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Properties of fuzzy absolute value on ℝ and Properties Finite Dimensional Fuzzy Normed Space

Jehad R. Kider*, Ahmed H. ALi
Department of Mathematics and Computer Applications, School of Applied Sciences, University of Technology, Baghdad, Iraq.

Abstract
The first aim in this paper is to introduce the definition of fuzzy absolute value on the vector space of all real numbers ℝ then basic properties of this space are investigated. The second aim is to prove some properties that finite dimensional fuzzy normed space have.
Keywords: Fuzzy Absolute value, Finite dimensional fuzzy normed space, Complete fuzzy Normed space, Compact fuzzy normed space.

خواص القيمة المطلقة الضبابية على ℝ وخواص فضاء القياس الضبابي المنتهي البعد

جهاد رمضان خضر * ، احمد حسن علي
فرع الرياضيات وتطبيقات الحاسوب، قسم العلوم التطبيقية، الجامعة التكنولوجية، بغداد، العراق.

الخلاصة
الهدف الاول في هذا البحث هو تقديم تعريف القيمة المطلقة الضبابية على فضاء المتجهات الذي يمثل مجموعة الاعداد الحقيقية وبعد ذلك تم البحث على الخواص الاساسية لهذا الفضاء. الهدف الثاني لهذا لبحث هو برهان بعض الخواص التي يمتلكها فضاء القياس الضبابي المنتهي البعد.

1. Introduction
Zadeh in his research in [1] introduced fuzzy logic and fuzzy set theory then this paper found it applications in variety branch of sciences. Many mathematicians at this time tried to translate the classical theory of various branches of mathematics in fuzzy context. Katsaras [2] in his study of fuzzy topological linear space was the first who introduced the notion of fuzzy norm. Another type of fuzzy norm was introduced by Felbin [3]. The idea of fuzzy norm corresponding to the fuzzy metric of type Kramosil and Michalek [4] was introduced by Cheng and Mordenson [5]. We believe that the suitable definition of fuzzy norm was introduced by Bag and Samanta [6] and they study properties of finite dimensional fuzzy normed space. Other approaches for fuzzy normed space can be found in [7-13].

Absolute value is the tool for real analysis to translate the classical results in this paper we introduce the notion of fuzzy absolute value and proved the famous result that ℝ with fuzzy absolute value is complete. After that we study completeness and compactness properties over finite dimensional fuzzy normed linear spaces.

2. Fuzzy normed space
In this section we recall basic properties of fuzzy normed space

Definition 2.1: [1]
Suppose that U is any set, a fuzzy set $\widetilde{A}$ in U is equipped with a membership function, $\mu_{\widetilde{A}}(u): U \to [0,1]$. Then $\widetilde{A}$ is represented by $\widetilde{A} = \{(u, \mu_{\widetilde{A}}(u): u \in U, 0 \le \mu_{\widetilde{A}}(u) \le 1\}$.

