Strong domination number of some path related graphs

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Abstract

Let $G = (V, E)$ be a graph and $uv \in E$ be an edge then $u$ strongly dominates $v$ if $\text{deg}(u) \geq \text{deg}(v)$. A set $S$ is a strong dominating set (sd-set) if every vertex $v \in V - S$ is strongly dominated by some $u$ in $S$. We investigate strong domination number of some path related graphs.

Keywords: Domination number, Independent domination number, strong domination number, independent strong domination number.

AMS Subject Classification(2010): 05C69, 05C76.

1 Introduction

We consider simple, finite, connected and undirected graph $G$ with vertex set $V$ and edge set $E$. For all standard terminology and notations we follow West [15] while the terms related to the theory of domination in graphs are used in the sense of Haynes et al. [6].

Definition 1.1. A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called a dominating set if every vertex $v \in V$ is either an element of $S$ or is adjacent to an element of $S$. A dominating set $S$ is a minimal dominating set if no proper subset $S' \subset S$ is a dominating set. The domination number $\gamma(G)$ of a graph $G$ is the minimum cardinality of a dominating set in graph $G$.

Definition 1.2. A set $S \subseteq V$ is an independent set of $G$, if $\forall u, v \in S$, $N(u) \cap \{v\} = \phi$. A dominating set which is independent, is called an independent dominating set. The minimum cardinality of an independent dominating set in $G$ is called the independent domination number $i(G)$ of a graph $G$.

The theory of independent domination was formalized by Berge [2] and Ore [8] in 1962. Allan and Laskar [1] have identified the graphs $G$ for which $\gamma(G) = i(G)$ whereas bounds on the independent domination number are determined by Goddard and Henning [5]. Vaidya and
Pandit [14] have investigated the exact value of independent domination number of some wheel related graphs.

We denote the degree of a vertex $v$ in a graph $G$ by $\text{deg}(v)$ while the maximum and minimum degree of the graph $G$ are denoted by $\Delta(G)$ and $\delta(G)$ respectively.

**Definition 1.3.** Let $G = (V, E)$ be a graph and $uv \in E$. Then, $u$ strongly dominates $v$ if $\text{deg}(u) \geq \text{deg}(v)$. A set $S$ is a strong dominating set ($sd$-set) if every vertex $v \in V - S$ is strongly dominated by some $u$ in $S$. Analogously, one can define a weak dominating set ($wd$-set).

The concept of strong (weak) domination was introduced by Sampathkumar and Pushpa Latha [11]. Rautenbach [10] has derived a new bound on $\gamma_{st}(G)$ and Meena et al. [7] have found the classes of graphs which are strong efficient. Domke et al. [4] have proved that the problems of computing $i_w$ and $i_{st}$ are NP-hard. Bounds on strong domination number are also reported by Rautenbach [9]. Swaminathan and Thangaraju [12] have established the relation between strong domination and maximum degree of the graph as well as weak domination and minimum degree of the graph.

**Definition 1.4.** The independent strong (weak) domination number of a graph $G$ is the minimum cardinality of a strongly (weak) dominating set which is independent. The independent strong domination number and the independent weak domination number are denoted by $i_{st}(G)$ and $i_w(G)$ respectively.

## 2 Main Results

**Proposition 2.1.** [13] If $S \subseteq V$ is a strong dominating set and $v \in V$ is the only vertex of maximum degree in $G$ then $v \in S$.

**Proposition 2.2.** [13] Let $v$ be a vertex with $\text{deg}(v) = \Delta(G) = k$ and $v$ is not adjacent to any other vertex of degree $k$ then $v$ must be in $sd$-set.

**Proposition 2.3.** [3] $\gamma_{st}(P_n) = \left\lceil \frac{n}{3} \right\rceil$.

**Definition 2.4.** Let $G$ be a graph with $V(G) = S_1 \cup S_2 \cup \ldots S_t \cup T$ where $S_i$ is the set having at least two vertices of same degree and $T = V(G) - \cup S_i$ where $i = 1, 2, \ldots, t$. The degree splitting graph $DS(G)$ is obtained from $G$ by adding vertices $w_1, w_2, \ldots, w_t$ and joining $w_i$ to each vertex of $S_i$ for $1 \leq i \leq t$.

