Some Properties of Operations on $\alpha O(X)$

Alias B. Khalaf$^1$, Hariwan Z. Ibrahim$^2$

$^1$ Department of Mathematics
Faculty of Science, University of Duhok
Kurdistan-Region, Iraq.
aliasbkhalaf@gmail.com

$^2$ Department of Mathematics
Faculty of Science, University of Zakho
Kurdistan-Region, Iraq.
hariwan_math@yahoo.com

Abstract

In this paper, we introduce the notions of $\alpha_\gamma$-interior, $\alpha_\gamma$-neighbourhood, $\alpha_\gamma$-derived, $\alpha_\gamma$-boundary, $\alpha_\gamma$-kernel and $\alpha_\gamma$-g.closed set defined by $\gamma$-operation on $\alpha O(X)$ and investigate some of their properties.

Keywords: operation, $\alpha$-open set, $\alpha_\gamma$-open set.

AMS Subject Classification(2010): Primary: 54A05, 54A10; Secondary: 54C05.

1 Introduction

The notion of $\alpha$-open sets was introduced by Njastad [6] and he denoted the family of all $\alpha$-open sets in a topological space $(X, \tau)$ by $\alpha O(X, \tau)$ or $\alpha O(X)$. Ibrahim [1] defined the concept of an operation on $\alpha O(X)$ and introduced the notion of $\alpha_\gamma$-open sets. Kasahara [2] defined the concept of an operation on topological spaces and introduced $\alpha_\gamma$-closed graphs of an operation. Ogata [7] called the operation $\alpha$ as $\gamma$ operation and introduced the notion of $\tau_\gamma$ which is the collection of all $\gamma$-open sets in a topological space $(X, \tau)$. The aim of this paper is to continue the study of topological properties by means of operations on $\alpha O(X)$.

2 Preliminaries

Let $(X, \tau)$ be a topological space and $A$ a subset of $X$. The closure and the interior of $A$ are denoted by $Cl(A)$ and $Int(A)$, respectively. A subset $A$ of a topological space $(X, \tau)$ is said to be $\alpha$-open [6] if $A \subseteq Int(Cl(Int(A)))$. The complement of an $\alpha$-open set is said to be $\alpha$-closed.

The intersection of all $\alpha$-closed sets containing $A$ is called the $\alpha$-closure of $A$ and is denoted by $\alpha Cl(A)$. An operation $\gamma : \alpha O(X, \tau) \to P(X)$ [1] is a mapping satisfying the condition, $V \subseteq V^\gamma$ for each $V \in \alpha O(X, \tau)$. We call the mapping $\gamma$ an operation on $\alpha O(X, \tau)$. A subset $A$ of $X$ is called an $\alpha_\gamma$-open set [1] if for each point $x \in A$, there exists an $\alpha$-open set $U$ of $X$ containing $x$ such that $U^\gamma \subseteq A$. The complement of an $\alpha_\gamma$-open set is said to be $\alpha_\gamma$-closed. We denote the set
of all $\alpha_\gamma$-open (resp., $\alpha_\gamma$-closed) sets of $(X, \tau)$ by $\alpha O(X, \tau)_\gamma$ (resp., $\alpha C(X, \tau)_\gamma$). The $\alpha_\gamma$-closure [1] of a subset $A$ of $X$ with an operation $\gamma$ on $\alpha O(X)$ is denoted by $\alpha Cl(A)$ and is defined to be the intersection of all $\alpha_\gamma$-closed sets containing $A$. A point $x \in X$ is in $\alpha Cl_\gamma$-closure [1] of a set $A \subseteq X$, if $U^\gamma \cap A \neq \emptyset$ for each $\alpha$-open set $U$ containing $x$. The $\alpha Cl_\gamma$-closure of $A$ is denoted by $\alpha Cl_\gamma(A)$. An operation $\gamma$ on $\alpha O(X, \tau)$ is said to be $\alpha$-open [1] if for every $\alpha$-open set $U$ of $X$ containing $x \in X$, there exists an $\alpha_\gamma$-open set $V$ of $X$ such that $x \in V$ and $V \subseteq U^\gamma$.

3 Some Properties of $\gamma$-operations on $\alpha O(X)$

Definition 3.1. Let $(X, \tau)$ be a topological space and $\gamma$ an operation on $\alpha O(X)$. A point $a \in A \subseteq X$ is said to be $\alpha_\gamma$-interior point of $A$ if there exists an $\alpha$-open set $N$ of $X$ containing $a$ such that $N^\gamma \subseteq A$. We denote the set of all such points by $\alpha Int_\gamma(A)$.

