Edge antimagic total labeling of isomorphic copies of subdivided stars

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Abstract

Enomoto, Llado, Nakamigawa and Ringel defined the concept of a super \((a,0)\)-edge-antimagic total labeling and proposed the conjecture that every tree is a super \((a,0)\)-edge-antimagic total labeling. In the support of this conjecture, the present paper deals with different results on antimagicness of isomorphic copies of subdivided stars.

Keywords: Super edge antimagic total graph, subdivided stars.

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1 Introduction

We begin with simple, finite, connected and undirected graph \(G(V,E)\) with \(V\) and \(E\) denote the vertex-set and the edge-set. A labeling of a graph is a mapping that carries the graph elements to numbers (usually to positive or non-negative integers). Some labelings use the vertex-set only or the edge-set. We shall call them vertex-labelings or edge-labelings, respectively. A general reference for graph-theoretic ideas can be found in [22]. For a detailed survey of the graph labeling we refer to Gallian [11]. In this paper the domain will be the set of all vertices and edges and such a labeling is called a total labeling. The notion of edge-magic total labeling of graphs has its origin in the works of Kotzig and Rosa [12, 13] on what they called magic valuations of graphs. The definition of \((a,d)\)-edge-antimagic total labeling was introduced by Simanjuntak, Bertault and Miller in [20] as a natural extension of edge-magic labeling defined by Kotzig and Rosa.

Conjecture 1.1. [8] Every tree admits a super edge-magic total labeling.

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To support this conjecture, many authors have considered super edge-magic total labeling for many particular classes of trees for example, [1–7, 9, 10, 15–21]. Lee and Shah [14] verified this conjecture by a computer search for trees with at most 17 vertices. However, this conjecture is still as an open problem. Ngurah et. al. [15] proved that $T(m, n, k)$ is also super edge-magic if $k = n + 3$ or $n + 4$. In [19], Salman et. al. found the super edge-magic total labeling of a subdivision of a star $S^m_n$ for $m = 1, 2$. However, super $(a, d)$-edge-antimagic total labelings of copies of subdivided star $G \cong mT(n_1, n_2, n_3, ..., n_r)$ for different $\{n_i : 1 \leq i \leq r\}$ and $m \geq 3$ is still an open problem.

**Definition 1.2.** A graph $G$ is called $(a, d)$-edge-antimagic total ($(a, d) - \text{EAT}$) if there exist integers $a > 0$, $d \geq 0$ and a bijection $\lambda : V(G) \cup E(G) \rightarrow \{1, 2, ..., v + e\}$ such that $W = \{w(xy) : xy \in E(G)\}$ forms an arithmetic progression starting from $a$ with the common difference $d$, where $w(xy) = \lambda(x) + \lambda(y) + \lambda(xy)$ for any $xy \in E(G)$. $W$ is called the set of edge-weights of the graph $G$.

**Definition 1.3.** A $(a, d)$-edge-antimagic total labeling $\lambda$ is called super $(a, d)$-edge-antimagic total labeling if $\lambda(V(G)) = \{1, 2, ..., v\}$.

**Lemma 1.4.** [7] If $g$ is a super edge-magic total labeling of $G$ with the magic constant $a$, then the function $g_1 : V(G) \cup E(G) \rightarrow \{1, 2, ..., v + e\}$ defined by

$$
g_1(x) = \begin{cases} 
  v + 1 - g(x), & \text{for } x \in V(G), \\
  2v + e + 1 - g(x), & \text{for } x \in E(G).
\end{cases}
$$

is also a super edge-magic total labeling of $G$ with the magic constant $a_1 = 4v + e + 3 - a$.

**Definition 1.5.** For $n_i \geq 1$ and $p \geq 3$, let $G \cong T(n_1, n_2, ..., n_p)$ be a graph obtained by inserting $n_i - 1$ vertices to each of the $i$-th edge of the star $K_{1,p}$, where $1 \leq i \leq p$.

2 Main Results

We consider the following proposition which will be used frequently in the main results.

**Proposition 2.1.** [3] If a $(v, e)$-graph $G$ has a $(s, d)$-EAV labeling then

- $G$ has a super $(s + v + 1, d + 1)$-EAT labeling,
- $G$ has a super $(s + v + e, d - 1)$-EAT labeling.