*Email: jehadkider@gmail.com

Definition 2.2: [2]
Let $*$: $[0,1]\times[0,1]\to[0,1]$ be a binary operation then $*$ is called a continuous **t -norm** (or **triangular norm**) if for all $\alpha,\beta,\gamma,\delta\in[0,1]$ it has the following properties
(1) $\alpha*\beta=\beta*\alpha$.
(2) $\alpha*1=\alpha$.
(3) $(\alpha*\beta)*\gamma=\alpha*(\beta*\gamma)$.
(4) If $\alpha\le\beta$ and $\gamma\le\delta$ then $\alpha*\gamma\le\beta*\delta$.
Remark 2.3: [14]
(1) If $\alpha>\beta$ then there is γ such that $\alpha*\gamma\ge\beta$.
(2) There is δ such that $\delta*\delta\ge\sigma$ where $\alpha,\beta,\gamma,\delta,\sigma\in[0,1]$.
Definition 2.4: [6]
The triple $(V,L,*)$ is said to be a **fuzzy normed space** if V is a vector space over the field $\mathbb{F}$,$*$ is a t-norm and $L: V\times[0,\infty)\to[0,1]$ is a fuzzy set has the following properties for all $a,b\in V$ and $\alpha,\beta>0$.
1- $L(a,\alpha)>0$.
2- $L(a,\alpha)=1$ *if and only if* $a=0$.
3- $L(ca,\alpha)=L\left(a,\frac{\alpha}{|c|}\right)$ *for all* $c\neq 0\in\mathbb{F}$.
4- $L(a,\alpha)*L(b,\beta)\le L(a+b,\alpha+\beta)$.
5- $L(a,.):[0,\infty)\to[0,1]$ is continuous function of α.
6- $lim_{\alpha\to\infty}L(a,\alpha)=1$.
Remark 2. 5: [14]
Assume that $(V,L,*)$ is a fuzzy normed space and let $a\in V$, $t>0$, $0<q<1$. If $L(a,t)>(1-q)$ then there is s with $0<s<t$ such that $L(a,s)>(1-q)$.
Definition 2.6:[6]
Suppose that $(V,L,*)$ is a fuzzy normed space. Put
$FB(a,p,t)=\{b\in X: L(a-b,t)>(1-p)\}$
$FB[a,p,t]=\{b\in X: L(a-b,t)\ge(1-p)\}$
Then $FB(a,p,t)$ and $FB[a,p,t]$ is called **open and closed fuzzy ball** with the center $a\in V$ and radius p, with $p>0$.
Definition 2.8: [6]
Assume that $(V,L,*)$ is a fuzzy normed space $W\subseteq V$ is called **fuzzy bounded** if we can find $t>0$ and $0<q<1$ such that $L(w,t)>(1-q)$ for each w $\in$ W.
Definition 2.9 :[6]
A sequence (v_n) in a fuzzy normed space $(V,L,*)$ is called **converges to** $v\in V$ if for each $q>0$ and $t>0$ we can find N with $L[v_n-v,t]>(1-q)$ for all $n\ge N$. Or in other word $lim_{n\to\infty}v_n=v$ or simply represented by $v_n\to v$, v is known the limit of (v_n) or $lim_{n\to\infty}L[v_n-v,t]=1$.
Definition 2. 10: [6]
A sequence (v_n) in a fuzzy normed space $(V,L,*)$ is said to be a **Cauchy sequence** if for all $0<q<1$, $t>0$ there is a number N with $L[v_m-v_n,t]>(1-q)$ for all $m,n\ge N$.
Definition 2.11: [4]
Suppose that $(V,L,*)$ is a fuzzy normed space and let W be a subset of V. Then the **closure of W** is written by $\overline{W}$ or $CL(W)$ and which is $\overline{W}=\bigcap\{W\subseteq B: B$ is closed in $V\}$.
Lemma 2.12: [14]
Assume that $(V,L,*)$ is a fuzzy normed space and suppose that W is a subset of V. Then $y\in\overline{W}$ if and only if there is a sequence (w_n) in W with (w_n) converges to y.
Definition 2.15: [14]
A fuzzy normed space $(V,L,*)$ is said to be **complete** if every Cauchy sequence in V converges to a point in V.
Definition 2.16: [15]
Let L_1 and L_2 be two fuzzy norms on V with for all $(v_n)\in V$ and v in V then $lim_{n\to\infty}L_1[v_n-v,s]=1$ if and only if $lim_{n\to\infty}L_2[v_n-v,t]=1$ for all $t>0,s>0$. *t*hen L_1 and

L_2 are said to be **equivalent fuzzy norms** on V. Also $(V, L_1,*)$ and $(V, L_2,*)$ are **equivalent fuzzy normed spaces.**

Theorem 2.17: [15]

Two fuzzy norms L_1 and L_2 on a vector space V are equivalent if we find $k \in \mathbb{R}$ with $\frac{1}{k}\, L_2(v,t) \leq\, L_1(v,s) \leq\, k\, L_2(v,t)$ for all $v\, \in\, V$ and $t > 0, s > 0$.

Theorem 2.18:[15]

The fuzzy normed space $(V, L,*)$ is compact if and only if every (v_n) in V contains (v_{n_k}) with $v_{n_k} \to v$.

Proposition 2.19:[15]

Suppose that $(V, L,*)$ a fuzzy normed space and $W \subset V$. If W is compact then W is closed.

3. Fuzzy Absolute Value

First we introduce the main definition in this section

Definition 3.1:

Let $\mathbb{R}$ the vector space of real numbers over the field $\mathbb{R}$ and $\odot, \otimes$ be a continuous t-norm. A fuzzy set $L_{\mathbb{R}}$: $\boldsymbol{R} \times [\mathbf{0}, \infty)$ is called a fuzzy absolute value on $\mathbb{R}$ if it satisfies the following conditions for all $a, b, \in\, \mathbb{R}$;

(L1) $0 \leq\, L_{\mathbb{R}}(a,t) < 1$ for all $t > 0$.

(L2) $L_{\mathbb{R}}(a,t) = 1$ for all $t > 0$ if and only if $a = 0$.

(L4) $L_{\mathbb{R}}(a+b, t+s)\, \geq L_{\mathbb{R}}(a,t) \odot L_{\mathbb{R}}(b,s)$.

(L5) $L_{\mathbb{R}}(ab, st)\, \geq L_{\mathbb{R}}(a,t) \otimes L_{\mathbb{R}}(b,s)$.

(L6) $L_{\mathbb{R}}(a,\cdot) :\, [0,\infty)\, \to\, [0,\infty)$ is a continuous function of t.

(L7) $lim_{t\to\infty} L_{\mathbb{R}}(a,t) = 1$.

