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Generalized-hollow $\mathbf{lifting}_g$modules

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Abstract

Let R be any ring with identity, and let M be a unitary left R-module. A submodule K of M is called generalized coessential submodule of N in M , if (N)/K⊆〖Rad〗_g (M/K) . A module M is called generalized hollow-〖lifting 〗_g module, if every submodule N of M with M/N is a G-hollow module, has a generalized coessential submodule of N in M that is a direct summand of M. In this paper, we study some properties of this type of modules .

Keywords: generalized coessential submodule, generalized strong supplement submodule, generalized hollow-lifting $_g$ module.

مقاسات (الرفع)g المجوفة المعممة

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الخلاصة

لتكن R حلقة ذات عنصر محايد وليكن M مقاسا أحاديا أيسر على R يقال على مقاس جزئي K معمم ضد جوهري في M اذا كان $\frac{N}{K} \subseteq Rad_g \frac{M}{K}$ يقال على المقاس M بانه مقاس مجوف –معم ،أذا كان $\frac{M}{N}$ لكل مقاس جزئي N في M بحيث أن أجوف-معمم فأن N له مقاس جزئي معمم رديف جوهريا و يكون جمع مباشر . في هذا االبحث سوف ندرس خواص هذا النوع من المقاسات و نبرهن بعض النتائج التي تعتبر تعميم لمقاسات(الرفع)g المجوفة.

1. Introduction

Throughout this paper R is a ring with identity , and every R-module is a unitary left R-module , N⊆M denotes N is a submodule of M .Let M be an R-module , and let N⊆M , N is called essential submodule of M (denoted by N⊆_eM) if every nonzero submodule B of M , we have B∩A≠0 [1]. A submodule N of M is called small submodule of M (denoted by N ≪M) , if for every K⊆M , M=N+K implies K = M [2]. Rad(M) is the sum of all small submodules of M [2] . A submodule N of M is called generalized–small submodule of M (for short ,G-small) (denoted by N ≪_G M) , if for every K⊆_eM , M= N+K implies K = M [3] . Rad_g (M) is the sum of all G-small of M[3] , It clear that Rad(M)⊆Rad_g (M) , but the converse is not true in general . A nonzero module M is called generalized-hollow (for short ,G-hollow) , if every proper submodule of M G-small (in [4] , it is denoted by e-hollow) . A Submodule K of M is called coessential submodule of N in M (denoted by K⊆_Ce N) , if N/K≪M/K. A module M is called lifting module or satisfies (D1) if for every submodule N of M there exists a direct summand K of M such that M = K⊕ K′ , K ⊆N , K′ ⊆ M and N ∩ K′≪ M[5] . M is called hollow lifting , if for every submodule N of M with M/Nis hollow has a coessential submodule in M that is a direct summand of M , [6] . Clearly every lifting module is hollow lifting , while the converse does not hold in general , see [6] . A submodule K of M is called G-

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coessential submodule of N in M (denoted by K⊆_GceN) , if N/K ≪_(G) M/K,[7] . An R-module M is called generalized lifting or satisfies (GD1) , if for every submodule N of M , there exists a direct summand K of M , such that K⊆_(GCe) in M [4] . It is clear that every lifting module is a generalized lifting module . An R-module M is called a generalized hollow lifting module (for short , G-hollow lifting module) , if for every submodule N of M , with M/N is hollow module , N has a generalized coessentioal submodule of M that is a direct summand of M , [7]
tIn this paper we introduce a generalized hollow 〖lifting〗_g module as a generalization of generalized hollow lifting module .
Let N , K⊆M , N is called supplement of K in M if M=N+K and N∩K≪N,[2] and N is called strong supplement of K if N is a supplement of K in M and N∩K is a direct summand of K [3] .
We introduce G-strong supplement submodule , let N , K⊆M we called K is G-strong supplement of N in M if M=N+K , N∩K⊆ 〖Rad〗 _g (M)and N∩K is a direct summand of N .
In fact , we prove for an indecomposable module M , M is G-hollow-lifting_g module if and only if M is G- hollow or else M has no G-hollow factor module . We also prove that for N⊆M , N has a generalized strong supplement in M if and only if N has a generalized coessential submodule that is a direct summand of M , therefore M is a G-hollow-lifting_g module if and only if for every submodule N of M , with M/N is G-hollow has a generalized strong supplement in M .

