On Some Limit Properties for \( (a_n, \phi(n)) \)-Asymptotic Circular Markov Chains

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Abstract The purpose of this article is to obtain some limit properties for \( (a_n, \phi(n)) \)-Asymptotic Circular Markov Chains. This paper firstly presents some limit theorems of delayed sums for finite \( (a_n, \phi(n)) \)-Asymptotic Circular Markov Chains and then establishes the generalized Shannon-McMillan-Breiman theorem [1, 2, 3, 4].

Keywords: \( (a_n, \phi(n)) \)-asymptotic circular Markov Chain, delayed sum, generalized Shannon-McMillan theorem.

1 Introduction

Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be the underlying probability space and \( (\xi_n)_{n=0}^{\infty} \) be a nonhomogeneous Markov chain taking values in \( \mathcal{X} = \{1, 2, \ldots, b\} \) with the transition matrices,

\[
P_n = (p_n(i,j)), \ i, j \in \mathcal{X}, n \geq 1, \tag{1}
\]

where \( p_n(i,j) = \mathbb{P}(\xi_n = j | \xi_{n-1} = i) \). For simplicity, \( \xi_{m,n} \) represents the random vector of \( (\xi_m, \xi_{m+1}, \ldots, \xi_{m+n}) \) and \( x_{m,n} = (x_m, x_{m+1}, \ldots, x_{m+n}) \), a realization of \( \xi_{m,n} \). Let the joint distribution of \( \xi_{m,n} \) be

\[
p(x_{m,n}) = \mathbb{P}(\xi_{m,n} = x_{m,n}), \ x_k \in \mathcal{X}, \ m \leq k \leq m+n \tag{2}
\]

Let \( f^{(0)} \) be a probability distribution on \( \mathcal{X} \) and let

\[
P^{(m,n)} := P_{m+1}P_{m+2}\cdots P_n, \tag{3}
\]

\[
f^{(k)} := f^{(0)}P_1P_2\cdots P_k. \tag{4}
\]

For convenience, let \( p^{(m,n)}(i,j) \) denote the \((i,j)\) element of \( P^{(m,n)} \) and \( f^{(k)}(j) \) be the \( j \) th element of \( f^{(k)} \). It is easy to see that

\[
p^{(m,n)}(i,j) = \mathbb{P}(\xi_n = j | \xi_m = i), \tag{5}
\]

\[
f^{(k)}(j) = \mathbb{P}(\xi_k = j). \tag{6}
\]

If the Markov chain is homogeneous, then \( \{P_n, n \geq 1\} \) will be denoted simply by \( P \) and \( P^{(m,m+k)} \) is \( P^k \).

Let \( A = (a_{ij}) \) be a matrix on \( \mathcal{X} \times \mathcal{X} \). We define the norm \( \| \cdot \| \) of \( A \) as follows:

\[
\|A\| := \sup_{i \in \mathcal{X}} \sum_{j \in \mathcal{X}} |a_{ij}|. \tag{7}
\]

If \( f = (f_1, f_2, \ldots) \) is a row vector, we define \( \|f\| = \sum_{j=1}^{\infty} |f_j| \), if \( g = (g_1, g_2, \ldots)^T \) is a column vector, we define \( \|g\| = \sup_{i \in \mathcal{X}} |g_i| \). The norms defined as above satisfy the following properties:

(a) \( \|AB\| \leq \|A\| \cdot \|B\| \) for all matrices \( A \) and \( B \);
(b) \( \|P\| = 1 \) for any stochastic matrix \( P \).

These two properties will be used repeatedly in this article.

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Definition 1. Let \( Q \) be a “constant” stochastic matrix (i.e. \( Q \) is a stochastic matrix each row of which is the same). The sequence \( \{P_n, n \geq 1\} \) is said to be strongly ergodic (with constant stochastic matrix \( Q \)) if for every \( m \geq 0 \)
\[
\lim_{n \to \infty} ||P^{(m,n)} - Q|| = 0. \tag{8}
\]

Throughout this paper we always assume that \( (a_n)_{n=0}^{\infty} \) and \( (\phi_n)_{n=0}^{\infty} \) are two sequences of nonnegative integers such that \( \phi(n) \) tends to infinity as \( n \to \infty \).

The sequence \( \{P_n, n \geq 1\} \) is called to be \((a_n, \phi(n))-\)strong ergodicity of Markov chains in the Cesáro sense (to constant stochastic matrix \( Q \)) if for every \( m \geq 0 \)
\[
\lim_{n \to \infty} \left| \left| \frac{1}{\phi(n)} \sum_{t=a_n+1}^{a_n+\phi(n)} P^{(m,t)} - Q \right| \right| = 0. \tag{9}
\]

If the Markov chain is homogeneous, (9) become
\[
\lim_{n \to \infty} \left| \left| \frac{1}{\phi(n)} \sum_{t=a_n+1}^{a_n+\phi(n)} P^t - Q \right| \right| = 0. \tag{10}
\]

An irreducible stochastic matrix \( P \), of period \( d(d \geq 1) \) partitions the state space \( X \) into \( d \) disjoint subspaces \( C_0, C_1, \ldots, C_{d-1} \), and \( P_d \) yields \( d \) stochastic matrices \( \{T_l, 0 \leq l \leq d-1\} \), where \( T_l \) is defined on \( C_l \). If the irreducible periodic stochastic matrix \( P \) is finite, then each \( T_l \) is automatically strongly ergodic, but if the irreducible periodic stochastic matrix \( P \) is infinite, the strong ergodicity of \( T_l \) is not guaranteed. If each \( T_l \) is strongly ergodic, then the stochastic matrix will be called periodic strongly ergodic [5].