**Theorem 2.5.** $\gamma_{st}(DS(P_n)) = i_{st}(DS(P_n)) = 2$ for $n \geq 5$.

**Proof:** The path $P_n$ has two pendant vertices and the remaining $n - 2$ vertices are of degree 2. Thus, $V(P_n) = \{v_i; 1 \leq i \leq n\} = S_1 \cup S_2$ where $S_1 = \{v_1, v_n\}$ and $S_2 = \{v_i; 2 \leq i \leq n - 1\}$.
To obtain $DS(P_n)$ from $P_n$, add two vertices $w_1$ and $w_2$ corresponding to $S_1$ and $S_2$ respectively. 
Thus, $V(DS(P_n)) = V(P_n) \cup \{w_1, w_2\}$ and $|V(DS(P_n))| = n + 2$. 
As $w_2$ is adjacent to $n - 2$ vertices of degree 3, it strongly dominates them and the vertex $w_1$ strongly dominates both the pendant vertices $v_1$ and $v_n$. Thus all the vertices of the graph including $w_1$ and $w_2$ are strongly dominated by $\{w_1, w_2\}$. Thus, $S = \{w_1, w_2\}$ is the strong dominating set as well as independent set of minimum cardinality. Hence $\gamma_{st}(DS(P_n)) = i_{st}(DS(P_n)) = 2$. 

**Definition 2.6.** A uniform t-ply is vertex disjoint union of $t$ paths of same length having common end points. A uniform ply has $t$ number of paths of length $k$. It is denoted by $P_t(k)$. The paths are called threads.

**Theorem 2.7.** $\gamma_{st}(P_t(k)) = 2 + t \left\lceil \frac{k - 3}{3} \right\rceil$ for $t \geq 3$ and $k \geq 3$.

**Proof:** Let $P_t(k)$ be the graph with $t$ number of paths of length $k$, where $u$ and $v$ are the common points with degree $t$. So, $|V(P_t(k))| = t(k - 1) + 2$.

By Proposition 2.2, the vertices $u$ and $v$ must be in strong dominating set. As there are $t$ paths between $u$ and $v$, the vertex $u$ strongly dominates $t$ vertices which are adjacent to it and the vertex $v$ strongly dominates $t$ vertices which are adjacent to it. Therefore, total $2t + 2$ vertices are strongly dominated by $u$ and $v$ including themselves. Now, there are $t(k - 1) + 2 - (2t + 2) = tk - 3t = (k - 3)t$ vertices which are not strongly dominated. That is, from each path there are $k - 3$ vertices which are not strongly dominated. Since the strongly domination number of the path $P_n$ is $\left\lceil \frac{n}{3} \right\rceil$, we need $\left\lceil \frac{k - 3}{3} \right\rceil$ vertices from each path to strongly dominate remaining vertices of the graph. Therefore, $t \left\lceil \frac{k - 3}{3} \right\rceil$ vertices are enough to strongly dominate remaining vertices of the graph. So, total $2 + t \left\lceil \frac{k - 3}{3} \right\rceil$ vertices are enough to strongly dominate all the vertices of the graph.

Hence, $\gamma_{st}(P_t(k)) = 2 + t \left\lceil \frac{k - 3}{3} \right\rceil$. \hfill \blacksquare

**Theorem 2.8.**

1. $\gamma_{st}(P_1(k)) = \left\lceil \frac{k + 1}{3} \right\rceil$ for $k \in \mathbb{N}$.

2. $\gamma_{st}(P_2(k)) = \left\lceil \frac{2k}{3} \right\rceil$ for $k \in \mathbb{N}$.

3. $\gamma_{st}(P_t(1)) = 1$

4. $\gamma_{st}(P_t(2)) = 2$
Theorem 2.10. \( G \) one is a vertex of \( E \) the graph. Hence, the minimal strong dominating set contains either \( v \) or \( \varepsilon \). Therefore, the result we consider following cases.

Case (i): \( n \) is odd.

If \( n = 5 \) then by the argument given in the beginning of the proof, \( e_{1}, e_{4} \in S \). Now \( e_{1} \) strongly dominates \( v_{1} \) and \( v_{2} \) other than itself whereas \( e_{4} \) strongly dominates \( v_{4} \) and \( v_{5} \) other than itself. The vertices \( e_{2}, e_{3} \) and \( v_{4} \) are not strongly dominated by \( e_{1} \) or \( e_{4} \). We know that \( v_{3} \) is strongly dominated by \( e_{2}, e_{3} \) or itself while \( e_{2} \) and \( e_{3} \) strongly dominate each other. Therefore, we must consider at least one vertex from \( \{ e_{2}, e_{3} \} \) in \( S \) to obtain strong dominating set. As \( S \) is an \( sd - set \) of minimum cardinality, \( \gamma_{st}(M(P_{5})) = 3 \).