In section three, we prove that for fully invariant submodule N of M, if M is G-hollow-lifting$_g$ module, then *M/N* is a G-hollow-lifting$_g$module. In fact, we give sufficient condition for direct sum of two G-hollow lifting module to be G-hollow lifting. We prove if $M = M_1 \oplus M_2$is a duo module, then M is a G-hollow-lifting$_g$module, if and only if M_1 and M_2 are G-hollow -lifting$_g$modules.

2. Some properties of G-hollow l$lifting_g$modules

In this section, we introduce G-hollow $lifting_g$ module as a generalization of hollow lifting module, and study some properties of this type of modules.

Definition 2.1[7]: A submodule *K* of *M* is called generalized coessential submodule of *N* in *M* denoted by

$$K \subseteq_{GCe} N, \text{ if } \frac{N}{K} \subseteq Rad\frac{M}{K}.$$

It is clear that, if K is coessential submodule of *N* in *M*, then K is generalized coessential submodule of N in M. However the converse in general is not true, for example $0 \subseteq_{GCe}$ Q as Z-module, but 0 is not coessential of *Q*.

Definition2.2[4]: An R-module M is called generalized lifting or satisfies (GD1), if for every submodule *N* of *M*, there exists a direct summand *K* of *M*, such that $K \subseteq_{GCe}$ *N* in *M*.
It is clear that every lifting module is a generalized lifting module. An R-module M is called hollow lifting, if every submodule N of M such that $\frac{M}{N}$hollow has a coessential submodule that is a direct summand of M [6].
It is know that *Rad(M)* ⊆ $Rad_g(M)$, [8].
The following gives the properties of $Rad_g(M)$ which appeared in [8].

Lemma2.3: The following assertions are holds:

1. If M be an R -module , then $Rm \ll M$ for every m∈ $Rad(M)$.
2. If f:$M \to N$ is an R-module homomorphism , then f($Rad_g(M) \subseteq Rad_g(N)$.
3. If N⊆ M , then $Rad_g(N) \subseteq Rad_g(M)$.
4. If K,L⊆ M , then $Rad_g(K) + Rad_g(L) \subseteq Rad_g(K + L)$.
5. If K,L⊆ M, then $Rad_g \frac{K+L}{L} \subseteq \frac{Rad_g(K+L)}{L}$.
6. If M=$\oplus_{i\in I} M_i$, then $Rad_g(M) = \oplus_{i\in I} Rad_g(M_i)$.

Lemma 2.4: Let N be a direct summand submodule of M. Then $Rad_g(N) = Rad_g(M) \cap N$.
Proof: See [8].
As a generalization of generalized of hollow lifting module we introduce the following:-
Definition 2.5: An R-module M is called G-hollow -lifting$_g$ module, if for every submodule N of M with $\frac{M}{N}$ is G-hollow has a G-coessential submodule in M that is a direct summand of M.
Examples and Remarks 2.6:

1- Z_4 as Z-module is G-hollow $lifting_g$ module.
2- M=Z_{12} as Z-module is not G-hollow $lifting_g$ module, since let N=<$\bar{2}$> and K=<$\bar{4}$> is a direct summand of M such that $\frac{N}{K} \nsubseteq Rad_g \frac{M}{N}$.

Proposition 2.7: Let M be a G-hollow $lifting_g$ module , then every submodule N of M such that $\frac{M}{N}$ G-hollow, can be written as N = K ⊕L , where K is a direct summand of M and N∩L⊆ Rad_g(M).