Definition 2. Let \( (\xi_n)_{n=0}^{\infty} \) be a non-homogeneous Markov Chain with the initial distribution \( f^{(0)} \) and the transition matrices of (1.1), \( T_l = (t_l(i,j)), (l = 1, 2, \ldots d) \) be \( d \) transition matrices. The following Markov chains is called an \((a_n, \phi(n))-\)asymptotic circular Markov chain of moving average if
\[
\lim_{n \to \infty} \frac{1}{\phi(n)} \sum_{t=a_n+1}^{a_n+\phi(n)} \|P_{td+t} - T_l\| = 0, \quad l = 1, 2, \ldots, d. \tag{11}
\]

In particular, if \( a_n = 0, \phi(n) = n \) and
\[
P_{td+t} = T_l, \quad l = 1, 2, \ldots, d, \quad t = 0, 1, 2, \ldots \tag{12}
\]
this Markov chain is called a circular Markov chain.

Circular Markov chains play an important role in the nearest neighbour random walks on inhomogeneous Markov chains periodic lattices, i.e[6]. Periodic lattices which consist of a periodically repeated unit cell, where unit cell contains a number of non-equivalent sites [7], an example of circular Markov process occurs in the evaluation system of M/M/C queueing system in which the servers, after an idle period, only restart work when enough customers arrived to the system.

Definition 3. Let \( (\xi_n)_{n=0}^{\infty} \) be a nonhomogeneous Markov chain taking on values in \( X \). Let
\[
f_{a_n, \phi(n)}(\omega) = -\frac{1}{\phi(n)} \log p(\xi_{a_n, \phi(n)}), \tag{13}
\]
where log is the natural logarithm. \( f_{a_n, \phi(n)}(\omega) \) is called generalized entropy density of \( \xi_{a_n, \phi(n)} \) and \( f_{a_n, \phi(n)}(\omega) \) can be rewritten as
\[
f_{a_n, \phi(n)}(\omega) = -\frac{1}{\phi(n)} \left\{ \log f^{(a_n)}(\xi_{a_n}) + \sum_{k=a_n+1}^{a_n+\phi(n)} \log p_k(\xi_{k-1}, \xi_k) \right\} \tag{14}
\]
Let $Q$ be another probability measure on $(\Omega, \mathcal{H})$ if $(\eta_n)_{n=0}^\infty$ is a nonhomogeneous Markov chain under $Q$ with initial distribution
\[(q(1), q(2), \ldots, q(b)),\] (15)
and transition matrices
\[Q_n = (q_n(i,j))_{d \times d}, \quad i, j \in \mathcal{X}, n \geq 1,\] (16)
where $q_n(i,j) = Q(\eta_n = j|\xi_{n-1} = i)$. Define
\[q(x_{a_n, \phi(n)}) = p(x_{a_n}) \prod_{k=a_n+1}^{a_n+\phi(n)} q_k(x_{k-1}, x_k).\] (17)
\[L_{a_n, \phi(n)}(\omega) = -\frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} \log \frac{q_k(\eta_{k-1}, \eta_k)}{p_k(\xi_{k-1}, \xi_k)}\] (18)
\[L(\omega) = \limsup_n L_{a_n, \phi(n)}(\omega) = -\liminf_n \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} \log \frac{q_k(\eta_{k-1}, \eta_k)}{p_k(\xi_{k-1}, \xi_k)}\] (19)
$L_{a_n, \phi(n)}(\omega)$ and $L(\omega)$ are called generalized sample relative entropy and generalized sample relative entropy rate between $P$ and $Q$ respectively.

Our paper is aim at extending the known results, the approach used in this paper different from [8] and [9], the essence of the method is first to construct a one parameter class of random variables with means of 1, then, using Borel-Cantelli lemma, to prove the existence of a.e. convergence of certain random variables [10]. Under the condition of Lemma 1 of [10], we first give the definition of an $(a_n, \phi(n))$-asymptotic circular Markov chain and prove some lemmas. Then, we prove generalized limit theorems for countable $(a_n, \phi(n))$-asymptotic circular Markov chains, as corollaries, we obtain the strong law of large number for non-homogeneous Markov chains which is known results of [10] as well as extending the results of [8]. Finally, we achieve the generalized Shannon-McMillan Breiman theorem for finite $(a_n, \phi(n))$-asymptotic circular Markov chains, which is to some extent, an extension of the result of [9].

The remaining paper is organized as follows. Section 2 provides some related Lemmas. Section 3 gives the main results and the proofs.

### 2 Some Lemmas

Before proving the main results, we firstly prove some related lemmas, which will play important roles in achieving our results.

**Lemma 1.** [11] Let $Q_i (i = 1, 2, \cdots, d)$ be $d$ stochastic matrices and let $R_1 = Q_1Q_2\cdots Q_d, R_2 = Q_dQ_1\cdots Q_{d-1}, \ldots, R_d = Q_dQ_1\cdots Q_{d-i}$. If $R_i$ is $(a_n, \phi(n))$-strongly ergodic with constant stochastic matrix $T_i$. Then $R_1, R_2, \ldots, R_d$ are also $(a_n, \phi(n))$-strongly ergodic with the constant stochastic matrices $T_2, T_3, \cdots, T_d$, resp., where $T_i = T_i \prod_{j=1}^{i-1} Q_l (l = 2, \cdots, d)$.