**Theorem 2.2.** Let $G \cong 2T(n + 2, n, n)$ be a graph with order $v$ and $n \equiv 1(\text{mod}2)$. Then $G$ admits super $(a, 0)$-edge-antimagic total labeling with $a = 2v + s - 1$ and super $(a, 2)$-edge-antimagic total labeling with $a = v + s + 1$, where $s = 3n + 7$.

**Proof:** We suppose the vertex-set and the edge-set of $G$, as follows:

$$
V(G) = \{c_j : 1 \leq j \leq 2\} \cup \{x_{ij}^{l_i} : 1 \leq i \leq 3; 1 \leq l_i \leq n_i; 1 \leq j \leq 2\}.
$$
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The order and size of the graph $G$ are $v = 6(n + 1)$ and $e = 2(3n + 2)$. Now, we define the labeling $\lambda : V(G) \to \{1, 2, ..., v\}$ as follows:

$$\lambda(c_j) = 2(n + 3) + \frac{3n + 1}{2} j, \quad j = 1, 2.$$

For odd $l_i \leq l_i \leq n_i$ and $i = 1, 2$ and $3$, we define

$$\lambda(u) = \begin{cases} 
\frac{n+4-l_i}{2} + \frac{3n+5}{2} (j-1), & \text{for } u = x_{1j}^l, \\
\frac{n+4+l_i}{2} + \frac{3n+5}{2} (j-1), & \text{for } u = x_{2j}^l, \\
\frac{3n+5}{2} (j-1), & \text{for } u = x_{3j}^l; \quad l_3 = 1, \\
\frac{3(n+2)-l_i}{2} + \frac{9(n+1)}{2} (j-1), & \text{for } u = x_{3j}^l; \quad l_i \geq 3.
\end{cases}$$

For even $l_i \leq l_i \leq n_i$ and $i = 1, 2$ and $3$, we define

$$\lambda(u) = \begin{cases} 
\frac{7n+13-l_i}{2} + \frac{3n+1}{2} (j-1), & \text{for } u = x_{1j}^l, \\
\frac{7n+13+l_i}{2} + \frac{3n+1}{2} (j-1), & \text{for } u = x_{2j}^l, \\
\frac{9n+13-l_i}{2} - \frac{3(n+1)}{2} (j-1), & \text{for } u = x_{3j}^l.
\end{cases}$$

The set of all edge-sums generated by the above formula forms a consecutive integer sequence $S = \{(3n + 6) + 1, (3n + 6) + 2, ..., (3n + 6) + e\}$. Let $s = \min(S)$. Therefore, by Proposition 2.1, $\lambda$ can be extended to a super $(a, 0)$-edge-antimagic total labeling and we obtain the magic constant $a = v + e + s = 15n + 17$. Similarly by Proposition 2.1, $\lambda$ can be extended to a super $(a, 2)$-edge-antimagic total labeling and we obtain the magic constant $a = v + 1 + s = 9n + 14$.

**Theorem 2.3.** Let $G \cong 2T(n + 2, n, n + 1)$ be a graph with order $v$ and $n \equiv 1 \pmod{2}$. Then $G$ admits super $(a, 1)$-edge-antimagic total labeling with $a = 2v + s - 1$ and super $(a, 3)$-edge-antimagic total labeling with $a = v + s + 1$, where $s = 4$.

**Proof:** We suppose the vertex-set and the edge-set of $G$, as follows:

$$V(G) = \{c_j \mid 1 \leq j \leq 2\} \cup \{x_{ij}^l \mid 1 \leq i \leq 3 ; 1 \leq l_i \leq n_i; \ 1 \leq j \leq 3\},$$

$$E(G) = \{c_j x_{ij}^l \mid 1 \leq i \leq 3 ; 1 \leq j \leq 2\} \cup \{x_{ij}^l x_{ij}^{l+1} \mid 1 \leq i \leq 3 ; 1 \leq l_i \leq n_i - 1 ; 1 \leq j \leq 2\}.$$