Proof : Let N⊆ M , with$\frac{M}{N}$ is G- hollow, since M be a G-hollow $lifting_g$ module, then ∃K⊆M , K⊆N and $\frac{N}{K} \subseteq Rad_g\left(\frac{M}{K}\right)$, let L⊆M with M = K ⊕L then N = K⊕(L ∩ N). Now $\frac{N}{K}=\frac{(K \oplus (L\cap N)}{K} \cong \frac{N\cap L}{K\cap(N\cap L)}$ But N/ K ⊆ Rad_g (M / K) = Rad_g (K ⊕L)/K≅ Rad_g (L / (L ∩ K)) Thus N∩ L⊆ Rad_g (L) ⊆ Rad_g (M).

Proposition 2.8: Let M_1 andM_2 be G- hollow modules, if M = $M_1 \oplus M_2$ then the following are equivalent:
1. M is G-hollow lifting$_g$.
2. M is G-lifting.

Proof : 1→ 2 Let N ⊆M , let π_1 : M →M_1 and π_2: M →M_2 . If π_1 (N) ≠ $M_1$and π_1 (N)≠ M_2, then π_1 (N)$\ll_G$ M_1 and π_2 (N)) $\ll_G$ M_2.Thus π_1 (N)⊕ π_2 (N) $\ll_G$ $M_1 \oplus M_2$.[9]
Now let n ϵ N, then nϵ M = $M_1 \oplus M_2$, hence n= $m_1 + m_2$, where m_1 ϵ M1, m_2ϵ M_2
π_1 (n) = π_1 $(m_1 + m_2)$= $m_1$and π_2 (n) = π_2 $(m_1 + m_2)$= m_2, thus n= π_1 (n) +π_2 (n) this implies that N ⊆ π_1 (N) ⊕π_2 (N)) therefore N $\ll_G$ M. Assume that π_1 (N) = M_1then M= N +M_2 , thus M / N = N + M_2 /N but M_2 is G- hollow, hence$M_2 + N$/ N is a G- hollow this implies that M / N is G-hollow, therefore ∃ $K \subseteq \oplus M$ such that N / K < Rad_g(M / K),hence M is a generalized lifting.
2 → 1 Clear .

Remark 2.9: It is clear that every module has no hollow factor module is a G-hollow $lifting_g$ module . However, if M is indecomposable we have the following:

Proposition 2.10: Let M be an indecomposable module, then the following are equivalent :
1. M is G-hollow $lifting_g$ module.
2. M is G-hollow or else M has no G-hollow factor module.

Proof: 1→2 Suppose that M has a G-hollow factor module, then ∃N⊆M, such that is $\frac{M}{N}$ G-hollow . Since M is G-hollow $lifting_g$ module., then ∃K ⊆ M, K$\subseteq_\oplus M$ such that $\frac{N}{K} \subseteq Rad_g(M)$.But M is indecomposable, then K = 0 and hence N⊆ Rad_g (M).
2→1 Clear.

Let R be any ring, and M is an R-module.Let N, K be two submodules of M, K is called strong supplement of N in M, if K is a supplement of N in M, and K ∩N is a direct summand of N ,[3].
As a generalization of strong supplement submodule, we introduce the following:

Definition 2.11: Let N, K be submodules of M. K is called a generalized strong supplement of N (for short G-strong supplement of N), if M = N+K with K∩N ⊆ Rad_g (K) and K∩ N$\subseteq_\oplus$N.

It is clear that if K is strong supplement submodule in M, then K is G-strong supplement submodule , but the converse in general is not true , for example : consider Z_{12} as Z-module , let N={$\bar{0}$,$\bar{4}$,$\bar{8}$}, it is clear that N is G-strong supplement since there exist a direct summand 0 of M , $N \ll_G M$, but N not small in M.

Remark 2.12: In semisimple modules, every submodule is G-strong supplement.

Proposition 2.13: Let N⊆M , then the following are equivalent:
1. N has a G-strong supplement in M.
2. N has a G-coessential submodule that is a direct summand of M.