Example 3.2:

Define $L_{\mathbb{R}}(a,t) = \frac{t}{t+|a|}$ for all $a \in \mathbb{R}$ then L is a fuzzy absolute value on $\mathbb{R}$ where $a \odot b\, = a \otimes b = a\, \cdot\, b$ for all $a, b \in \mathbb{R}$.

Definition 3.3:

Let $\{a_n\}_{n=1}^{\infty}$be a sequence in $\mathbb{R}$, we say that $\{a_n\}_{n=1}^{\infty}$ fuzzy approaches the limit a as n approaches to ∞ if for every $\varepsilon \in\, (0,1)$ there exists $N\, \in\, \mathbb{R}$ such that $L_{\mathbb{R}}(a_n - a, t)\, >\, (1-\varepsilon)$ for all $t > 0$ and for all $n \geq N$. If a_n fuzzy approaches the limit a we write or $lim_{n\to\infty} a_n = a$, $a_n \to a$ or $lim_{n\to\infty} L_{\mathbb{R}}(a_n - a, t) = 1$.

Theorem 3.4:

If $\{a_n\}_{n=1}^{\infty}$is a sequence in $\mathbb{R}$ such that $lim_{n\to\infty} a_n = a$ and $lim_{n\to\infty} a_n = b$ then $a = b$.

Theorem 3.5:

If the sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ fuzzy approaches the limit a then any subsequence of it is also fuzzy approaches to a.

Proof:

Since $a_n \to a$ then $\lim_{n\to\infty} L_{\mathbb{R}}\left[a_n - a, \frac{t}{2}\right] = 1$. Also (a_n) is a Cauchy sequence then

$\lim\limits_{n\to\infty}\, L_{\mathbb{R}}\left[a_n - a_m, \frac{t}{2}\right] =\, 1$ when $n \to \infty$ and $m \to \infty$. Now

$L_{\mathbb{R}}\left[a_{n_k} - a, t\right] = L_{\mathbb{R}}\left[a_{n_k} - a_n + a_n - a, t\right] \geq L_{\mathbb{R}}\left[a_{n_k} - a_n, \frac{t}{2}\right] *\, L_{\mathbb{R}}\left[a_n - a, \frac{t}{2}\right]$

Now $lim_{n\to\infty} L_{\mathbb{R}}\left[a_{n_k} - a, t\right] \geq 1 * 1 = 1$. Hence $\left(a_{n_k}\right)$converges to a.

Definition 3.6:

The sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ is said to be fuzzy bounded if there exists $q\, \in\, [0,1]$ such that $L_{\mathbb{R}}\, (a_n, t) > (1-q)$ for all $t > 0$.

Theorem 3.7:

If the sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ fuzzy approaches the limit a then it is fuzzy bounded.

Proof:

Suppose that $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ fuzzy approaches the limit a then for every $\varepsilon \in\, (0,1)$ there exists $N \in \mathbb{R}$ such that $L_{\mathbb{R}}(a_n - a, \frac{t}{2}) > (1-\varepsilon)$ for all $t > 0$ and for all $n \geq N$.

This implies

$L_{\mathbb{R}}(a_n, t) = L_{\mathbb{R}}(a - a_n - a, t) \geq L_{\mathbb{R}}(a, \frac{t}{2}) \odot L_{\mathbb{R}}(a_n - a, \frac{t}{2}) \geq L_{\mathbb{R}}(a, \frac{t}{2}) \odot (1-\varepsilon)$

Now put $(1-q) = min\{L_{\mathbb{R}(}a_1, t_1), L_{\mathbb{R}}: (a_2, t_2), \dots, L_{\mathbb{R}}(a_{N-1}, t_{N-1})\}$.
Then $L_{\mathbb{R}}(a_n, t) \geq (1-q) \odot L(a, \frac{t}{2}) \odot (1-\varepsilon) > (1-p)$ for some $q \in [0,1]$ ■

Theorem 3.8:

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences in $\mathbb{R}$ if $\{a_n\}_{n=1}^{\infty}$ fuzzy approaches the limit a and $\{b_n\}_{n=1}^{\infty}$ fuzzy approaches the limit b then $\{a_n + b_n\}_{n=1}^{\infty}$ fuzzy approaches the limit $a + b$.

Proof:

Since $\{a_n\}_{n=1}^{\infty}$ approaches the limit a then for every $r \in (0,1)$ there exists $N \in \mathbb{R}$ such that $L_{\mathbb{R}}(a_n - a, t) > (1-r)$ for all $t > 0$ and for all $n \geq N$. Also Since $\{b_n\}_{n=1}^{\infty}$ approaches the limit b then for every $p \in (0,1)$ there exists $N \in \mathbb{R}$ such that $L_{\mathbb{R}}(b_n - b, t) > (1-p)$ for all $t > 0$ and for all $n \geq N$. Now

$L_{\mathbb{R}}[a_n + b_n - (a+b), t+s] \geq L_{\mathbb{R}}(a_n - a, t) \odot L_{\mathbb{R}}(b_n - b, s) \geq (1-r) \odot (1-p)$

Put $(1-r) \odot (1-p) = (1-q)$ for some $q \in [0,1]$ then $L_{\mathbb{R}}[a_n + b_n - (a+b), t+s] \geq (1-q)$ for all $n \geq N$. ■

Theorem 3.9:

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ and $c \neq 0 \in \mathbb{R}$. If $\{a_n\}_{n=1}^{\infty}$ is fuzzy approaches the limit a then $\{ca_n\}_{n=1}^{\infty}$ is fuzzy approaches the limit ca.