**Proof.** Since $T_1$ is a constant stochastic matrix, which implies that $T_i (i = 2, \cdots, d)$ are also constant stochastic matrices. For any stochastic matrix $P$ and a constant stochastic matrix $Q$ we have $PQ = Q$. Since $R_1$ is $(a_n, \phi(n))$-strongly ergodic with a constant stochastic matrix $T_1$, then
\[\lim_{n \to \infty} \left\| \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} R_i^k - T_1 \right\| = 0.\] (20)
when \( l \geq 2 \), note that
\[
\frac{1}{\phi(n)} - 1 \sum_{k=a_n+1}^{a_n + \phi(n)} R^k_l - T_l = \frac{1}{\phi(n)} - 1 \sum_{k=a_n+1}^{a_n + \phi(n)} (R^k_l - T_l)
\]
\[
= \frac{1}{\phi(n)} - 1 \sum_{k=a_n+1}^{a_n + \phi(n)} \left( \prod_{i=1}^{d} Q_i \right) R^{k-1}_l \left( \prod_{i=1}^{d} Q_i \right) - \left( \prod_{i=1}^{d} Q_i \right) T_l \left( \prod_{i=1}^{d} Q_i \right)
\]
\[
= \left( \prod_{i=1}^{d} Q_i \right) \left( \frac{1}{\phi(n)} - 1 \sum_{k=a_n+1}^{a_n + \phi(n)} R^{k-1}_l - T_l \left( \prod_{i=1}^{d} Q_i \right) \right)
\]
we have
\[
\frac{1}{\phi(n)} - 1 \sum_{k=a_n+1}^{a_n + \phi(n)} R^k_l - T_l \leq \frac{1}{\phi(n)} - 1 \sum_{k=a_n+1}^{a_n + \phi(n)} R^{k-1}_l - T_l \| 1
\]
(20) and (22) imply that
\[
\lim_{n \to \infty} \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n + \phi(n)} R^k_l - T_l = 0.
\]
This means \( R_l(1,2,\ldots,d) \) are \((a_n,\phi(n))\)-strongly ergodic.

**Lemma 2.** Let \( Q_l(l=1,2,\ldots,d) \) be \( d \) stochastic matrices. Let \( \{Q_n,n \geq 1\} \) be a sequence of stochastic matrices satisfying
\[
Q_l(t+d+l) = Q_l(l=1,2,\ldots,d), t = 0,1,2\ldots
\]
Let \( P^{(m,n)} \) be defined as in (3). If (11) holds, then, for any positive integer \( k \),
\[
\frac{1}{\phi(n)} \sum_{a_n+1}^{a_n + \phi(n)} \| P^{(td+l,td+l+k)} - Q^{(td+l,td+l+k)} \| = 0, l = 1,2,\ldots,d.
\]

**Proof.** For \( k = 2 \), we have by (24)
\[
\| P^{(td+l,td+l+2)} - Q^{(td+l,td+l+2)} \| = \| P_{td+l+1} + P_{td+l+2} - Q_{td+l+1} + Q_{td+l+2} \|
\]
\[
= \| P_{td+l+1} P_{td+l+2} - Q_{td+l+1} Q_{td+l+2} \| + \| P_{td+l+2} - Q_{td+l+2} \|
\]
By (11), we have
\[
\frac{1}{\phi(n)} \sum_{a_n+1}^{a_n + \phi(n)} \| P^{(td+l,td+l+2)} - Q^{(td+l,td+l+2)} \| = 0, l = 1,2,\ldots,d.
\]
Similarly, for \( k \geq 2 \), (25) holds by induction.

**Lemma 3.** [10] Assume that \( (\xi_n)_{n=0}^{\infty} \) is a nonhomogeneous Markov chain taking values in \( X = \{1,2,\ldots,b\} \) with initial distribution (15) and the transition matrices as (16). Let \( (g_n(x,y))_{n=0}^{\infty} \) be a sequence of real functions defined on \( X \times X \). If for every \( \varepsilon > 0 \)
\[
\sum_{n=1}^{\infty} \exp[-\varepsilon \phi(n)] < \infty,
\]
and there exists a real number \( 0 < \gamma < \infty \) such that

\[
\limsup_n \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} E\|g_k(\xi_{k-1}, \xi_k)\|^2 \exp(\gamma\|g_k(\xi_{k-1}, \xi_k)\|\xi_{k-1}) = c(\gamma; \omega) < \infty \text{ a.e.,}
\] (29)

then, we have

\[
\lim_n \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} \{g_k(\xi_{k-1}, \xi_k) - E[g_k(\xi_{k-1}, \xi_k)|\xi_{k-1}]\} = 0 \text{ a.e.}
\] (30)

Proof. See Lemma 1 of [10]. □

**Lemma 4.** Let \( \{t_k\}_{k=0}^{\infty} \) be a bounded sequence of points in the plane, \( \|t_k\| \leq M \), \( \delta \) be a positive number, and let \( N_n(\delta) \) be the number of terms which not belong to \( U(0, \delta) \) in the first \( n \) terms of the sequence. Then

\[
\lim_n \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} t_k = 0
\] (31)

holds if and only if

\[
\lim_n \frac{1}{\phi(n)} N_n(\delta) = 0, \quad \forall \delta > 0.
\] (32)

**Lemma 5.** Let \( \varphi(x) \) be a bounded function defined on area \( D \), \( a \) be an interior point in \( D \), and \( \varphi(x) \) be continues at \( x = t \), and let \( \{t_k, k \geq 1\} \) be a collection of points in \( D \). If

\[
\lim_n \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} \|t_k - t\| = 0
\]

holds, then

\[
\lim_n \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} \|\varphi(t_k) - \varphi(t)\| = 0.
\] (33)