Proof: 1→2 Let K be a G-strong supplement of N in M, then M = N+K, N∩K⊆ Rad_g(M) and N ∩K $\subseteq_\oplus$ N , hence ∃ L ⊆N such that (N ∩ K)⊕ L = N , then M = L ⊕ K . Now $\frac{N}{L}=\frac{(N\cap L)\oplus K}{L} \subseteq \frac{Rad_g(\,M\,)+L}{L} \subseteq Rad_g\left(\frac{M}{L}\right)$.

2 →1 Let N⊆ M ,then by (2),∃ $K \subseteq N$ such that $\frac{N}{K} \subseteq Rad_g\left(\frac{M}{K}\right)$ and K is a direct summand of M,hence M=K⊕L for L⊆ $M.$ Thus N=N∩ $(K⊕L)$= $K⊕(N \cap L$) thus $N \cap L$ *is a direct summand of N.*

$now\ \frac{N}{K} = \frac{K+(N\cap L)}{K} \cong \frac{(N\cap L)}{N\cap L\cap K} = \frac{(N\cap L)}{L\cap K} \cong N \cap L.$

But $\frac{N}{K} \subseteq Rad_g\left(\frac{M}{K}\right)$, *then* $N \cap L \subseteq Rad_g(M)$.

Thus N has a G-strong supplement in M.

Corollary 2.14: Let M be any R-module, then the following are equivalent:

1. M is a G-hollow $lifting_g$ module.
2. Every submodule N of M, with $\frac{M}{N}$is G-hollow , has a G-strong supplement in M.

Proposition 2.15: Let M be a G-hollow module , Then the following are equivalent:

1. M is a G-hollow $lifting_g$module.
2. M is a G-lifting module.

Proof : 1→2 by [4], for any N⊂ M,$\frac{M}{N}$ is G-hollow and by (1) M is G-lifting.

2→1 Clear.

3. The direct sum of G-hollow $lifting_g$ module

In this section we study the quotient and the direct sum of G-hollow $lifting_g$ module , we prove under certain condition the quotient and the direct summand of G-hollow $lifting_g$module is G-hollow $lifting_g$module.

Remark 3.1: the quotient module of G-hollow $lifting_g$module needn't be G-hollow $lifting_g$ the following example shows:

Example3.2: Consider the Z-module M =$\frac{Z}{4Z} \oplus \frac{Z}{8Z}$, let N =$\frac{2Z}{4Z} \oplus < 0 >$, clearly that M is G-hollow $lifting\ _g$module, since it is lifting but $\frac{M}{N}$ is not, since $\frac{M}{N} = \frac{\frac{Z}{4Z}\oplus\frac{Z}{8Z}}{\frac{2Z}{4Z}\oplus\ <0>} \cong \frac{\frac{Z}{4Z}}{\frac{2Z}{4Z}} \oplus \frac{Z}{8Z}$. Then$\frac{M}{N} \cong \frac{Z}{2Z} \oplus \frac{Z}{8Z}$ which is not G-hollow $lifting_g$.

Recall that a submodule N of M is called fully invariant if $f(N) \subseteq N$ for every f∈ $End(\ M\ )$, and an R-module M is called duo module, if every submodule of M is fully invariant,[10].

Proposition3.3 : Let M be any R-module, if M is a G-hollow $lifting_g$ module, then $\frac{M}{N}$is a G-hollow $lifting_g$ module, for every fully invariant submodule N of M.