Thus $\alpha Int_\gamma(A) = \{x \in A : x \in N \in \alpha O(X) \text{ and } N^\gamma \subseteq A\} \subseteq A$.

Theorem 3.2. Let $(X, \tau)$ be a topological space and $\gamma$ an operation on $\alpha O(X)$. If $A$ and $B$ are two subsets of $X$, then the following statements are true:

1. If $A \subseteq B$, then $\alpha Int_\gamma(A) \subseteq \alpha Int_\gamma(B)$.
2. $\alpha Int_\gamma(A) \cup \alpha Int_\gamma(B) \subseteq \alpha Int_\gamma(A \cup B)$.
3. If $\gamma$ is $\alpha$-regular, then $\alpha Int_\gamma(A) \cap \alpha Int_\gamma(B) = \alpha Int_\gamma(A \cap B)$.

Proof: Follows from Definition 3.1 and 2.14 [1].

Theorem 3.3. Let $(X, \tau)$ be a topological space and $\gamma$ an operation on $\alpha O(X)$. If $A$ is a subset of $X$, then

1. $\alpha Int_\gamma(X \setminus A) = X \setminus \alpha Cl_\gamma(A)$.
2. $\alpha Cl_\gamma(X \setminus A) = X \setminus \alpha Int_\gamma(A)$.
3. $\alpha Int_\gamma(A) = X \setminus \alpha Cl_\gamma(X \setminus A)$.
4. $\alpha Cl_\gamma(A) = X \setminus \alpha Int_\gamma(X \setminus A)$.

Proof: We prove (1) only and the other parts can be proved similarly.

Let $x \in \alpha Int_\gamma(X \setminus A)$, then there exists an $\alpha$-open sets $U$ containing $x$ such that $U^\gamma \subseteq X \setminus A$. This implies that $U^\gamma \cap A = \emptyset$. This gives that $x \notin \alpha Cl_\gamma(A)$ and so $x \in X \setminus \alpha Cl_\gamma(A)$.

Conversely, let $x \in X \setminus \alpha Cl_\gamma(A)$ implies that $x \notin \alpha Cl_\gamma(A)$, then there exists an $\alpha$-open sets $V$ containing $x$ such that $V^\gamma \cap A = \emptyset$ implies that $x \in V \subseteq V^\gamma \subseteq X \setminus A$. It follows that $x \in \alpha Int_\gamma(X \setminus A)$.

The proof of the following theorem is obvious and hence omitted.
**Theorem 3.4.** Let \((X, \tau)\) be a topological space and \(\gamma\) an operation on \(\alpha O(X)\). Then for \(A \subseteq X\), we have

1. \(\alpha Int_\gamma(A)\) is an \(\alpha\)-open set.
2. \(A\) is \(\alpha\)-\(\gamma\)-open if and only if \(\alpha Int_\gamma(A) = A\).

**Theorem 3.5.** If a subset \(A\) of \(X\) is \(\alpha\)-\(\gamma\)-open, then there exists an \(\alpha\)-open set \(O\) such that \(O \subseteq A \subseteq O^\gamma\).

**Proof:** If \(A\) is an \(\alpha\)-\(\gamma\)-open set, then \(\alpha Int_\gamma(A) = A\). By taking \(O = \alpha Int_\gamma(A)\), we obtain that \(O \subseteq A \subseteq O^\gamma\).

**Definition 3.6.** [3] A topological space \((X, \tau)\) is said to be \(\alpha\)-\(\gamma\)-regular if for each \(x \in X\) and for each \(\alpha\)-open set \(V\) in \(X\) containing \(x\), there exists an \(\alpha\)-open set \(U\) in \(X\) containing \(x\) such that \(U^\gamma \subseteq V\).

**Theorem 3.7.** Let \((X, \tau)\) be a topological space and \(\gamma\) an operation on \(\alpha O(X)\). Then the following statements are equivalent.

1. \(\alpha O(X, \tau) = \alpha O(X, \tau)_\gamma\).
2. \((X, \tau)\) is an \(\alpha\)-\(\gamma\)-regular space.
3. For every \(x \in X\) and every \(\alpha\)-open set \(U\) of \(X\) containing \(x\) there exists an \(\alpha\)-\(\gamma\)-open set \(W\) of \(X\) such that \(x \in W\) and \(W \subseteq U\).