Proof. By the continuity of the function, \( \forall \varepsilon > 0, \exists \delta > 0 \) satisfying \( U(t, \delta) \subset D \), whenever \( t_1 \in U(t, \delta) \), we have \( |\varphi(t_1) - \varphi(t)| \leq \varepsilon \). Let \( N_n(\delta) \) be the number of terms which not belong to \( U(0, \delta) \) in the first \( n \) terms of sequence \( \{t_k - t, k \geq 1\} \), and \( M_n(\varepsilon) \) be the number of terms which are greater than \( \varepsilon \) in the first \( n \) terms of sequence \( \{\varphi(t_k) - \varphi(t)\}_{k=1}^{\infty} \). Then,

\[
M_n(\varepsilon) \leq N_n(\delta)
\] (34)

It follows from (1.11), Lemma 1, and (2.15) that

\[
\lim_n \frac{1}{\phi(n)} M_n(\varepsilon) = 0, \quad \forall \varepsilon > 0.
\] (35)

Since the sequence \( \{\|\varphi(t_k) - \varphi(t)\|\}_{k=1}^{\infty} \) is bounded, (2.14) follows from (2.16). □

**Lemma 6.** Let \( (\xi_n)_{n=0}^{\infty} \) be a nonhomogeneous Markov chain with initial distribution (2) and transition matrices (16) under measure \( P \), \( (\eta_n)_{n=0}^{\infty} \) also be a nonhomogeneous Markov chain with initial distribution (15) and transition matrices (16)

\[
Q_n = (q_n(i, j), \quad q_n(i, j) > \tau, \quad 0 < \tau < 1, \quad i, j \in \mathcal{X}, n \geq 0,
\] (36)

then

\[
\lim_n \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} \left\{ \log \frac{p_k(\xi_{k-1}, \xi_k)}{q_k(\xi_{k-1}, \xi_k)} - \sum_{j=1}^{i} p_k(\xi_{k-1}, j) \log \frac{p_k(\xi_{k-1}, j)}{q_k(\xi_{k-1}, j)} \right\} = 0, \quad P - a.e.
\] (37)
Proof. Letting \(g_k(s, t) = \log q_k(s, t)\) in Lemma 3, we have

\[
EP[|g_k(\xi_{k-1}, \xi_k)|^2 \exp(\gamma|g_k(\xi_{k-1}, \xi_k)|\xi_{k-1})] = \sum_{j=1}^{b} \left( \log q_k(\xi_{k-1}, j) \right)^2 (q_k(\xi_{k-1}, j))^\gamma \rho_k(\xi_{k-1}, j) \\
\leq \sum_{j=1}^{b} (\log \tau)^2 \tau^\gamma \leq b(\log \tau)^2 \tau^\gamma
\]

By (40), (41) and Lemma 3, we have

\[
\lim_{n \to \infty} \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} \left\{ \log \frac{p_k(\xi_{k-1}, \xi_k)}{q_k(\xi_{k-1}, \xi_k)} - \sum_{j=1}^{b} \left( \log q_k(\xi_{k-1}, j) \right)^2 (q_k(\xi_{k-1}, j))^\gamma \rho_k(\xi_{k-1}, j) \right\} = 0, \text{ } P \text{-a.e.}
\]

3 The Main Results

In this section, we will present our main results based on previous Lemmas.

**Theorem 1.** Let \((\xi_n)_{n=0}^{\infty}\) be an \((a_n, \phi(n))-\text{asymptotic circular Markov chain defined by Definition 1. Let} \((g_n(x, y))_{n=0}^{\infty}\) be a sequence of real functions defined on \(X \times X\). If, for every \(\epsilon > 0\),

\[
\sum_{n=1}^{\infty} \exp[-\epsilon \phi(n)] < \infty,
\]

and there exists a real number \(0 < \gamma < \infty\) such that

\[
\limsup_{n} \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} E[|g_k(\xi_{k-1}, \xi_k)|^2 \exp(\gamma|g_k(\xi_{k-1}, \xi_k)|\xi_{k-1})] = c(\gamma; \omega) < \infty \text{ a.e.}
\]

Let

\[
h_n(i) = \sum_{j \in X} g_n(i, j)p_n(i, j)
\]

\(h_n\) be a column vector with ith element \(h_n(i)\) and \(h^l(l = 1, 2, \cdots, d)\) be \(d\) column vectors with ith elements \(h^l(i)\). Let \(R_l, T_l(l = 1, 2, \cdots, d)\) be the same as in Lemma 2 and \(R_1\) be \((a_n, \phi(n))-\text{strongly ergodic. If}

\[
\lim_{n \to \infty} \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} \|h_{n+d+l} - h^l\| = 0, \text{ } l = 1, 2, \cdots, d,
\]

and \(\|h_n\|\) and \(h^l\) are finite, then

\[
\lim_{n \to \infty} \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} g_k(\xi_{k-1}, \xi_k) = \frac{1}{d} \sum_{l=1}^{d} \sum_{1 \leq l \leq d} h^l(i) \pi^l(i) \text{ a.e.,}
\]

where \(\pi^l = (\pi^l(1), \pi^l(2), \cdots)\) is the the common row vector of \(T_1\) and also the unique stationary distribution determined by \(R_l(l = 1, 2, \cdots, d)\).