Proof:

Since $\{a_n\}_{n=1}^{\infty}$ approaches the limit a then for every $r \in (0,1)$ there exists $N \in \mathbb{R}$ such that $L_{\mathbb{R}}(a_n - a, t) \geq (1-r)$ for all $t > 0$ and for all $n \geq N$. Now $L_{\mathbb{R}}(ca_n - ca, t) = L_{\mathbb{R}}[c(a_n - a), t] = L_{\mathbb{R}}(c, t) \otimes L_{\mathbb{R}}(a_n - a, t) \geq L_{\mathbb{R}}(c, t) \otimes (1-r)$. Let $L_{\mathbb{R}}(c, t) = (1-\sigma)$. Now choose α where $0 < \alpha < 1$ such that $(1-\sigma) \otimes (1-r) > (1-\alpha)$ so $L_{\mathbb{R}}(ca_n - ca, t) > (1-\alpha)$.

Hence $\{ca_n\}_{n=1}^{\infty}$ is fuzzy approaches to the limit ca.■

Theorem 3.10:

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences in $\mathbb{R}$ if $\{a_n\}_{n=1}^{\infty}$ fuzzy approaches the limit a and $\{b_n\}_{n=1}^{\infty}$ fuzzy approaches the limit b then $\{a_n \cdot b_n\}_{n=1}^{\infty}$ fuzzy approaches the limit a·b.

Proof:

Since $\{a_n\}_{n=1}^{\infty}$ fuzzy approaches the limit a then for every $r \in (0,1)$ there exists $N \in \mathbb{R}$ such that $L_{\mathbb{R}}(a_n - a, t) > (1-r)$ for all $t>0$ and for all $n \geq N$. Also Since $\{b_n\}_{n=1}^{\infty}$ approaches the limit b then for every $p \in (0,1)$ there exists $N \in \mathbb{N}$ such that $L(b_n - b, t) > (1-p)$ for all $t>0$ and for all $n \geq N$. Now

$$L_{\mathbb{R}}[a_n b_n - ab, t) = L_{\mathbb{R}}[a_n b_n - ab_n + ab_n - ab, t]$$
$$\geq L_{\mathbb{R}}[a_n b_n - ab_n, \tfrac{t}{2}] \odot L_{\mathbb{R}}[ab_n - ab, \tfrac{t}{2}]$$
$$\geq L_{\mathbb{R}}(b_n, \sqrt{\tfrac{t}{2}}) \otimes L_{\mathbb{R}}[a_n - a, \sqrt{\tfrac{t}{2}}] \odot L_{\mathbb{R}}(a, \sqrt{\tfrac{t}{2}}) \otimes L_{\mathbb{R}}[b_n - b, \sqrt{\tfrac{t}{2}}]$$
$$\geq L_{\mathbb{R}}(b_n, \sqrt{\tfrac{t}{2}}) \otimes (1-r) \odot L_{\mathbb{R}}(a, \sqrt{\tfrac{t}{2}}) \otimes (1-p)$$

Put $L_{\mathbb{R}}(b_n, \sqrt{\frac{t}{2}}) = (1-\alpha)$ and $L_{\mathbb{R}}(a, \sqrt{\frac{t}{2}}) = (1-\delta)$ for some $0 < \alpha, \delta < 1$. Now let $(1-q)$ for some $q \in [0,1]$ be choose so that $(1-\alpha) \otimes (1-r) \odot (1-\delta) \otimes (1-p) > (1-q)$.

Hence $L_{\mathbb{R}}[a_n b_n - ab, t) \geq (1-q)$ for all $n \geq N$. ■

Definition 3.11:

Let $\{a_n\}_{n=1}^{\infty}$ be a sequences in $\mathbb{R}$ that is fuzzy bounded above and let $M_n = l.u.b\{a_n, a_{n+1}, \dots\}$. If $\{M_n\}$ approaches we define $\lim_{n\to\infty} \sup a_n = \lim_{n\to\infty} M_n$. If $\{a_n\}_{n=1}^{\infty}$ is not bounded above we write $\lim_{n\to\infty} \sup a_n = \infty$.