**Proof:**

(1) \(\Rightarrow\) (2): Let \(x \in X\) and \(V\) be an \(\alpha\)-open set containing \(x\). Then by assumption, \(V\) is an \(\alpha\)-\(\gamma\)-open set. This implies that for each \(x \in V\), there exists an \(\alpha\)-open set \(U\) such that \(U^\gamma \subseteq V\). Therefore \((X, \tau)\) is an \(\alpha\)-\(\gamma\)-regular space.

(2) \(\Rightarrow\) (3): Let \(x \in X\) and \(U\) be an \(\alpha\)-open set containing \(x\). Then by (2), there is an \(\alpha\)-open set \(W\) containing \(x\) and \(W \subseteq U^\gamma \subseteq U\). Applying (2) to set \(W\) shows that \(W\) is \(\alpha\)-\(\gamma\)-open. Hence \(W\) is an \(\alpha\)-\(\gamma\)-open set containing \(x\) such that \(W \subseteq U\).

(3) \(\Rightarrow\) (1): By (3) and [[1], Proposition 2.13], it follows that every \(\alpha\)-open set is \(\alpha\)-\(\gamma\)-open, that is, \(\alpha O(X, \tau) \subseteq \alpha O(X, \tau)_\gamma\). Also from [[1], Remark 2.6], \(\alpha O(X, \tau)_\gamma \subseteq \alpha O(X, \tau)\). Hence we have the result.

**Remark 3.8.** For any topological space \((X, \tau)\), we have

1. If \(\alpha O(X)\) is indiscrete, then \(\alpha O(X)_\gamma\) is also indiscrete.
2. If \(\alpha O(X)_\gamma\) is discrete, then \(\alpha O(X)\) is discrete.

**Remark 3.9.** Let \((X, \tau)\) be a topological space and \(x \in X\). If \(\{x\} \in \alpha O(X)_\gamma\), then \(\{x\}^\gamma = \{x\}\).
Definition 3.10. Let \((X, \tau)\) be a topological space and \(x \in X\), then a subset \(N\) of \(X\) is said to be \(\alpha_\gamma\)-neighbourhood (resp., \(\alpha\)-neighbourhood [4]) of \(x\), if there exists an \(\alpha_\gamma\)-open (resp., \(\alpha\)-open) set \(U\) in \(X\) such that \(x \in U \subseteq N\).

Proposition 3.11. In a topological space \((X, \tau)\), a subset \(A\) of \(X\) is \(\alpha_\gamma\)-open if and only if it is an \(\alpha_\gamma\)-neighbourhood of each of its points.

Proof: Let \(A \subseteq X\) be an \(\alpha_\gamma\)-open set, since for every \(x \in A\), \(x \in A \subseteq A\) and \(A\) is \(\alpha_\gamma\)-open. This shows \(A\) is an \(\alpha_\gamma\)-neighbourhood of each of its points. Conversely, suppose that \(A\) is an \(\alpha_\gamma\)-neighbourhood of each of its points. Then for each \(x \in A\), there exists \(B_x \in \alpha O(X)_\gamma\) such that \(B_x \subseteq A\). Then \(A = \cup\{B_x : x \in A\}\). Since each \(B_x\) is \(\alpha_\gamma\)-open. It follows that \(A\) is \(\alpha_\gamma\)-open set. □

Proposition 3.12. If \(A \subseteq B\) in a topological space \((X, \tau)\) and \(A\) is an \(\alpha_\gamma\)-neighbourhood of a point \(x \in X\), then \(B\) is also \(\alpha_\gamma\)-neighbourhood of the same point \(x\).

Proof: Obvious. □

Remark 3.13. Since every \(\alpha_\gamma\)-open set is \(\alpha\)-open, then every \(\alpha_\gamma\)-neighbourhood of a point is an \(\alpha\)-neighbourhood of the same point.

Definition 3.14. Let \(A\) be a subset of a topological space \((X, \tau)\) and \(\gamma\) be an operation on \(\alpha O(X)_\gamma\). The union of all \(\alpha_\gamma\)-open sets contained in \(A\) is called the \(\alpha_\gamma\)-interior of \(A\) and denoted by \(\alpha_\gamma\text{Int}(A)\).