**Proof.** By (40), (41) and Lemma 3, we have

\[
\lim_{n} \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} \left\{ g_k(\xi_{k-1}, \xi_k) - E[g_k(\xi_{k-1}, \xi_k)|\xi_{k-1}] \right\} = 0 \text{ a.e.}
\]
Now, we consider
\[ \frac{1}{\phi(n)} \sum_{k=1}^{\phi(n)} E[g_{a_n+k}(\xi_{a_n+k}, \xi_{a_n+k+1})]\xi_{a_n} \]
\[ = \frac{1}{\phi(n)} \sum_{k=1}^{\phi(n)} \sum_{i} \sum_{j} g_{a_n+k}(i, j)p_{a_n+k+1}(i, j)p_{(a_n, a_n+\phi(n))}(\xi_{a_n}, i) \]
\[ = \frac{1}{\phi(n)} \sum_{k=1}^{\phi(n)} \sum_{i} h_{a_n+k}(i)p_{(a_n, a_n+\phi(n))}(\xi_{a_n}, i) \]
\[ = \frac{1}{\phi(n)} \sum_{k=1}^{\phi(n)} \sum_{i} h_{a_n+k}(i)p_{(t_{d+1}, t_{d+1}+a_n)}(\xi_{t_{d+1}}, i) \]
\[ + \frac{1}{\phi(n)} \sum_{k=1}^{\phi(n)} \sum_{i} h_{k+a_n}(i)p_{(k, k+a_n)}(\xi_k, i) \]
(46)

where \[ \lceil \cdot \rceil \] represents the greatest integer not more than \( x \). The second term of (46) is defined to be zero if \( \phi(n)/d \) is a positive integer. Obviously,
\[ \frac{1}{\phi(n)} \sum_{k=1}^{\phi(n)} g_{k+a_n}p_{(k, k+a_n)}(\xi_k, i) = 0 \]
(47)

Let \( \{Q_{n, n} \geq 1\} \) be the same as in Lemma 2. And \( q^{(m,n)}(i, j) \) be the \((i, j)\) element of \( Q^{(m,n)} \), \( M \) be the upper bound of \( \|y'\| (l = 1, 2, \cdots, d) \). Let \( v \) be a positive integer, \( a_n = vd \) and \( h^{d+1} = h^1 \). Since
\[ 1 \sum_{i}^{d} \sum_{l=0}^{\phi(n)-1} h_{td+1}+vd+1(i)p_{(td+1, td+1+vd)}(\xi_{td+1}, i) \]
\[ - \frac{1}{\phi(n)} \sum_{i}^{d} \sum_{l=0}^{\phi(n)-1} h^{l+1}(i)q_{(td+1, td+1+vd)}(\xi_{td+1}, i) \]
\[ \leq \frac{1}{\phi(n)} \sum_{i}^{d} \sum_{l=0}^{\phi(n)-1} h_{td+1}+vd+1(i)p_{(td+1, td+1+vd)}(\xi_{td+1}, i) \]
\[ - \frac{1}{\phi(n)} \sum_{i}^{d} \sum_{l=0}^{\phi(n)-1} h^{l+1}(i)p_{(td+1, td+1+vd)}(\xi_{td+1}, i) \]
\[ + \frac{1}{\phi(n)} \sum_{i}^{d} \sum_{l=0}^{\phi(n)-1} h^{l+1}(i)p_{(td+1, td+1+vd)}(\xi_{td+1}, i) \]
\[ - \frac{1}{\phi(n)} \sum_{i}^{d} \sum_{l=0}^{\phi(n)-1} h^{l+1}(i)q_{(td+1, td+1+vd)}(\xi_{td+1}, i) \]
\[ \leq \frac{1}{\phi(n)} \sum_{i}^{d} \sum_{l=0}^{\phi(n)-1} h_{td+1}+vd+1(i) - h^{l+1}(i)p_{(td+1, td+1+vd)}(\xi_{td+1}, i) \]
\[ + \frac{1}{\phi(n)} \sum_{i}^{d} \sum_{l=0}^{\phi(n)-1} h^{l+1}(i) - \left| h^{l+1}(i) \right| p_{(td+1, td+1+vd)}(\xi_{td+1}, i) - q_{(td+1, td+1+vd)}(\xi_{td+1}, i) \]
\[
\leq \frac{1}{\phi(n)} \sum_{l=1}^{d} \sum_{i=1}^{\left\lfloor \frac{\alpha(n)}{l} \right\rfloor - 1} \sum_{t=0}^{h_{l+1}(i)} \left\| h_{l+1}(i)q_{l+t+1}^{(l)}(\xi_{l+t+1}, i) \right\| + \frac{M}{\phi(n)} \sum_{l=1}^{d} \sum_{t=0}^{\left\lfloor \frac{\alpha(n)}{l} \right\rfloor - 1} \left\| P_{l+t+1}^{(l)} - Q_{l+t+1}^{(l)} \right\| \tag{48}
\]

It follows from (43) that the first term of (48) goes to zero when \( n \to \infty \). By Lemma 2, the second term of (48) also tends to zero when \( n \to \infty \). Combining (45)-(48), we have

\[
\lim_{n \to \infty} \left\{ \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} \sum_{i=1}^{n} \frac{1}{\phi(n)} \sum_{l=1}^{d} \sum_{t=0}^{\left\lfloor \frac{\alpha(n)}{l} \right\rfloor - 1} h_{l+1}(i)q_{l+t+1}^{(l)}(\xi_{l+t+1}, i) \right\} = 0 \quad \text{a.e.} \tag{49}
\]

and for any positive integer \( N \)

\[
\lim_{n \to \infty} \left\{ \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} \sum_{i=1}^{n} \frac{1}{\phi(n)} \sum_{l=1}^{d} \sum_{t=0}^{\left\lfloor \frac{\alpha(n)}{l} \right\rfloor - 1} h_{l+1}(i)q_{l+t+1}^{(l)}(\xi_{l+t+1}, i) \right\} = 0 \quad \text{a.e.} \tag{50}
\]