If \( n = 7 \) then by the argument given in the beginning of the proof, \( e_{1}, e_{6} \in S \). Now \( e_{1} \) strongly dominates \( v_{1} \) and \( v_{2} \) other than itself while \( e_{6} \) strongly dominates \( v_{6} \) and \( v_{7} \) other than itself. We observe that the vertices \( e_{2}, e_{3}, e_{4}, e_{5}, e_{6} \) are not strongly dominated by \( e_{1} \) or \( e_{6} \). If \( e_{3} \in S \) then the vertices of the set \( \{ e_{1}, e_{3}, e_{6} \} \) strongly dominate all the vertices except \( v_{5} \) and \( e_{5} \) while if \( e_{4} \in S \) then the vertices of the set \( \{ e_{1}, e_{4}, e_{6} \} \) strongly dominate all the vertices except \( v_{3}, e_{2} \). If \( e_{3} \) is included in \( S \) then \( e_{5} \) or \( e_{4} \) must be in \( S \) or if \( e_{4} \) is included in \( S \) then \( e_{2} \) or \( e_{3} \) must be in \( S \). Hence, we need to include two vertices in \( S \) other than \( e_{1} \) and \( e_{7} \) to obtain

**Proof:** In \( \gamma_{st}(P_{i}(k)) \), if \( t = 1, k \in N \) then \( P_{1}(k) \) becomes path of length \( k \) with \( k + 1 \) vertices. Hence, \( \gamma_{st}(P_{1}(k)) = \left[ \frac{k + 1}{3} \right] \).

If \( t = 2, k \in N \) then \( P_{2}(k) \) becomes cycle with \( (k + 1) + (k + 1) - 2 = 2k \) vertices. Hence, \( \gamma_{st}(P_{2}(k)) = \left[ \frac{2k}{3} \right] \) for \( k \in N \).

If \( k = 1, t \in N \) then there are only two vertices \( u \) and \( v \) which can strongly dominate each other. Therefore, the minimal strong dominating set contains either \( u \) or \( v \). Hence, \( \gamma_{st}(P_{1}(1)) = 1 \).

If \( k = 2, t \in N \) then from \( P_{2}(2) \) the vertices \( u \) and \( v \) strongly dominate all the vertices of the graph. Hence, \( \gamma_{st}(P_{2}(2)) = 2 \). \( \blacksquare \)

**Definition 2.9.** The middle graph \( M(G) \) of a graph \( G \) is the graph whose vertex set is \( V(G) \cup E(G) \) and two vertices in \( M(G) \) are adjacent whenever either they are adjacent edges of \( G \) or one is a vertex of \( G \) and the other is an edge incident with it.

**Theorem 2.10.** \( \gamma_{st}(M(P_{n})) = \left\lceil \frac{n}{2} \right\rceil \) for \( n \geq 5 \).

**Proof:** Let \( v_{1}, v_{2}, \ldots, v_{n} \) be the vertices and \( e_{1}, e_{2}, \ldots, e_{n-1} \) be the edges of path \( P_{n} \). Then, \( V(M(P_{n})) = \{v_{1}, v_{2}, v_{3}, \ldots, v_{n}, e_{1}, e_{2}, e_{3}, \ldots, e_{n-1}\} \). Therefore, \( |V(M(P_{n}))| = 2n - 1 \). Let \( V(M(P_{n})) = P \cup P' \) where \( P = \{v_{1}, v_{2}, v_{3}, \ldots, v_{n}\} \) and \( P' = \{e_{1}, e_{2}, e_{3}, \ldots, e_{n-1}\} \). Every \( v_{i}(i = 1, 2, \ldots, n) \) strongly dominates itself and every \( e_{i}(i = 2, 3, \ldots, n - 2) \) strongly dominates four vertices \( (e_{i-1}, e_{i+1}, v_{i}, v_{i+1}) \) other than itself. We can observe that for any \( n \in N \), \( v_{1} \) and \( v_{n} \) are strongly dominated by only \( e_{1} \) and \( e_{n-1} \) respectively except themselves. Therefore, to strongly dominate \( v_{1} \) and \( v_{n} \) the vertices \( e_{1} \) and \( e_{n-1} \) must be in \( sd - set \ \ S \). In order to prove the result we consider following cases.

**Case (i):** \( n \) is odd.

If \( n = 5 \) then by the argument given in the beginning of the proof, \( e_{1}, e_{4} \in S \). Now \( e_{1} \) strongly dominates \( v_{1} \) and \( v_{2} \) other than itself whereas \( e_{4} \) strongly dominates \( v_{4} \) and \( v_{5} \) other than itself. The vertices \( e_{2}, e_{3} \) and \( v_{4} \) are not strongly dominated by \( e_{1} \) or \( e_{4} \). We know that \( v_{3} \) is strongly dominated by \( e_{2}, e_{3} \) or itself while \( e_{2} \) and \( e_{3} \) strongly dominate each other. Therefore, we must consider at least one vertex from \( \{ e_{2}, e_{3} \} \) in \( S \) to obtain strong dominating set. As \( S \) is an \( sd - set \) of minimum cardinality, \( \gamma_{st}(M(P_{5})) = 3 \).