Theorem 3.15. Let \((X, \tau)\) be a topological space and \(\gamma\) be an operation on \(\alpha O(X)\). For any subsets \(A, B\) of \(X\) we have the following:

1. \(\alpha_\gamma\text{Int}(A)\) is an \(\alpha_\gamma\)-open set in \(X\).
2. \(A\) is \(\alpha_\gamma\)-open if and only if \(A = \alpha_\gamma\text{Int}(A)\).
3. \(\alpha_\gamma\text{Int}(\alpha_\gamma\text{Int}(A)) = \alpha_\gamma\text{Int}(A)\).
4. \(\alpha_\gamma\text{Int}(\phi) = \phi\) and \(\alpha_\gamma\text{Int}(X) = X\).
5. \(\alpha_\gamma\text{Int}(A) \subseteq A\).
6. If \(A \subseteq B\), then \(\alpha_\gamma\text{Int}(A) \subseteq \alpha_\gamma\text{Int}(B)\).
7. \(\alpha_\gamma\text{Int}(A \cup B) \supseteq \alpha_\gamma\text{Int}(A) \cup \alpha_\gamma\text{Int}(B)\).
8. \(\alpha_\gamma\text{Int}(A \cap B) \subseteq \alpha_\gamma\text{Int}(A) \cap \alpha_\gamma\text{Int}(B)\).

Proof: Straight forward. □
**Definition 3.16.** Let \((X, \tau)\) be a topological space with an operation \(\gamma\) on \(\alpha O(X)\). A point \(x \in X\) is said to be \(\alpha_\gamma\)-limit point of a set \(A\) if for each \(\alpha_\gamma\)-open set \(U\) containing \(x\), then \(U \cap (A \setminus \{x\}) \neq \emptyset\). The set of all \(\alpha_\gamma\)-limit points of \(A\) is called an \(\alpha_\gamma\)-derived set of \(A\) and is denoted by \(\alpha_\gamma D(A)\).

Some properties of \(\alpha_\gamma\)-derived sets are stated in the following proposition.

**Proposition 3.17.** Let \(A, B\) be any two subsets of a space \(X\), and \(\gamma\) be an operation on \(\alpha O(X)\). Then we have the following properties:

1. \(\alpha_\gamma D(\emptyset) = \emptyset\).
2. If \(x \in \alpha_\gamma D(A)\), then \(x \in \alpha_\gamma D(A \setminus \{x\})\).
3. \(\alpha_\gamma D(A \cup B) \supseteq \alpha_\gamma D(A) \cup \alpha_\gamma D(B)\).
4. \(\alpha_\gamma D(A \cap B) \subseteq \alpha_\gamma D(A) \cap \alpha_\gamma D(B)\).
5. \(\alpha_\gamma D(\alpha_\gamma D(A)) \setminus A \subseteq \alpha_\gamma D(A)\).
6. \(\alpha_\gamma D(A \cup \alpha_\gamma D(A)) \subseteq A \cup \alpha_\gamma D(A)\).

**Proof:** Obvious.

The proofs of Propositions 3.18 and 3.19 are clear.

**Proposition 3.18.** A subset \(A\) of a topological space \(X\) is \(\alpha_\gamma\)-closed if and only if it contains the set of its \(\alpha_\gamma\)-limit points.

**Proposition 3.19.** Let \(A\) be any subset of a topological space \((X, \tau)\) and \(\gamma\) be an operation on \(\alpha O(X)\), then \(\alpha_\gamma Cl(A) = A \cup \alpha_\gamma D(A)\).

**Proposition 3.20.** Let \(A\) be any subset of a topological space \((X, \tau)\) and \(\gamma\) be an operation on \(\alpha O(X)\). Then \(\alpha_\gamma Int(A) = A \setminus \alpha_\gamma D(X \setminus A)\).

**Proof:** If \(x \in A \setminus \alpha_\gamma D(X \setminus A)\), then \(x \notin \alpha_\gamma D(X \setminus A)\) and so there exists an \(\alpha_\gamma\)-open set \(U\) containing \(x\) such that \(U \cap (X \setminus A) = \emptyset\). Then \(x \in U \subseteq A\) and hence \(x \in \alpha_\gamma Int(A)\), that is \(A \setminus \alpha_\gamma D(X \setminus A) \subseteq \alpha_\gamma Int(A)\). On the other hand, if \(x \in \alpha_\gamma Int(A)\), then \(x \notin \alpha_\gamma D(X \setminus A)\) since \(\alpha_\gamma Int(A)\) is \(\alpha_\gamma\)-open and \(\alpha_\gamma Int(A) \cap (X \setminus A) = \emptyset\). Hence \(\alpha_\gamma Int(A) = A \setminus \alpha_\gamma D(X \setminus A)\).