Set \( R_{d+1} = R_1, a_n = v_d \). It is easy to see that for \( l = 1, 2, \cdots, d \),

\[
Q_{l+t+1}^{(l)} = Q_{l+t+1}^{(l)}Q_{l+t+2} \cdots Q_{l+v_d} = Q_{l+1}^{(l)}Q_{l+2} \cdots Q_{l+v_d} = R_{l+1}^{(l)} \tag{51}
\]

Let \( r_{l+1}^{(v)}(i, j) \) be the v-step probabilities determined by \( R_{l+1} \). Set \( \pi_{l+1} = \pi_1 \) and \( T_{d+1} = T_1 \), we have

\[
\left| \frac{1}{\phi(n)} \sum_{l=1}^{d} \sum_{i=1}^{n} \frac{1}{\phi(n)} \sum_{l=1}^{d} \sum_{t=0}^{\left\lfloor \frac{\alpha(n)}{l} \right\rfloor - 1} h_{l+1}(i)q_{l+t+1}^{(l)}(\xi_{l+t+1}, i) - \frac{1}{d} \pi_{l+1}^{(l)}(i) \right| = 0
\]

\[
\sum_{i=1}^{d} \left| h_{l+1}(i) \right| \cdot \left| \frac{1}{\phi(n)} \sum_{l=1}^{d} \sum_{t=0}^{\left\lfloor \frac{\alpha(n)}{l} \right\rfloor - 1} \pi_{l+1}^{(l)}(\xi_{l+t+1}, i) - \frac{1}{d} \pi_{l+1}^{(l)}(i) \right| \leq \frac{1}{d} \left| h_{l+1}(i) \right| \cdot \left| \frac{1}{\phi(n)} \sum_{l=1}^{d} \sum_{t=0}^{\left\lfloor \frac{\alpha(n)}{l} \right\rfloor - 1} \pi_{l+1}^{(l)}(\xi_{l+t+1}, i) - \frac{1}{d} \pi_{l+1}^{(l)}(i) \right| \tag{52}
\]

Given \( \varepsilon > 0 \), by Lemma 1, we can choose a fixed \( N \) large enough so that the first term of (52) does not exceed \( \varepsilon \). The second term of (52) tends to zero as \( n \) goes to infinity. By (50), (52) and the arbitrariness of \( \varepsilon \), (44) follows. These complete the proof of Theorem 1.
Corollary 1. Let \((\xi_n)_{n=0}^\infty\) be an \((a_n, \phi(n))\)-asymptotic circular Markov chain defined by Definition 2. Let \(R_l(l=1,2, \cdots, d)\) be the same as in Lemma 1. Assume that \(R_1\) is \((a_n, \phi(n))\)-strongly ergodic. Let \(g(x)\) be a bounded function defined on \(\mathcal{X}\). Then
\[
\lim_{n \to \infty} \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} g(\xi_k) = \sum_{i \in \mathcal{X}} \frac{1}{d} \sum_{l=1}^{d} g(i) \pi^l(i) \quad \text{a.e.,}
\]
(53)

Proof. Let \(g_n(x,y) = g(x)\) in Theorem 1, then
\[
h_{td+l}(i) = \sum_j g_{td+l}(i,j)p_{td+l}(i,j) = \sum_j g_{td+l}(i)p_{td+l}(i,j) = g(i)
\]
(54)

Let \(h^l(i) = \sum_j g(i)q(i,j) = g(i)\), where \(q(i,j)\) is the \((i,j)\) element of transition matrix \(Q_l\), therefore (43) holds. Since \(g(x)\) is bounded, thus \(\|h_n\|\) and \(\|h^l\|\) are finite and (41) also follows. Note that
\[
\frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} g(\xi_{k-1}, \xi_k) = \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} g(\xi_{k-1})
\]
(55)

and
\[
\sum_i \frac{1}{d} \sum_{l=1}^{d} h^l(i) \pi^l(i) = \sum_i \frac{1}{d} \sum_{l=1}^{d} g(i) \pi^l(i)
\]
(56)

This corollary follows from Theorem 1 directly. \(\square\)

Define the indicator function \(1_i(j)\) on \(\mathcal{X}\) as follows:
\[
1_i(j) = \begin{cases} 
1, & \text{if } j = i; \\
0, & \text{if } j \neq i.
\end{cases}
\]
(57)

where \(i = 1, 2, \cdots\)

Corollary 2. Let \((\xi_n)_{n=0}^\infty\) be an \((a_n, \phi(n))\)-asymptotic circular Markov chain defined by Definition 2. Let \(R_l(l=1,2, \cdots, d)\) be the same as in Lemma 1. Assume that \(R_1\) is \((a_n, \phi(n))\)-strongly ergodic. Let \(S_{a_n, \phi(n)}(c, \omega)\) be the number of \(c\) in the sequence of \(\xi_{a_n+1}(\omega), \xi_{a_n+2}(\omega), \cdots, \xi_{a_n+\phi(n)}(\omega)\), i.e. \(S_{a_n, \phi(n)}(c, \omega) = \sum_{m=a_n+1}^{a_n+\phi(n)} 1_c(\xi_m)\). Then
\[
S_{a_n, \phi(n)}(c, \omega) = \frac{1}{d} \sum_{l=1}^{d} \pi^l(c) \quad \text{a.e.,}
\]
(58)