The order and size of the graph $G$ are $v = 2(3n + 4)$ and $e = 6(n + 1)$. Now, we define the labeling $\lambda : V(G) \to \{1, 2, ..., v\}$ as follows:

$$\lambda(c_j) = (2n + 4) + j, \quad j = 1, 2.$$
For all \( l_i \quad 1 \leq l_i \leq n_i \) and \( i = 1, 2 \) and 3, we define

\[
\lambda(u) = \begin{cases} 
(2n + 4) + j - 2l_1, & \text{for } u = x_{1j}^{l_1}, \\
(2n + 4) + j + 2l_2, & \text{for } u = x_{2j}^{l_2}, \\
(6n + 8) + j - 2l_3, & \text{for } u = x_{3j}^{l_3}.
\end{cases}
\]

The set of all edge-sums generated by the above formula forms a consecutive integer sequence \( S = \{4, 4 + 2, \ldots, 4 + 2(e - 1)\} \). Let \( s = \text{min}(S) \). Therefore, by Proposition 2.1, \( \lambda \) can be extended to a super \((a, 1)\)-edge-antimagic total labeling and we obtain the magic constant \( a = v + e + s = 6(2n + 3) \). Similarly by Proposition 2.1, \( \lambda \) can be extended to a super \((a, 3)\)-edge-antimagic total labeling and we obtain the magic constant \( a = v + 1 + s = (6n + 13) \). 

**Theorem 2.4.** Let \( G \cong 2T(n + 2, n, n + 1, 2n + 1) \) be a graph with order \( v \) and \( n \equiv 1(\text{mod}2) \). Then \( G \) admits super \((a, 0)\)-edge-antimagic total labeling with \( a = 2v + s - 1 \) and super \((a, 2)\)-edge-antimagic total labeling with \( a = v + s + 1 \), where \( s = 5n + 9 \).

**Proof:** We suppose the vertex-set and the edge-set of \( G \), as follows:

\[ V(G) = \{c_j \mid 1 \leq j \leq 2\} \cup \{x_{ij}^{l_i} \mid 1 \leq i \leq 4 \ ; \ 1 \leq l_i \leq n_i \ ; \ 1 \leq j \leq 2\}, \]

\[ E(G) = \{c_jx_{ij}^{l_i} \mid 1 \leq i \leq 4 \ ; \ 1 \leq j \leq 2\} \cup \{x_{ij}^{l_i}x_{ij}^{l_i+1} \mid 1 \leq i \leq 4 \ ; \ 1 \leq l_i \leq n_i - 1 \ ; \ 1 \leq j \leq 2\} \]

The order and size of the graph \( G \) are \( v = 10(n + 1) \) and \( e = (10n + 8) \). Now, we define the labeling \( \lambda : V(G) \to \{1, 2, \ldots, v\} \) as follows:

\[ \lambda(c_j) = 3n + 7 + \frac{5n + 3}{2}j, \quad j = 1, 2. \]

For odd \( l_i \quad 1 \leq l_i \leq n_i \) and \( i = 1, 2, 3 \) and 4, we define

\[
\lambda(u) = \begin{cases} 
\frac{n + 4 - l_1}{2} + \frac{5n + 7}{2}(j - 1), & \text{for } u = x_{1j}^{l_1}, \\
\frac{n + 4 + l_2}{2} + \frac{5n + 7}{2}(j - 1), & \text{for } u = x_{2j}^{l_2}, \\
\frac{3(n+2)-l_3}{2} + \frac{(5n+7)}{2}(j - 1), & \text{for } u = x_{3j}^{l_3}, \\
\frac{5n+7}{2}j, & \text{for } u = x_{4j}^{l_4}; \ l_4 = 1, \\
\frac{5n+8-l_4}{2} + \frac{15(n+1)}{2}(j - 1), & \text{for } u = x_{3j}^{l_4}; \ l_4 \geq 3.
\end{cases}
\]

For even \( l_i \quad 1 \leq l_i \leq n_i \) and \( i = 1, 2, 3 \) and 4, we define
Theorem 2.5. Let \( G \cong 2T(n+2, n, n+1, 2n+2) \) be a graph with order \( v \) and \( n \equiv 1(\text{mod}2) \). Then \( G \) admits super \((a, 1)\)-edge-antimagic total labeling with \( a = v2 + s - 1 \) and super \((a, 3)\)-edge-antimagic total labeling with \( a = v + s + 1 \), where \( s = 4 \).