Theorem 3.12:

If $\{a_n\}_{n=1}^{\infty}$ is sequence in $\mathbb{R}$ fuzzy approaches the limit a then $\lim_{n\to\infty} \sup a_n = \lim\limits_{n\to\infty} a_n = a$

Proof:

Since $\{a_n\}_{n=1}^{\infty}$ fuzzy approaches the limit a then for every $r \in (0, 1)$ there exists $N \in \mathbb{N}$ such that $L(a_n - a, t) \geq (1-r)$ for all $t > 0$ and for all $n \geq N$.Or

$a + (1-r) \leq a_n \leq a - (1-r)$. Thus if $n \geq N$, $a - (1-r)$ is an upper bound for the set $\{a_n, a_{n+1}, \dots\}$ and $a + (1-r)$ is not an upper bound . Hence

$a + (1-r) \leq M_n = l.u.b\{a_n, a_{n+1}, \dots\} \leq a - (1-r)$ that is

$a + (1-r) \leq lim_{n\to\infty} M_n \leq a - (1-r)$ but $lim_{n\to\infty} M_n = lim_{n\to\infty} sup\, a_n$. Thus

$a + (1 - r) \leq lim_{n\to\infty} sup a_n \leq a - (1 - r)$ since $(1 - r)$ was arbitrary this implies $lim_{n\to\infty} sup a_n = a$.■

Definition 3.13:

Let $\{a_n\}_{n=1}^{\infty}$ be a sequences in $\mathbb{R}$ that is fuzzy bounded below and let $m_n = g.l.b\{a_n, a_{n+1}, \dots\}$. If $\{m_n\}$ approaches we define $lim_{n\to\infty} inf a_n = lim_{n\to\infty} m_n$. If $\{a_n\}_{n=1}^{\infty}$is not bounded below we write $lim_{n\to\infty} inf a_n = -\infty$.

Theorem 3.14:

If $\{a_n\}_{n=1}^{\infty}$ is sequence in $\mathbb{R}$ fuzzy approaches the limit a then $lim_{n\to\infty} inf a_n = lim_{n\to\infty} a_n = a$.

Theorem 3.15:

If $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ then $\{a_n\}_{n=1}^{\infty}$ is fuzzy bounded.

Proof:

Since $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ then for every $r \in (0, 1)$ there exists $N \in \mathbb{N}$ such that $L_{\mathbb{R}}(a_m - a_n, \frac{t}{2}) > (1 - r)$ for all $t > 0$ and for all $n, m \geq N$. Then

$L_{\mathbb{R}}(a_m - a_N, \frac{t}{2}) \geq (1 - r)$ for all $t > 0$ and for all $m \geq N$. Hence if $m \geq N$ we have

$L_{\mathbb{R}}(a_m, t) = L_{\mathbb{R}}(a_m - a_N + a_N, t) \geq L_{\mathbb{R}}\left(a_m - a_N, \frac{t}{2}\right) \odot L_{\mathbb{R}}(a_N, \frac{t}{2})$ and so

$L_{\mathbb{R}}(a_m, t) \geq (1 - r) \odot L_{\mathbb{R}}(a_N, \frac{t}{2})$, Now put $(1 - p) = min\{ L_{\mathbb{R}}(a_1, t), L_{\mathbb{R}}(a_2, t), \dots, L_{\mathbb{R}}(a_{N-1}, t)\}$

Then $L_{\mathbb{R}}(a_m, t) \geq (1 - r) \odot L_{\mathbb{R}}(a_N, \frac{t}{2}) \odot (1 - p)$. Hence $\{a_n\}_{n=1}^{\infty}$ is fuzzy bounded.■

The following is the main results in this section

Theorem 3.16:

Every Cauchy $\{a_n\}_{n=1}^{\infty}$ sequence in $\mathbb{R}$ is fuzzy approaches the limit $a \in \mathbb{R}$. That is $(\mathbb{R}, L_{\mathbb{R}})$ is complete fuzzy normed space.

Proof:

Let be a Cauchy $\{a_n\}_{n=1}^{\infty}$ sequence in $\mathbb{R}$ then $\{a_n\}_{n=1}^{\infty}$has a monotonic subsequence $\{a_{n_j}\}_{n=1}^{\infty}$ but $\{a_n\}_{n=1}^{\infty}$is fuzzy bounded hence $\{a_{n_j}\}_{n=1}^{\infty}$is fuzzy bounded. Thus $\{a_{n_j}\}_{n=1}^{\infty}$ fuzzy approaches the limit $a \in \mathbb{R}$. Then for every $r \in (0,1)$ there exists $N \in \mathbb{R}$ such that $L_{\mathbb{R}}(a_{n_j} - a, \frac{t}{2}) \geq (1 - r)$ for all $t > 0$ and for all $n \geq N$. Since $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ then for every $r \in (0,1)$ there exists $N \in \mathbb{R}$ such that $L_{\mathbb{R}}(a_m - a_n, \frac{t}{2}) > (1 - r)$ for all $t > 0$ and for all $n, m \geq N$. Then

$L_{\mathbb{R}}(a_m - a_N, \frac{t}{2}) \geq (1 - r)$ for all $t > 0$ and for all $m \geq K$. We may choose $K \geq N$.

