathematical experiences, encourage mathematical engagement through experience and inquisitiveness, develop the learner's ability to cope with the problems taken from their daily environments, reduce anxiety of the subject and create opportunities for geometric activities for pupils who often find geometry difficulty (Patkin & Lavenberg, 2012).

Misconceptions of Learners

The research uncovered many misconceptions in geometry held by the learners. As asserted by Swan (2001), misconceptions are an integral part of learning a concept. As learners develop a more robust understanding of the concept, the misconceptions will be replaced by more sound and appropriate conception in line with the curriculum requirements. However it is important for teachers to help learners become aware of their misconceptions so that these can be confronted and resolved.

Visualisation Skills

The fact that learners struggled to answer items without the diagrammatic representation, emphasizes the role played by visualization in the teaching and learning of mathematics. Most geometric concepts are learnt using diagrams and shapes. Visualisation is the ability to interpret and reflect upon pictures, images and diagrams in minds, with the purpose of depicting information (Arcavi, 2003). It is an aid to an understanding or means towards an end. Visualisation refers to mental images of a problem, and to visualise a problem means to understand a problem in terms of a diagram or visual image (Presmeg, 2006). According to Presmeg (2006), the visualisation process is one which involves visual imagery with or without a diagram, as an essential part of the solution. Teaching mathematics especially geometry, should include the use of diagrams or visual images to help develop an understanding of conceptual knowledge.

Difficulties in Changing from One Semiotic Representation to Another

It was found that when the learners were given properties and definitions in the natural language, most of them were unable to relate it to the iconic representation (geometric figures). So, they were not able to work with the properties of figures when the diagram was not in front of them. Geometric figures arise in a register of multifunctional representation; in this case the learners were given properties and definitions in the natural language (discursive representation) and were required to relate it to the non-discursive representation (geometric figures). The learners struggled to interpret the information in the natural language in terms of the properties of geometric figures. Duval (2006) argues that the characteristic feature of mathematical activity is the simultaneous mobilisation of at least two registers of representation, or the possibility of changing from one register to another at any moment. If one wishes to analyse the difficulties in learning mathematics, it is of paramount importance to study the conversion of representation (Duval, 2006). Hence, teachers need to ensure that their learners are exposed to tasks which require them to work within different registers of representation.

Problems with Proving

As revealed in the learners' written responses to the multiple choice items and the questions form the worksheet, the learners had not developed sufficient skills in proof. Students also revealed difficulties with reasoning about class inclusion and differentiating between properties which are necessarily satisfied by a special figure and properties which are sufficient for a general figure to exhibit in order to be part of the class of the special figures. Many researchers have found that learners have difficulties with solving proof problems (de Villiers, 2004; Healy, & Hoyles, 2000; Moore, 1994; Weber, 2004). If a learner lacks knowledge of definitions, he is likely to face challenges with proof questions (Moore, 1994). These results support the findings of Clements and Battista (1992), who found that in the United States, elementary and middle school learners fail to learn basic geometry concepts and geometry problem solving techniques, making them woefully under-prepared for the study of more sophisticated geometric concepts and proofs. One of the reasons why learners experience problems in proof questions is because proofs are mainly given as finished products in textbooks and this does not challenge learners to think deductively (de Villiers, 2004).

Conclusions

The research found that the most learners' geometric understanding was limited to the first and second van Hiele level, because they had not developed formal or even informal deduction skills. It was shown that more than 1/3 of the group were limited to reasoning skills at the Visual level of van Hiele's model. This means that they can only see shapes as wholes and cannot discern the properties within a figure and interrelations between figures. The results also revealed serious problems with proving skills. These results help explain why geometry is perceived as a difficult section of mathematics. The learners have not been given sufficient opportunities to develop the necessary reasoning skills at the higher van Hiele levels. The participants also revealed difficulties with reasoning about class inclusion and differentiating between properties which are necessarily satisfied by a special figure and properties which are sufficient for a general figure to exhibit in order to be part of the class of the special figures. The role of the teacher is crucial in this process since it is the teacher who needs to identify which levels of reasoning the learners have access to. The teacher can then design suitable activities for the learners that can help them progress through the levels of geometric thinking. Without the appropriate interventions, the learners will not be able to cope with tasks that require higher levels of understanding.

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The research also identified that students had particular problems with making connections between the verbal and the visual representations. It is also important that teachers use a diversity of representations when teaching geometry, instead of showing an over-reliance on verbal explanations or definitions only. This can help learners to switch easily from one representation to another and to make connections between the representations.

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