Proof: We suppose the vertex-set and the edge-set of \( G \), as follows:
\[
V(G) = \{c_j \mid 1 \leq j \leq 2\} \cup \{x_{ij}^l \mid 1 \leq i \leq 4; 1 \leq l_i \leq n_i; 1 \leq j \leq 2\},
\]
\[
E(G) = \{c_jx_{ij}^1 | 1 \leq i \leq 4; 1 \leq j \leq 2\} \cup \{x_{ij}^1x_{ij}^{l_i+1} | 1 \leq i \leq 4; 1 \leq l_i \leq n_i - 1; 1 \leq j \leq 2\}.
\]
The order and size of the graph \( G \) are \( v = 2(5n + 6) \) and \( e = 10(n + 1) \). Now, we define the labeling \( \lambda : V(G) \rightarrow \{1, 2, \ldots, v\} \) as follows:
\[
\lambda(c_j) = (2n + 4) + j, \quad j = 1, 2.
\]
For all \( l_i \quad 1 \leq l_i \leq n_i \), where \( i = 1, 2 \) and \( 3 \), we define
\[
\lambda(u) = \begin{cases} 
(2n + 4) + j - 2l_1, & \text{for } u = x_{1j}^1, \\
(2n + 4) + j + 2l_2, & \text{for } u = x_{2j}^2, \\
(6n + 8) + j - 2l_3, & \text{for } u = x_{3j}^3, \\
(10n + 12) + j - 2l_4, & \text{for } u = x_{4j}^4.
\end{cases}
\]
The set of all edge-sums generated by the above formula forms a consecutive integer sequence \( S = \{4, 4 + 2, \ldots, 4 + 2(e - 1)\} \). Let \( s = \text{min}(S) \). Therefore, by Proposition 2.1, \( \lambda \) can be extended to a super \((a, 1)\)-edge-antimagic total labeling and we obtain the magic constant.
a = v + e + s = 2(10n + 13). Similarly by Proposition 2.1, λ can be extended to a super (a, 3)-
edge-antimagic total labeling and we obtain the magic constant a = v + 1 + s = 10n + 17.

**Theorem 2.6.** Let $G \cong 2T(n + 2, n, n + 1, 2(n + 1), 4n + 3)$ be a graph with order v and
$n \equiv 1(\text{mod} 2)$. Then G admits super (a, 0)-edge-antimagic total labeling with $a = 2v + s - 1$
and super (a, 2)-edge-antimagic total labeling with $a = v + s + 1$, where $s = 9n + 13$.

**Proof:** We suppose the vertex-set and the edge-set of $G$, as follows:
$V(G) = \{c_j \mid 1 \leq j \leq 2\} \cup \{x_{ij}^l \mid 1 \leq i \leq 5 ; 1 \leq l_i \leq n_i; 1 \leq j \leq 2\},$
$E(G) = \{c_jx_{ij}^l \mid 1 \leq i \leq 5 ; 1 \leq j \leq 2\} \cup \{x_{ij}^l x_{ij}^{l+1} \mid 1 \leq i \leq 5 ; 1 \leq l_i \leq n_i - 1 ; 1 \leq j \leq 2\}.$
The order and size of the graph $G$ are $v = 18(n + 1)$ and $e = 2(9n + 8)$. Now, we define the
labeling $\lambda : V(G) \to \{1, 2, ..., v\}$ as follows:

$$
\lambda(c_j) = 5n + 9 + \frac{9n + 7}{2}j, \quad j = 1, 2.
$$

For odd $l_i \quad 1 \leq l_i \leq n_i$ and $i = 1, 2, 3, 4$ and 5, we define

$$
\lambda(u) = \begin{cases}
\frac{n+4-l_1}{2} + \frac{9n+11}{2} (j - 1), & \text{for } u = x_{ij}^{l_1}, \\
\frac{n+4+2}{2} + \frac{9n+11}{2} (j - 1), & \text{for } u = x_{ij}^{l_2}, \\
\frac{3(n+2)-l_3}{2} + \frac{9n+11}{2} (j - 1), & \text{for } u = x_{ij}^{l_3}, \\
\frac{5n+8-l_4}{2} + \frac{9n+11}{2} (j - 1), & \text{for } u = x_{ij}^{l_4}, \\
\frac{9n+11}{2} + \frac{9n+11}{2}, & \text{for } u = x_{ij}^{l_5} ; l_5 = 1, \\
\frac{9n+12-l_5}{2} + \frac{27(n+1)}{2}, & \text{for } u = x_{ij}^{l_5} ; l_5 \geq 3.
\end{cases}
$$