If \( n = 7 \) then by the argument given in the beginning of the proof, \( e_{1}, e_{6} \in S \). Now \( e_{1} \) strongly dominates \( v_{1} \) and \( v_{2} \) other than itself while \( e_{6} \) strongly dominates \( v_{6} \) and \( v_{7} \) other than itself. We observe that the vertices \( e_{2}, e_{3}, e_{4}, v_{4}, e_{5}, e_{6} \) are not strongly dominated by \( e_{1} \) or \( e_{6} \). If \( e_{3} \in S \) then the vertices of the set \( \{e_{1}, e_{3}, e_{6}\} \) strongly dominate all the vertices except \( v_{5} \) and \( e_{5} \) while if \( e_{4} \in S \) then the vertices of the set \( \{e_{1}, e_{4}, e_{6}\} \) strongly dominate all the vertices except \( v_{3}, e_{2} \). If \( e_{3} \) is included in \( S \) then \( e_{5} \) or \( e_{4} \) must be in \( S \) or if \( e_{4} \) is included in \( S \) then \( e_{2} \) or \( e_{3} \) must be in \( S \). Hence, we need to include two vertices in \( S \) other than \( e_{1} \) and \( e_{7} \) to obtain
a strong dominating set of minimum cardinality. Hence, $\gamma_{st}(M(P_7)) = 4$.

In general, if $n = 2k + 1$ where $k = 2, 3, \ldots$ then $e_1, e_{n-1} \in S$. That is, only six vertices are strongly dominated by $e_1$ and $e_{n-1}$. Therefore, to strongly dominate remaining vertices of $M(P_n)$ we consider $\frac{n-3}{2}$ alternate vertices from $P'$. Therefore, to obtain a strong dominating set $S$ of minimum cardinality, we have to include $e_1, e_{n-1}$ and $\frac{n-3}{2}$ vertices from $P'$. Hence, 

$$\gamma_{st}(M(P_n)) = 2 + \frac{n-3}{2} = \frac{n+1}{2}.$$ 

**Case (ii):** $n$ is even.

If $n = 6$ then by the argument given in the beginning of the proof, $e_1, e_5 \in S$. Now $e_1$ strongly dominates $v_1$ and $v_2$ other than itself and $e_5$ strongly dominates $v_5$ and $v_6$ other than itself. The vertices $v_2, v_3, v_3, v_4$ and $v_4$ are not strongly dominated by $e_1$ or $e_5$. By including $e_3$ in $S$ the remaining vertices $e_2, v_3, e_3, v_4$ and $v_4$ are strongly dominated by $e_3$. Therefore, $e_3$ must be in $S$. So, $S = \{e_1, e_3, e_5\}$ becomes a strong dominating set of minimum cardinality. Hence, $\gamma_{st}(M(P_6)) = 3$.

If $n = 8$ then by the argument given in the beginning of the proof, $e_1, e_7 \in S$. Now $e_1$ strongly dominates $v_1$ and $v_2$ other than itself and $e_7$ strongly dominates $v_7$ and $v_8$ other than itself. The vertices $e_2, v_3, e_3, v_4, e_4, v_5, e_5, v_6, e_6$ are not strongly dominated by $e_1$ or $e_7$. If we consider any one vertex from $\{e_2, e_3, e_4, e_5, e_6\}$ then some vertices of $M(P_8)$ are not strongly dominated.

Therefore, we must include at least two vertices in $S$ from $\{e_3, e_4, e_5\}$ such that $S$ will become a strong dominating set of minimum cardinality. If we consider any two successive vertices $e_3, e_4$ then $v_6, e_6$ are not dominated and $e_4, e_5$ are considered then $v_3, e_3$ are not dominated. In the same way we can not consider $e_2$ and $e_4$. Therefore, we must consider $e_3$ and $e_5$ in $S$. So, $S = \{e_1, e_3, e_5, e_7\}$ and $S$ becomes an $sd-set$ of minimum cardinality. Hence, $\gamma_{st}(M(P_8)) = 4$.