Proof: Let N be a fully invariant submodule of M, and let $\frac{K}{N} \subseteq \frac{M}{N}$such that$\frac{M/N}{K/N} \cong \frac{M}{K}$ is G-hollow . Since M is G-hollow $lifting_g$, then ∃L⊆ ⊕M, such that L⊆K, $\frac{K}{L} \subseteq Rad_g$ ($\frac{M}{L}$) and M =K_1⊕L for K_1 ⊆M, clearly N+L⊆K, then $\frac{L+N}{N} \subseteq \frac{K}{N}$. Define f: $\frac{M}{L} \to \frac{M}{N+L}$ by f(m+ L)=m+(L+N), ∀m⊂M. It is clear that f is an epimorphism, f($\frac{K}{L}$)⊆ Rad_g($\frac{M}{N+L}$), then K+(L+N)⊆ Rad_g ($\frac{M}{N+L}$), hence$\frac{K}{N+L} \subseteq_{Gce} \frac{M}{N+L}$. Now$\frac{M}{N}$ =$\frac{K_1+N}{N} \oplus \frac{L+N}{N}$,hence L+N/ N$\subseteq_{\oplus}$ $\frac{M}{N}$, thus $\frac{M}{N}$ is a G-hollow $lifting_g$ module.

Corollary3.4: The direct summand of duo G-hollow $lifting_g$ module is again G-hollow $lifting_g$ module

Remark 3.5: The direct sum of two G-hollow $lifting_g$ modules need not be a G-hollow lifting as the following example shows:

Example3.6: The modules $Z_4 and\ Z_3$as Z-module are G-hollow $lifting_g$ modules.

While the module Z_4⊕Z_3 ≅ Z_{12}which is not G-hollow $lifting_g$ module.

The following shows under curtain condition the direct sum of two G-hollow $lifting\ _g$ is again G-hollow $lifting\ _g$module.

Proposition 3.7: Let M be a duo module such that M = M_1⊕M_2 , if M_1 and M_2 are G-hollow $lifting\ _g$modules, then M is a G-hollow $lifting\ _g$ module.

Proof: Let N⊆M with $\frac{M}{N}$is G-hollow, then N∩ M = (N∩ M_1)⊕(N∩ M_2) by [9]. Hence$\frac{M}{N}$ =$\frac{M_2 \oplus M_2}{(N\cap M_1)\oplus(N\cap M_2)} \cong \frac{M_1}{N\cap M_1} \oplus \frac{M_2}{N\cap M_2}$, thus $\frac{\frac{M}{N}}{\frac{M_1}{N\cap M_1}} \cong \frac{M_2}{N\cap M_2}$is G-hollow, and similarly $\frac{M_1}{N\cap M_1}$is G-hollow. SinceM_1 and M_2 are G-hollow $lifting_g$module, then $\exists k_1 \subseteq_{\oplus} M_1$ with $k_1 \subseteq N \cap M_1 and\ \frac{N\cap M_1}{K} \subseteq Rad_g\left(\frac{M_1}{K_1}\right)$, $M_1 = K_1 \oplus L_1$, $L_1 \subseteq M_1$ and $\exists K_2 \subseteq_{\oplus} M_2$ with $K_2 \subseteq N \cap M_2$ and $\frac{N\cap M}{K_2} \subseteq Rad_g\left(\frac{M_2}{K_2}\right)$, $M_2 = K_2 \oplus L_2$, $L_2 \subseteq M_2$. Thus $K_1 + K_2 \subseteq (N \cap M_1) + (N \cap M_2) = N$and $K_1 + K_2 \oplus L_1 + K_2 = M_1 \oplus M_2 = M$.Thus $K_1 \oplus K_2 \subseteq_{\oplus}$M.

Now,$\frac{N}{K_1+K_2} = \frac{(N\cap M_1)\oplus(N\cap M_2)}{K_1 \oplus K_2} \cong \frac{N\cap M_1}{K_1} \oplus \frac{N\cap M_2}{K_2} \subseteq Rad_g\left(\frac{M_1}{K_1}\right) + Rad_g\left(\frac{M_2}{K_2}\right) \subseteq Rad_g\left(\frac{M}{K_1+K_2}\right)$. Then $K_1 + K_2 \subseteq_{Gce} N$, and hence M is G-hollow $lifting_g$ module.

Corollary3.8: let M=$M_1 \oplus M_2 \oplus \ldots . \oplus M_n$ be a duo module if $\forall i = 1,2, \ldots . , n , M_i$ is a G-hollow $lifting\ _g$module, then M is M is a G-hollow $lifting\ _g$module.

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