For even $l_i \quad 1 \leq l_i \leq n_i$ and $i = 1, 2, 3, 4$ and 5, we define

$$
\lambda(u) = \begin{cases}
\frac{19n+25-l_1}{2} + \frac{9n+7}{2} (j - 1), & \text{for } u = x_{ij}^{l_1}, \\
\frac{19n+25+l_2}{2} + \frac{9n+7}{2} (j - 1), & \text{for } u = x_{ij}^{l_2}, \\
\frac{21n+27-l_3}{2} - \frac{9n+7}{2} (j - 1), & \text{for } u = x_{ij}^{l_3}, \\
\frac{23n+29-l_4}{2} + \frac{9n+7}{2} (j - 1), & \text{for } u = x_{ij}^{l_4}, \\
\frac{27n+31-l_5}{2} - \frac{9(n+1)}{2} (j - 1), & \text{for } u = x_{ij}^{l_5}.
\end{cases}
$$
The set of all edge-sums generated by the above formula forms a consecutive integer sequence \( S = \{(9n + 12) + 1, (9n + 12) + 2, ..., (9n + 12) + e\} \). Let \( s = \min(S) \). Therefore, by Proposition 2.1, \( \lambda \) can be extended to a super \((a,0)\)-edge-antimagic total labeling and we obtain the magic constant \( a = v + e + s = 55n + 47 \). Similarly by Proposition 2.1, \( \lambda \) can be extended to a super \((a,2)\)-edge-antimagic total labeling and we obtain the magic constant \( a = v + 1 + s = 45n + 47 \).

**Theorem 2.7.** Let \( G \cong 2T(n + 2, n, n + 1, 2(n + 1), 4(n + 1)) \) be a graph with order \( v \) and \( n \equiv 1(\text{mod}2) \). Then \( G \) admits super \((a,1)\)-edge-antimagic total labeling with \( a = 2v + s - 1 \) and super \((a,3)\)-edge-antimagic total labeling with \( a = v + s + 1 \), where \( s = 4 \).

**Proof:** We suppose the vertex-set and the edge-set of \( G \), as follows: \( V(G) = \{c_j \mid 1 \leq j \leq 2\} \cup \{x_{ij}^l \mid 1 \leq i \leq 5 ; 1 \leq l \leq n_i ; 1 \leq j \leq 2\} \), \( E(G) = \{c_jx_{ij} \mid 1 \leq i \leq 5 ; 1 \leq j \leq 2\} \cup \{x_{ij}^lx_{ij}^{l+1} \mid 1 \leq i \leq 5 ; 1 \leq l \leq n_i - 1 ; 1 \leq j \leq 2\} \).

The order and size of the graph \( G \) are \( v = 2(9n + 8) \) and \( e = 2(9n + 7) \). Now, we define the labeling \( \lambda : V(G) \rightarrow \{1, 2, ..., v\} \) as follows:

\[
\lambda(c_j) = (2n + 4) + j, \quad j = 1, 2.
\]

For all \( l \) \( 1 \leq l \leq n_i \), where \( i = 1, 2 \) and \( 3 \), we define

\[
\lambda(u) = \begin{cases} 
(2n + 4) + j - 2l_1, & \text{for } u = x_{1j}^{l_1}, \\
(2n + 4) + j + 2l_2, & \text{for } u = x_{2j}^{l_2}, \\
(6n + 9) + j - 2l_3, & \text{for } u = x_{3j}^{l_3}, \\
(10n + 12) + j - 2l_4, & \text{for } u = x_{4j}^{l_4}, \\
(18n + 20) + j - 2l_5, & \text{for } u = x_{5j}^{l_5}.
\end{cases}
\]

The set of all edge-sums generated by the above formula forms a consecutive integer sequence \( S = \{4, 4 + 2, ..., 4 + 2(e - 1)\} \). Let \( s = \min(S) \). Therefore, by Proposition 2.1, \( \lambda \) can be extended to a super \((a,1)\)-edge-antimagic total labeling and we obtain the magic constant \( a = v + e + s = 36n + 34 \). Similarly by Proposition 2.1, \( \lambda \) can be extended to a super \((a,3)\)-edge-antimagic total labeling and we obtain the magic constant \( a = v + 1 + s = 18n + 19 \).