In general, if $n = 2k$ where $k = 3, 4, \ldots$ then $e_1, e_{n-1} \in S$. That is, only eight vertices are strongly dominated by $e_1$ and $e_{n-1}$. Therefore, to strongly dominate remaining vertices of $M(P_n)$ we consider $\frac{n-4}{2}$ alternate vertices from $P'$. Therefore, $S = \{e_1, e_3, e_5, \ldots e_{n-1}\}$ is an $sd-set$. As $S - e_i$ will no longer remain an $sd-set$, $S$ is minimal $sd-set$. Since $S$ is the only strong dominating set of minimum cardinality, 

$$\gamma_{st}(M(P_n)) = 2 + \frac{n-4}{2} = \frac{n}{2}.$$ 

Hence, in each case $\gamma_{st}(M(P_n)) = \left\lceil \frac{n}{2} \right\rceil$.

**Definition 2.11.** [11] Let $G = (V, E)$ be a graph and $D \subseteq V$. Then,

1. $D$ is full if every $u \in D$ is adjacent to some $v \in V - D$.

2. $D$ is $s$-full ($w$-full) if every $u \in D$ strongly (weakly) dominates some $v \in V - D$.

**Definition 2.12.** [11] A graph $G$ is domination balanced ($d$-balanced) if there exists an $sd-set$ $D_1$ and a $wd-set$ $D_2$ such that $D_1 \cap D_2 = \emptyset$. 
Proposition 2.13. [11] For a graph $G$, the following statements are equivalent.

1. $G$ is $d$-balanced.
2. There exists an $sd$-set $D$ which is $s$-full.
3. There exists an $wd$-set $D$ which is $w$-full.

Theorem 2.14. For the complete bipartite graph $K_{m,n}$,

1. $i_s(K_{m,n}) + i_w(K_{m,n}) = m + n$, for $m \neq n$.
2. $\gamma(K_{m,n}) + i_w(K_{m,n}) = m + n$, for $m \neq n$.

Proof: (1) If $m > n$ then by the definition of independent strong domination number and independent weak domination number $i_s(K_{m,n}) = n$ and $i_w(K_{m,n}) = m$. Therefore, $i_s(K_{m,n}) + i_w(K_{m,n}) = m + n$.

If $m < n$ then by the definition of independent strong domination number and independent weak domination number $i_s(K_{m,n}) = m$ and $i_w(K_{m,n}) = n$. Therefore, $i_s(K_{m,n}) + i_w(K_{m,n}) = m + n$.

(2) If $m > n$ then by the definition of domination number and independent weak domination number $\gamma(K_{m,n}) = n$ and $i_w(K_{m,n}) = m$. Therefore, $\gamma(K_{m,n}) + i_w(K_{m,n}) = m + n$.

If $m < n$ then by the definition of domination number and independent weak domination number $\gamma(K_{m,n}) = m$ and $i_w(K_{m,n}) = n$. Therefore, $\gamma(K_{m,n}) + i_w(K_{m,n}) = m + n$.

Theorem 2.15. The degree splitting graph $DS(P_n)$ is $d$-balanced for $n \geq 4$.

Proof: In Theorem 2.5, we obtained an $sd$-set $S = \{w_1, w_2\}$ for $DS(P_n)$ for $n \geq 4$ which is $s$-full as every vertex of $S$ strongly dominates some $v \in V - S$. Therefore, by Proposition 2.13, $DS(P_n)$ is a $d$-balanced graph $n \geq 4$.

Proposition 2.16. [13] If there exists an isolated vertex in graph $G$ then $G$ is not $d$-balanced.

Definition 2.17. The switching of a vertex $v$ of $G$ means removing all the edges incident to $v$ and adding edges joining $v$ to every vertex which is not adjacent to $v$ in $G$. The resultant graph is denoted by $\tilde{G}$.

Theorem 2.18. Let $G$ be any graph with order $p$ and there is at least one vertex $v$ such that $\triangle(v) = p - 1$. If $\tilde{G}$ is the graph obtained by switching of a vertex $v$ of degree $p - 1$ from $G$ then, $\tilde{G}$ is not $d$-balanced.

Proof: Since $G$ has a vertex with degree $p - 1$, there is an isolated vertex in $\tilde{G}$. Hence, by Proposition 2.16 it is not a $d$-balanced graph.
3 Concluding Remarks

The concept of strong domination in graphs relates dominating sets and the degree of vertices. The strong domination numbers of some standard graphs are already available in the literature while we investigate the strong domination number for the larger graphs obtained from path $P_n$ by means of some graph operations to derive similar results for other graph families as well as in the context of various domination models are open areas of research.

Acknowledgement

The authors are highly thankful to the anonymous referees for their critical comments and constructive suggestions for the improvement in the first draft of this paper.

References