Proof. Let \(g(x) = 1_c(x)\) in Corollary 1. Obviously, \(|g(x)| \leq 1\). Noticing that
\[
\frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} g(\xi_k) = \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} 1_c(\xi_k) = \frac{S_{a_n, \phi(n)}(c, \omega)}{\phi(n)}
\]
(59)

\[
\sum_i \frac{1}{d} \sum_{l=1}^{d} g(i) \pi^l(i) = \sum_i \frac{1}{d} \sum_{l=1}^{d} 1_c(i) \pi^l(i) = \frac{1}{d} \sum_{l=1}^{d} \pi^l(c)
\]
(60)

(3.19) follows from Corollary 1. \(\square\)

Corollary 3. Let \((\xi_n)_{n=0}^\infty\) be a non-homogeneous Markov chain. Let \(\{g_n(x,y), n \geq 1\}\) and \(h_n\) be the same as in Theorem 1. Let \(P\) be a stochastic matrix and be periodic strongly ergodic. Let \(h(i)\) be another function defined on \(\mathcal{X}\), \(h\) be column vector with \(ith\) element \(h(i)\). If conditions (40) and (41) hold resp., and
\[
\lim_{n \to \infty} \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} \|P_k - P\| = 0
\]
(61)
$$\lim_{n \to \infty} \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_{n+\phi(n)}} \|h_k - h\| = 0$$  \hfill (62) 

if \(\|h_n\|\) and \(\|g\|\) are finite, then

$$\lim_{n \to \infty} \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_{n+\phi(n)}} g_n(\xi_{k-1}, \xi_k) = \sum_i h(i)\pi(i) \quad a.e.$$  \hfill (63)

where \(\pi^l = (\pi^l(1), \pi^l(2), \cdots)\) is the unique distribution determined by \(P\).

**Proof.** Since periodic strong ergodicity implies \((a_n, \phi(n))-\)strong ergodicity, Let \(d = 1\) in Theorem 1, it follows.

Now we consider, based on theorem 1, the generalized Shannon-McMillan theorem, we give the following theorem:

**Theorem 2.** Let \((\eta_n)_{n=0}^\infty\) be an \((a_n, \phi(n))-\)asymptotic circular Markov chain on the state \(\mathcal{X} = \{1, 2, \cdots, b\}\) with the following the initial distribution and the transition matrices resp.,

$$f^{(0)} = (p(1), p(2), \cdots, p(b)), \quad \mathcal{P}_n = (p_n(i, j))_{b \times b}, \quad n \geq 1.$$  \hfill (64)

and \(q^l(i, j)\) be the \((i, j)\) element of \(Q^l(l = 1, 2, \cdots, d)\). Denote

$$h_n(i) = -\sum_{j \in \chi} p_n(i, j) \log p_n(i, j) \quad \mathcal{h}^l(i) = -\sum_{j \in \chi} q(i, j) \log q(i, j)$$  \hfill (66)

Let \(h_n\) be column vector with ith element \(h_n(i)\), \(h_l(l = 1, 2, \cdots, d)\) be d column vectors with ith elements \(h_l(i)\). Let \(R_l(l = 1, 2, \cdots, d)\) be the same as in Lemma 1 and \(R_1\) be \((a_n, \phi(n))-\)strongly ergodic. If

$$\lim_{n \to \infty} \frac{1}{\phi(n)} \sum_{t=a_n+1}^{a_{n+\phi(n)}} \|h_{td+l} - \mathcal{h}^l\| = 0, \quad l = 1, 2, \cdots, d,$$  \hfill (68)

if \(\|h_n\|\) and \(\|\mathcal{h}^l\|\) is finite. Then

$$\lim_n f_{a_n, \phi(n)}(\omega) = -\sum_{i=1}^b \frac{1}{d} \sum_{l=1}^d \pi^l(i) \sum_{j=1}^b q(i, j) \log(q(i, j))$$  \hfill (69)

where \(\pi^l = (\pi^l(1), \pi^l(2), \cdots, \pi^l(b))\) is the the unique stationary distribution determined by \(R_l(l = 1, 2, \cdots, d)\).

**Proof.** Let \(g_n(x, y) = -\log p_n(x, y)\) in Theorem 1, By (11) and the Lemma 2 of[10], we have for \(l = 1, 2, \cdots, d\)

$$\lim_{n \to \infty} \frac{1}{\phi(n)} \sum_{t=a_n+1}^{a_{n+\phi(n)}} p_{td+l}(i, j) \log p_{td+l}(i, j) - q(i, j) \log q(i, j) = 0, \quad \forall i, j \in \chi.$$  \hfill (70)

By (70), (68) holds.

$$\lim_{n \to \infty} \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_{n+\phi(n)}} |p_{kd+l}(i, j) - q(i, j)| = 0$$  \hfill (71)
(71) is equivalent to (11). Since
\[
\max_{0 \leq x \leq 1} \{x(\log x)^2\} = 4e^{-2}
\] (72)
thus
\[
E[\log p_n(\xi_{k-1}, \xi_k)]^2 = \sum_{i=1}^{b} \sum_{j=1}^{b} p_n(i, j) [\log p_n(i, j)]^2 P(\xi_{n-1} = i) \leq 4be^{-2}
\] (73)

By (73), (41) and Theorem 1, (69) follows.