**Theorem 2.8.** Let \( G \cong T(n + 2, n, n + 1, 2(n + 1), 4(n + 1), ..., n_p) \) be a graph with order \( v \) and \( n \equiv 1(\text{mod}2) \). Then \( G \) admits super \((a,0)\)-edge-antimagic total labeling with \( a = 2v + s - 1 \) and super \((a,2)\)-edge-antimagic total labeling with \( a = v + s + 1 \), \( s = (n + 5) + 2^{p-2}(n + 1) \), \( n_i = 2^{i-3}(n + 1) \) for \( i = 4, 5, 6, ..., p - 1 \) and \( n_p = 2^{p-3}(n + 1) - 1 \).
Proof: We suppose that the vertex-set and the edge-set of $G$, as follows:
\[ V(G) = \{c_j \mid 1 \leq j \leq 2\} \cup \{x_{ij} \mid 1 \leq i \leq p; \ 1 \leq l_i \leq n_i; \ 1 \leq j \leq 2\}, \]
\[ E(G) = \{c_jx_{ij} \mid 1 \leq i \leq p; \ 1 \leq j \leq 2\} \cup \{x_{ij}x_{ij+1} \mid 1 \leq i \leq p; \ 1 \leq l_i \leq n_i - 1; \ 1 \leq j \leq 2\}. \]
The order and size of the graph $G$ are $v = 2(n + 1) + 2^{p-1}(n + 1)$ and $e = 2n + 2^{p-1}(n + 1)$.

Now, we define the labeling $\lambda : V(G) \to \{1, 2, \ldots, v\}$ as follows:
\[ \lambda(c_j) = \frac{2(n + 2) + 2^{p-2}(n + 1)}{2} + \frac{(n - 1) + 2^{p-2}(n + 1)}{2}j, \quad j = 1, 2. \]

For odd $l_i$, $1 \leq l_i \leq n_i$, where $i = 1, 2, 3, 4$ and 5, we define
\[ \lambda(u) = \begin{cases} 
\frac{n+4-l_i}{2} + \frac{(n+3)+2^{p-2}(n+1)}{2}(j - 1), & \text{for } u = x_{ij}^1, \\
\frac{n+4+l_k}{2} + \frac{(n+3)+2^{p-2}(n+1)}{2}(j - 1), & \text{for } u = x_{ij}^2, \\
\frac{3(n+4)-l_i}{2} + \frac{(n+3)+2^{p-2}(n+1)}{2}(j - 1), & \text{for } u = x_{ij}^3, \\
\frac{(n+4)+2^{k-2}(n+1)-l_k}{2} + \frac{(n+3)+2^{p-2}(n+1)}{2}(j - 1), & \text{for } u = x_{ij}^4, \\
\frac{(n+4)+2^{k-2}(n+1)-l_p}{2} + \frac{3(n+1)(1+2^{p-2})}{2}(j - 1), & \text{for } u = x_{ij}^5, \\
\text{for } 1 \leq l_k \leq n_k, & \text{for } k = 4, 5, \ldots, p - 1, \\
\text{for } u = x_{ij}^l, & \text{for } l_p = 1, \\
\text{for } 2 \leq l_p \leq n_p. & \text{for } 1 \leq l_i \leq n_i \text{ and } i = 1, 2, 3, 4 \text{ and } 5, \text{ we define}
\end{cases} \]

For even $l_i$, $1 \leq l_i \leq n_i$ and $i = 1, 2, 3, 4$ and 5, we define
\[ \lambda(u) = \begin{cases} 
\frac{3(n+1)+2^{p-1}(n+1)-l_k}{2} + \frac{(n+1)+2^{p-2}(n+1)}{2}(j - 1), & \text{for } u = x_{ij}^1, \\
\frac{3(n+1)+2^{p-1}(n+1)+l_k}{2} + \frac{(n+1)+2^{p-2}(n+1)}{2}(j - 1), & \text{for } u = x_{ij}^2, \\
\frac{3(n+3)+(2^{p-1}+2^{k-2})(n+1)-l_k}{2} + \frac{(n+1)+2^{p-2}(n+1)}{2}(j - 1), & \text{for } u = x_{ij}^3, \\
\frac{(3n+7)+(2^{p-1}+2^{k-2})(n+1)-l_k}{2} + \frac{(n+1)+2^{p-2}(n+1)}{2}(j - 1), & \text{for } 2 \leq l_k \leq n_k \text{, } k = 3, 4, \ldots, p - 1, \\
\text{for } 2 \leq l_p \leq n_p. & \text{for } 2 \leq l_p \leq n_p. \\
\end{cases} \]