**Proposition 3.21.** Let \(A\) be any subset of a topological space \((X, \tau)\) and \(\gamma\) be an operation on \(\alpha O(X)\). Then the following statements are true:

1. \(X \setminus \alpha_\gamma Int(A) = \alpha_\gamma Cl(X \setminus A)\).
2. \(X \setminus \alpha_\gamma Cl(A) = \alpha_\gamma Int(X \setminus A)\).
3. \( \alpha O \text{Int}(A) = X \setminus \alpha O \text{Cl}(X \setminus A) \).

4. \( \alpha O \text{Cl}(A) = X \setminus \alpha O \text{Int}(X \setminus A) \).

**Proof:** We only prove (1), the other parts can be proved similarly.

\( X \setminus \alpha O \text{Int}(A) = X \setminus (A \setminus \alpha O \text{D}(X \setminus A)) = (X \setminus A) \cup \alpha O \text{D}(X \setminus A) = \alpha O \text{Cl}(X \setminus A) \).

**Definition 3.22.** Let \( A \) be a subset of a space \( X \), then the \( \alpha O \)-boundary of \( A \) is defined as \( \alpha O \text{Cl}(A) \setminus \alpha O \text{Int}(A) \) and is denoted by \( \alpha O \text{Bd}(A) \).

Some properties of \( \alpha O \)-boundary sets are stated in the following proposition.

**Proposition 3.23.** Let \( A \) be any subset of a topological space \( (X, \tau) \) and \( \gamma \) be an operation on \( \alpha O(X) \). Then the following statements hold:

1. \( \alpha O \text{Cl}(A) = \alpha O \text{Int}(A) \cup \alpha O \text{Bd}(A) \).

2. \( \alpha O \text{Int}(A) \cap \alpha O \text{Bd}(A) = \phi \).

3. \( \alpha O \text{Bd}(A) = \alpha O \text{Cl}(A) \cap \alpha O \text{Cl}(X \setminus A) \).

4. \( \alpha O \text{Bd}(A) = \alpha O \text{Bd}(X \setminus A) \).

5. \( \alpha O \text{Bd}(A) \) is an \( \alpha O \)-closed set.

**Proof:** Obvious.

**Definition 3.24.** Let \( (X, \tau) \) be a topological space. A mapping \( \gamma : \alpha O(X) \to P(X) \) is said to be:

1. \( \alpha \)-monotone on \( \alpha O(X) \) if for all \( A, B \in \alpha O(X) \), \( A \subseteq B \) implies \( A^\gamma \subseteq B^\gamma \).

2. \( \alpha \)-idempotent on \( \alpha O(X) \) if \( A^{\gamma \gamma} = A^\gamma \) for all \( A \in \alpha O(X) \).

3. \( \alpha \)-additive on \( \alpha O(X) \) if \( (A \cup B)^\gamma = A^\gamma \cup B^\gamma \) for all \( A, B \in \alpha O(X) \).

If \( \bigcup_{i \in I} A_i^\gamma \subseteq (\bigcup_{i \in I} A_i)^\gamma \) for any collection \( \{A_i\}_{i \in I} \subseteq \alpha O(X) \), then \( \gamma \) is said to be \( \alpha \)-subadditive on \( \alpha O(X) \).

**Proposition 3.25.** Let \( (X, \tau) \) be a topological space and \( \gamma \) an operation on \( \alpha O(X) \). Then, \( \gamma \) is \( \alpha \)-monotone on \( \alpha O(X) \) if and only if \( \gamma \) is \( \alpha \)-subadditive on \( \alpha O(X) \).

**Proof:** Let \( \gamma \) be \( \alpha \)-monotone on \( \alpha O(X) \) and \( \{A_i\}_{i \in I} \subseteq \alpha O(X) \). Then for each \( i \in I \), \( A_i^\gamma \subseteq (\bigcup_{i \in I} A_i)^\gamma \) and thus \( \bigcup_{i \in I} A_i^\gamma \subseteq (\bigcup_{i \in I} A_i)^\gamma \).