**Theorem 3.** Let \(\{\xi_n\}_{n=1}^{\infty}\) be an \((a_n, \phi(n))\)-asymptotic circular Markov chain, and let \(R_t\) be defined as in Lemma 7. \(R_t\) be \((a_n, \phi(n))\)-strong ergodicity, \((\pi_1^t, \pi_2^t, \ldots, \pi_d^t)\) is the unique stationary distribution determined by the stochastic matrix \(R_t\). Let \(\{r_n, n \geq 1\}\) be an asymptotic circular Markov chain with initial distribution (15) and transition matrices (16) under measure \(Q\). If \(H_t = (h_t(i, j)), l = 1, 2, \ldots, d, i, j \in X\) are strictly positive transition matrices, then
\[
\mathcal{L}(\omega) = \sum_{i=1}^{d} \sum_{j=1}^{b} \sum_{k=1}^{b} \pi_l^i(i, j) \log \frac{t_l(i, j)}{h_l(i, j)}
\] (74)

**Proof.** By (1.20) and (2.21), we have
\[
\left| \mathcal{L}_{a_n, \phi(n)}(\omega) - \sum_{l=1}^{d} \sum_{i=1}^{b} \sum_{j=1}^{b} \pi_l^i(i, j) \log \frac{t_l(i, j)}{h_l(i, j)} \right|
\]
\[
= \left| \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} \sum_{j=1}^{b} p_k(\xi_{k-1}, j) \log \frac{p_k(\xi_{k-1}, j)}{q_k(\xi_{k-1}, j)} \sum_{l=1}^{d} \sum_{i=1}^{b} \sum_{j=1}^{b} \pi_l^i(i, j) \log \frac{t_l(i, j)}{h_l(i, j)} \right|
\]
\[
= \left| \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} \sum_{j=1}^{b} p_k(\xi_{k-1}, j) \log \frac{p_k(\xi_{k-1}, j)}{q_k(\xi_{k-1}, j)} + \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} \sum_{j=1}^{b} p_k(\xi_{k-1}, j) \log \frac{p_k(\xi_{k-1}, j)}{q_k(\xi_{k-1}, j)} \right|
\]
\[
- \sum_{l=1}^{d} \sum_{i=1}^{b} \sum_{j=1}^{b} \pi_l^i(i, j) \log \frac{t_l(i, j)}{h_l(i, j)}
\]
\[
\leq \left| \frac{1}{\phi(n)} \sum_{j=1}^{b} \sum_{l=1}^{d} \sum_{t=a_n}^{a_n+\phi(n)-1} p_{t+1}(\xi_{t+1}, j) \log \frac{p_k(\xi_{t+1}, j)}{q_k(\xi_{t+1}, j)} \right|
\]
\[
+ \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} \sum_{j=1}^{b} p_k(\xi_{k-1}, j) \log \frac{p_k(\xi_{k-1}, j)}{q_k(\xi_{k-1}, j)}
\]
\[
- \sum_{l=1}^{d} \sum_{i=1}^{b} \sum_{j=1}^{b} \pi_l^i(i, j) \log \frac{t_l(i, j)}{h_l(i, j)}
\]
\[
+ \frac{1}{\phi(n)} \sum_{j=1}^{b} \sum_{l=1}^{d} \sum_{t=a_n}^{a_n+\phi(n)-1} \sum_{i=1}^{b} 1(\xi_{t+l}, i) \log \frac{p_{t+l}(i, j)}{q_{t+l}(i, j)}
\]
\[
- \sum_{l=1}^{d} \sum_{i=1}^{b} \sum_{j=1}^{b} \pi_l^i(i, j) \log \frac{t_l(i, j)}{h_l(i, j)}
\]
\[
+ \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} \sum_{j=1}^{b} p_k(\xi_{k-1}, j) \log \frac{p_k(\xi_{k-1}, j)}{q_k(\xi_{k-1}, j)}
\]
Corollary 4. Let $P$ and $Q$ be two measure on $(\Omega, \mathcal{F})$. Let $(\xi_n)_{n=1}^{\infty}$ be a nonhomogeneous Markov chain under measure $P$. Let $P = (p(i,j)), i, j \in \mathcal{X}$ be a transition matrix and let $P$ be irreducible. If

$$\lim_{n \to \infty} \frac{1}{\phi(n)} \sum_{j=1}^{b} a_{n+\phi(n)} \sum_{i=1}^{b} \sum_{t=a_n}^{t-d} |P_{d+t}(i,j) - p_{d+t}(i,j) - t_l(i,j) \log t_l(i,j) - h_l(i,j) - t_l(i,j) \log t_l(i,j) - h_l(i,j)| = 0,$$

(79) follows from (33), (36) and (37).

\[ \square \]
(π₁, π₂, . . . , πₙ) is the unique stationary distribution determined by the stochastic matrix \( \tilde{P} \). Let \( (\xi_n)_{n=0}^{\infty} \) be a nonhomogeneous Markov chain with initial distribution (15) and transition matrices (16) under measure \( \tilde{Q} \), (2.17) holds. Let
\[
\tilde{Q} = (q(i,j)), \quad q(i,j) > 0, \quad i, j \in \mathcal{X},
\]
be another transition matrix, if for any \( i, j \in \mathcal{X} \), \( \{q_n(i,j), n \geq 0\} \) absolute mean converges to \( q(i,j) \), that is,
\[
\lim_{n \to \infty} \frac{1}{\phi(n)} \sum_{k=a_n}^{a_n+\phi(n)} |q_k(i,j) - q(i,k)| = 0, \forall i, j \in S
\]
then
\[
\mathcal{L}(\omega) = \sum_{i=1}^{b} \sum_{j=1}^{b} \pi(i)p(i,j) \log \frac{p(i,j)}{q(i,j)} \quad a.e..
\]

Proof. It is easy to see that irreducible implies \( (a_n, \phi(n)) \)-strong-strong ergodicity. Letting \( d = 1 \) in Theorem 3, this corollary follows.

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