Now suppose $k \geq K$ then $k \geq N$, $L_{\mathbb{R}}(a_{n_k} - a_k, \frac{t}{2}) \geq (1 - r)$.

$$L_{\mathbb{R}}(a_k - a, t) = L_{\mathbb{R}}(a_k - a_{n_k} + a_{n_k} - a, t) \geq L_{\mathbb{R}}(a_k - a_{n_k}, \frac{t}{2}) \odot L_{\mathbb{R}}(a_{n_k} - a, \frac{t}{2})$$
$$\geq (1 - r) \odot (1 - r) \geq (1 - \varepsilon)$$

For some $\varepsilon \in (0,1)$. Thus $\{a_{n_j}\}_{n=1}^{\infty}$ fuzzy approaches the limit $a \in \mathbb{R}$.

Definition 3.17:

We say that $f(x)$ approaches d where $d \in \mathbb{R}$ as x approaches a if for any given $r \in (0,1)$ and $t > 0$ there exists $p \in (0,1)$ and $s > 0$ such that $L_{\mathbb{R}}(f(x) - d, t) \geq (1 - r)$ whenever $L_{\mathbb{R}}(x - a, s) \geq (1 - p)$. In this case we write $F\lim_{x\to a} f(x) = d$ or $f(x) \to d$ as $x \to a$.

Theorem 3.18:

If $F\, lim_{x\to a} f(x) = d$ and $F\, lim_{x\to a} g(x) = b$ then $F\, lim_{x\to a}[f(x) + g(x)] = d + b$.

Definition 3.19:

We say that $f(x)$ approaches d where $d \in \mathbb{R}$ as x approaches ∞ if for any given $r \in (0,1)$ and $t > 0$ there exists $p \in (0,1)$ and $s > 0$ such that $L_{\mathbb{R}}(f(x) - d, t) \geq (1 - r)$ when ever $x \to \infty$. In this case we write $F\, lim_{x\to\infty} f(x) = d$ or $f(x) \to d$ as $x \to \infty$.

Definition 3.20:

We say that $f(x)$ from the right approaches d where $d \in \mathbb{R}$ if for any given $r \in (0,1)$ and $t > 0$ there exists $p \in (0,1)$ and $s > 0$ such that $L(f(x) - d, t) \geq (1 - r)$ for all $a < x < a + p$. In this case we write $F\, lim_{x\to a^+} f(x) = d$. Also we say that $f(x)$ from the left approaches d where $d \in \mathbb{R}$ if for any given $r \in (0,1)$ and $t > 0$ there exists $p \in (0,1)$ and $s > 0$ such that $L_{\mathbb{R}}(f(x) - d, t) \geq (1 - r)$ for all $a - p < x < a$. In this case we write $F\, lim_{x\to a^-} f(x) = d$.

Theorem 3.21:
$F\ lim_{x\to a} f(x) = d$ if and only if $F\ lim_{x\to a^+} f(x) = F\ lim_{x\to a^-} f(x) = d$.
Example 3.22:
Show that $F\ lim_{x\to\infty} \frac{1}{x^2} = 0$.
Proof:
Given $r \in (0,1)$ and for all $t > 0$ we must find $d \in \mathbb{R}$ such that $L_{\mathbb{R}}\left(\frac{1}{x^2} - 0, t\right) \geq (1-r)$ ... (1) for all $x > d$ this is equivalent to $\frac{1}{\mathrm{x}} \geq \sqrt{(1-r)}$ for all $x > d$. Choose $d = \frac{1}{\sqrt{1-\mathrm{r}}}$ it is clear that (1) hold.