The set of all edge-sums generated by the above formula forms a consecutive integer sequence $S = \{(n + 4) + 2^{p-2}(n + 1) + 1, \ (n + 4) + 2^{p-2}(n + 1) + 2, \ldots, \ (n + 4) + 2^{p-2}(n + 1) + e\}$. Let $s = \min(S)$. Therefore, by Proposition 2.1, $\lambda$ can be extended to a super $(a, 0)$-edge-antimagic total labeling and we obtain the magic constant $a = v + e + s = 5n + 7 + 2^{p-2}5(n + 1)$. Similarly by Proposition 2.1, $\lambda$ can be extended to a super $(a, 2)$-edge-antimagic total labeling and we
obtain the magic constant \( a = v + 1 + s = 3n + 8 + 2p^{-2}3(n + 1) \).

**Theorem 2.9.** Let \( G \cong T(n + 2, n, n + 1, 2(n + 1), 4(n + 1), \ldots, n_p) \) be a graph with order \( v \) and \( n \equiv 1(\text{mod}2) \). Then \( G \) admits super \((a, 1)\)-edge-antimagic total labeling with \( a = 2v + s - 1 \) and super \((a, 3)\)-edge-antimagic total labeling with \( a = v + s + 1 \), where \( s = 4 \).

**Proof:** We suppose the vertex-set and the edge-set of \( G \), as follows:

\[
V(G) = \{c_j \mid 1 \leq j \leq 2\} \cup \{x_{ij}^l \mid 1 \leq i \leq p ; 1 \leq l_i \leq n_i; 1 \leq j \leq 2\},
\]

\[
E(G) = \{c_jx_{ij}^l | 1 \leq i \leq p ; 1 \leq j \leq 2\} \cup \{x_{ij}^l x_{ij}^{l+1} | 1 \leq i \leq p ; 1 \leq l_i \leq n_i - 1 ; 1 \leq j \leq 2\}.
\]

The order and size of the graph \( G \) are

\[
v = 2(n + 2) + 2p - 1(n + 1) \quad \text{and} \quad e = 2(n + 1) + 2p^{-1}(n + 1)
\]

Now, we define the labeling \( \lambda : V(G) \rightarrow \{1, 2, \ldots, v\} \) as follows:

\[
\lambda(c_j) = (2n + 4) + j, \quad j = 1, 2.
\]

For all \( l_i \quad 1 \leq l_i \leq n_i \), we define

\[
\lambda(u) = \begin{cases} 
(2n + 4) + j - 2l_1, & \text{for } u = x_{ij}^{l_1}, \\
(2n + 4) + j + 2l_2, & \text{for } u = x_{ij}^{l_2}, \\
(6n + 8) + j - 2l_3, & \text{for } u = x_{ij}^{l_3}, \\
(10n + 12) + j - 2l_4, & \text{for } u = x_{ij}^{l_4}, \\
(10n + 12) + j + \sum_{m=5}^{i} [2^{m-2}(n + 1)] - 2l_i, & \text{for } u = x_{ij}^{l_i}, \ i \geq 5.
\end{cases}
\]

The set of all edge-sums generated by the above formula forms a consecutive integer sequence

\[
S = \{4, 4 + 2, \ldots, 4 + 2(e - 1)\}.
\]

Let \( s = \min(S) \). Therefore, by Proposition 2.1, \( \lambda \) can be extended to a super \((a, 1)\)-edge-antimagic total labeling and we obtain the magic constant \( a = v + e + s = 2(2n + 5) + 2p(n + 1) \). Similarly by Proposition 2.1, \( \lambda \) can be extended to a super \((a, 3)\)-edge-antimagic total labeling and we obtain the magic constant \( a = v + 1 + s = 2n + 9 + 2p^{-1}(n + 1) \).

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References