Conversely, if \( \gamma \) is \( \alpha \)-subadditive on \( \alpha O(X) \) and \( A, B \in \alpha O(X) \) with \( A \subseteq B \), then \( A^\gamma \subseteq A^\gamma \cup B^\gamma \subseteq (A \cup B)^\gamma = B^\gamma \). Thus \( \gamma \) is \( \alpha \)-monotone on \( \alpha O(X) \).
**Remark 3.26.** The $\alpha$-regularity of operation $\gamma$ in [1] follows from the $\alpha$-monotonicity of operation $\gamma$.

**Remark 3.27.** It is easy to verify that if $\gamma$ is $\alpha$-additive on $\alpha O(X)$ then $\gamma$ is $\alpha$-monotone on $\alpha O(X)$.

The following result shows that the family of $\alpha_{\gamma}$-open sets may be a topology on $X$.

**Theorem 3.28.** Let $(X, \tau)$ be a topological space. If $\gamma$ is an $\alpha$-monotone operation on $\alpha O(X)$, then the family of $\alpha_{\gamma}$-open is a topology on $X$.

**Proof:** Clearly $\phi, X \in \alpha O(X)$ and by [[1], Theorem 2.11], the union of any family of $\alpha_{\gamma}$-open sets is $\alpha_{\gamma}$-open. To complete the proof it is enough to show that the finite intersection of $\alpha_{\gamma}$-open sets is $\alpha_{\gamma}$-open. Let $A$ and $B$ be two $\alpha_{\gamma}$-open sets and let $x \in A \cap B$, then $x \in A$ and $x \in B$, so there exist $\alpha$-open sets $U$ and $V$ such that $x \in U \subseteq U^\gamma \subseteq A$ and $x \in V \subseteq V^\gamma \subseteq B$, since $\gamma$ is an $\alpha$-monotone operation and $U \cap V$ is $\alpha$-open set such that $U \cap V \subseteq U$ and $U \cap V \subseteq V$, this implies that $(U \cap V)^\gamma \subseteq U^\gamma \cap V^\gamma \subseteq A \cap B$. Thus $A \cap B$ is $\alpha_{\gamma}$-open set. This completes the proof. ■

**Theorem 3.29.** Let $(X, \tau)$ be a topological space and $\gamma$ an operation on $\alpha O(X)$. If $(\bigcup_{i \in I} W_i)^\gamma \subseteq \bigcup_{i \in I} W_i^\gamma$ for any collection $\{W_i\}_{i \in I} \subseteq \alpha O(X)$, then for every $\alpha_{\gamma}$-open set $U$ we have $U^\gamma = U$.

**Proof:** Let $U$ be an $\alpha_{\gamma}$-open set. Then for every $x \in U$ there exists an $\alpha$-open set $W$ containing $x$ such that $W \subseteq W^\gamma \subseteq U$. Therefore $\bigcup_{x \in U} W \subseteq \bigcup_{x \in U} W^\gamma \subseteq U$, so $\bigcup_{x \in U} W \subseteq (\bigcup_{x \in U} W)^\gamma \subseteq U$. Therefore, $U \subseteq U^\gamma \subseteq U$ and so $U^\gamma = U$. ■

**Theorem 3.30.** For any operation $\gamma$ on $\alpha O(X)$, the map $\alpha_{\gamma} Cl : \alpha O(X) \rightarrow P(X)$ is an operation on $\alpha O(X)$ satisfying (1) and (2) of Definition 3.24, but not (3).

**Proof:** By [[1], Theorem 2.22 (5)], $\alpha_{\gamma} Cl$ is an $\alpha$-monotone on $\alpha O(X)$. We show that $\alpha_{\gamma} Cl$ is $\alpha$-idempotent on $\alpha O(X)$. Given $A \in \alpha O(X)$, it is obvious that $\alpha_{\gamma} Cl(A) \subseteq \alpha_{\gamma} Cl(\alpha_{\gamma} Cl(A))$. Let $x \in \alpha_{\gamma} Cl(\alpha_{\gamma} Cl(A))$ and $V$ be any $\alpha_{\gamma}$-open set containing $x$, then by [[1], Theorem 2.23], there is $z \in V \cap \alpha_{\gamma} Cl(A)$. Since $z \in \alpha_{\gamma} Cl(A)$ and $V$ is an $\alpha_{\gamma}$-open set containing $z$, we have that $V \cap A \neq \phi$, thus $x \in \alpha_{\gamma} Cl(A)$. Therefore $\alpha_{\gamma} Cl(\alpha_{\gamma} Cl(A)) \subseteq \alpha_{\gamma} Cl(A)$ and hence $\alpha_{\gamma} Cl(\alpha_{\gamma} Cl(A)) = \alpha_{\gamma} Cl(A)$. In general, $\alpha_{\gamma} Cl$ is not an $\alpha$-additive operation on $\alpha O(X)$ as shown in the following example. ■