4. Finite Dimensional Fuzzy Normed Space

In this section we deal with finite dimensional vector spaces with fuzzy norm.
Definition 4.1:
Let $(V, L_V, *)$ and $(\mathbb{R}, L_{\mathbb{R}}, *)$ be two fuzzy normed spaces. Define $\boldsymbol{L_V(\alpha v, t) \geq L_{\mathbb{R}}(\alpha, t) * L_V(v, t)}$ for all $v \in V$ and $\alpha \neq 0 \in \mathbb{R}$.
The following theorem plays the key role in the studying properties of finite dimensional fuzzy normed linear spaces.
Theorem 4.2:
Let $\{v_1, v_2 \ldots, v_n\}$ be linearly independent set in a fuzzy normed space $(V, L_V, *)$. Then there is $0 < r < 1$ such that $L_V[\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n, t] \leq r * L_{\mathbb{R}}(\alpha_j, t)$ for some $1 \leq j \leq n$.
Proof:
Suppose that this is not true, then we can find a sequence (v_m) in V where $v_m = \alpha_{1m} v_1 + \alpha_{2m} v_2 + \cdots + \alpha_{nm} v_n$ such that $L_V(v_m, t) \to 1$ as $m \to \infty$. Now for each fixed j we have a sequence $\alpha_{jm} = (\alpha_{j1}, \alpha_{j2} \ldots, \alpha_{jm}, \ldots)$ is fuzzy bounded since $0 \leq L_{\mathbb{R}}(\alpha_{jm}, t) \leq 1$ so (α_{jm}) has a convergent subsequence. Let α_j denote the limit of the subsequence (α_{jm}) for each $1 \leq j \leq n$. Let (v_{jm}) denote the corresponding subsequence of (v_m) where the corresponding subsequence of scalar α_{jm} converges to α_j for each $1 \leq j \leq n$.
Now put $v = \sum_{j=1}^{n} \alpha_j v_j$ then (v_m) has a subsequence (v_{jm}) converges to v since $\{v_1, v_2 \ldots, v_n\}$ is linearly independent set so $v \neq 0$. Now $v_{jm} \to v$ implies $L_V(v_{jm}, t) \to L(v, t)$ by fuzzy continuity of the fuzzy norm. But $L(v_m, 1) \to 1$ by our assumption and (v_{jm}) is a subsequence of (v_m). Thus $L_V(v_{jm}, t) \to 1$. Hence $L(v, t) = 1$ so $v = 0$. This contradicts $v \neq 0$.
Theorem 4.3:
Let $(V, L_V, *)$ be a fuzzy normed space. If W is a finite dimensional subspace of V then W is complete where $a * b = a \cdot b$ for all $, b \in [0,1]$.
Proof:
Suppose that (v_m) is a Cauchy sequence in W. Let $\dim W = n$ and $B = \{w_1, w_2, \ldots, w_n\}$ be any basis for W. Then each v_m has a unique representation as $v_m = \beta_{1m} w_1 + \beta_{2m} w_1 + \cdots + \beta_{nm} w_n$ since (v_m) is Cauchy sequence so for every $0 < \alpha < 1$ and $t > 0$ there is N such that $L_V(v_m - v_n, t) > (1-\alpha)$ for every $m, n \geq N$. Now by Theorem 4.2 we have some $0 < r < 1$ such that $(1-\alpha) < L_V(v_m - v_n, t) = L_V\left[\sum_{j=1}^{n} (\beta_{jm} - \beta_{jn}) w_j, t\right] \leq r * L_{\mathbb{R}}(\beta_{jm} - \beta_{jn}, t)$
dividing by r we get $L_{\mathbb{R}}(\beta_{jm} - \beta_{jn}, t) > \frac{(1-\alpha)}{r}$. This show that $(\beta_{jm}) = (\beta_{j1}, \beta_{j2}, \ldots)$ is Cauchy sequence in $\mathbb{R}$ or $\mathbb{C}$ Hence $\beta_{jm} \to \beta_j$ for each $1 \leq j \leq n$. Put $= \sum_{\mathrm{j}=1}^{\mathrm{n}} \beta_{\mathrm{j}} \mathrm{w}_{\mathrm{j}}$. Clearly $\in W$.
Also now for all $m > N$

$$L_V(v_m - v, t) = L_V\left[\sum_{j=1}^{n} (\beta_{jm} - \beta_{jn}) w_j, t\right]$$
$$\geq L_V\left(w_1, \frac{t}{n|\beta_{1m} - \beta_1|}\right) * L_V\left(w_2, \frac{t}{n|\beta_{2m} - \beta_2|}\right) \ldots * L_V\left(w_n, \frac{t}{n|\beta_{nm} - \beta_n|}\right)$$

$L_V(v_m - v) \geq (1-r_1) * (1-r_2) * \ldots * (1-r_n)$ where $L_V\left(w_j, \frac{t}{n|\beta_{jm} - \beta_j|}\right) = (1-r_j)$ for some $0 < (1-r_j) < 1$ $j = 1, 2, \ldots n$. Let $0 < (1-\gamma) < 1$ be such that $(1-r_1) * (1-r_2) * \ldots * (1-r_n) > (1-\gamma)$ so $L(v_m - v, t) > (1-\gamma)$ for all $m > N$. Hence $v_m \to v$.
Theorem 4.4:
Let V be a finite dimensional vector space if L_1 and L_2 are two fuzzy norms on V then L_1 is equivalent to L_2.

Proof:

Let dim $V = n$ and $B = \{v_1, v_2, \dots, v_n\}$ be any basis for V. Then for any $v \in V$, $v = \sum_{j=1}^{n} \alpha_j v_j$. Now $L_1(v,t) = L_1\left(\sum_{j=1}^{n} \alpha_j v_j, t\right) \geq L_1(\alpha_j v_j, t) \geq L_{\mathbb{R}}(\alpha_j, t) * L_1(v_j, t)$ put $L_1(v_j, t) = \alpha$ for some $0 < \alpha < 1$ and for some $0 < j < n$ we get $\frac{1}{\alpha} L_1(v,t) \geq L_{\mathbb{R}}(\alpha_j, t) \dots (1)$.

Also $L_2(v,t) = L_2\left(\sum_{j=1}^{n} \alpha_j v_j, t\right) \leq r * L_{\mathbb{R}}(\alpha_j, t)$ for some $0 < j < n$. $\frac{1}{r} L_2(v,t) \leq L_{\mathbb{R}}(\alpha_j, t) \dots (2)$
from 1 and 2 we get
$\frac{1}{r} L_2(v,t) \leq L_{\mathbb{R}}(\alpha_j, t) \leq \frac{1}{\alpha} L_1(v,t)$ or $\frac{1}{r} L_2(v,t) \leq \frac{1}{\alpha} L_1(v,t)$ or
$\frac{\alpha}{r} L_2(v,t) \leq L_1(v,t)$. Similarly we can get $L_1(v,t) \leq \frac{r}{\alpha} L_2(v,t)$
$\frac{\alpha}{r} L_2(v,t) \leq L_1(v,t) \leq \frac{r}{\alpha} L_2(v,t)$. Hence L_1 is equivalent to L_2.

Theorem 4.5:

Let $(V, L_V, *)$ be a finite dimensional fuzzy normed space and $W \subset V$. If W is closed and fuzzy bounded then W is compact.

Proof:

Let dim $W = n$ and $\mathrm{B} = \{\mathrm{w}_1, \mathrm{w}_2, \dots, \mathrm{w}_\mathrm{n}\}$ be any basis for V. Consider the sequence $(v_m) = \alpha_{1m} w_1 + \alpha_{2m} w_2 + \cdots + \alpha_{nm} w_n$ since W is fuzzy bounded so is (v_m) that is $L(v_m, t) \geq (1 - \alpha)$ for all m and for $t > 0$, some $0 < \alpha < 1$. Now by Theorem 4.2 $(1 - \alpha) \leq L\left(\sum_{j=1}^{n} \alpha_{jm} w_j, t\right) \leq r * L_{\mathbb{R}}(\alpha_{jm}, t)$ or $L_{\mathbb{R}}(\alpha_{jm}, t) \geq \frac{(1-\alpha)}{r}$. Hence the sequence (α_{jm}) for fixed j is fuzzy bounded so it has a limit point α_j for each $0 < j < n$. We see that (v_m) has a subsequence (z_m) which converge to $z = \sum_{j=1}^{n} \alpha_j w_j$. Since W is closed so $\in W$. Since (v_m) was an arbitrary sequence in W. Hence W is compact.

Lemma 4.6:

Let $(V, L_V, *)$ be a fuzzy normed space and let W and Z two subspace of V with $W \subseteq Z$ and W is closed. Then for every $r \in (0.1)$ there is $z \in Z$ such that $L(z - w, t) \leq r$ for all $w \in W$.

Proof:

Let $v \in z - w$ and put $a = sup_{w \in W} L(v - w, t)$. Clearly $a > 0$ since W is closed. Take $r \in (0.1)$ with $r > a$ then by definition of suprimum there is $w_0 \in W$ such that $\frac{a}{r} \leq L(v - w_0) \leq a$. put $z = v - w_0$. Now $L(z - w, t) = L(v - w_0 - w, t) = L(v - w_1, t)$ where $w_1 = w_0 + w$. Hence $L(z - w, t) = L(v - w_1, t) \leq a < r$.

Theorem 4.7:

Let $(V, L_V, *)$ be a fuzzy normed space and let $M = \{v \in V : 0 < L(v,t) \leq 1\}$ be a closed fuzzy ball in V which is compact then V must be finite dimension.

Proof:

Suppose that M is compact and dim$V$ is not finite. Choose $v_1 \in M$ and let V_1 be the subspace of V with basis $\{v_1\}$ so it is closed by Proposition 2.19. But $V_1 \neq V$ since dim$V$ is not finite. Now by Lemma 4.5 there is $v_2 \in M$ such that $L(v_2 - v_1, t) \leq r = \frac{1}{2}$. Let V_2 be the subspace of V with basis $\{v_1, v_2\}$ since $V_2 \neq V$ so there is $v_3 \in M$ such that $L(v_3 - v_2, t) \leq \frac{1}{2}$ and $(v_3 - v_1, t) \leq \frac{1}{2}$. Continue in this way by induction we obtain a sequence $(v_n) \in M$ such that $(v_m - v_n, t) \leq \frac{1}{2}$ $(m \neq n)$. This implies that (v_n) dose not contains a subsequence which is converges. This contradicts the compactness of M. Hence dim$V$ must be finite.

5. Conclusion

The principle goal of this research is to introduce the definition of fuzzy absolute values in order to tried to translate all the classical real analysis to a fuzzy real analysis in this paper we translate some of these properties. The second goal is study completeness and compactness properties of finite dimensional fuzzy normed space by proving a theorem which is the key role in studying properties of finite dimensional fuzzy normed spaces.

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