**Example 3.31.** Let $X = \{a, b, c\}$ equipped with the discrete topology on $X$. We define an operation $\gamma$ on $\alpha O(X)$ by

$$A^\gamma = \begin{cases} A & \text{if } A = \{a, b\} \text{ or } \{a, c\} \\ X & \text{otherwise.} \end{cases}$$
Then, the $\alpha_\gamma$-open subsets of $(X, \tau)$ are $\phi, \{a, b\}, \{a, c\}$ and $X$. Now, if we let $A = \{b\}$ and $B = \{c\}$, then $\alpha_\gamma Cl(A) = A$, $\alpha_\gamma Cl(B) = B$ and $\alpha_\gamma Cl(A \cup B) = X$, where $A \cup B = \{b, c\}$, this implies that $\alpha_\gamma Cl(A \cup B) = X \neq \{b, c\} = \alpha_\gamma Cl(A) \cup \alpha_\gamma Cl(B)$.

Suppose that $A$ is a subset of a topological space $(X, \tau)$, then we have the following properties:

**Theorem 3.32.** Let $\gamma : \alpha O(X) \to P(X)$ be an operation on $\alpha O(X)$, $A$ and $B$ subsets of a topological space $(X, \tau)$. Then, we have the following properties:

1. $A \subseteq \alpha Cl_{\gamma}(A)$.
2. $\alpha Cl_{\gamma}(\phi) = \phi$ and $\alpha Cl_{\gamma}(X) = X$.
3. $A$ is $\alpha_\gamma$-closed (that is, $X \setminus A$ is $\alpha_\gamma$-open) in $(X, \tau)$ if and only if $\alpha Cl_{\gamma}(A) = A$ holds.
4. If $A \subseteq B$, then $\alpha Cl_{\gamma}(A) \subseteq \alpha Cl_{\gamma}(B)$.
5. $\alpha Cl_{\gamma}(A) \cup \alpha Cl_{\gamma}(B) \subseteq \alpha Cl_{\gamma}(A \cup B)$ holds.
6. If $\gamma$ is $\alpha$-regular, then $\alpha Cl_{\gamma}(A \cup B) = \alpha Cl_{\gamma}(A) \cup \alpha Cl_{\gamma}(B)$ holds.
7. $\alpha Cl_{\gamma}(A \cap B) \subseteq \alpha Cl_{\gamma}(A) \cap \alpha Cl_{\gamma}(B)$ holds.

**Proof:** (1), (2), (4): Obviously, by [[1], Definition 2.20], we have $A \subseteq \alpha Cl_{\gamma}(A)$.

(3): Suppose that $X \setminus A$ is $\alpha_\gamma$-open in $(X, \tau)$. We claim that $\alpha Cl_{\gamma}(A) \subseteq A$. Let $x \notin A$. There exists an $\alpha$-open set $U$ containing $x$ such that $U^\gamma \subseteq X \setminus A$, that is, $U^\gamma \cap A = \phi$. Hence, using [[1], Definition 2.20], we have $x \notin \alpha Cl_{\gamma}(A)$ and so $\alpha Cl_{\gamma}(A) \subseteq A$. By (1), it is proved that $A = \alpha Cl_{\gamma}(A)$.

Conversely, suppose that $A = \alpha Cl_{\gamma}(A)$. Let $x \in X \setminus A$. Since $x \notin \alpha Cl_{\gamma}(A)$, there exists an $\alpha$-open set $U$ containing $x$ such that $U^\gamma \cap A = \phi$, that is, $U^\gamma \subseteq X \setminus A$. Namely, $X \setminus A$ is $\alpha_\gamma$-open in $(X, \tau)$ and so $A$ is $\alpha_\gamma$-closed.

(5), (7): Followed from (4).

(6): Let $x \notin \alpha Cl_{\gamma}(A) \cup \alpha Cl_{\gamma}(B)$. Then, there exist two $\alpha$-open sets $U$ and $V$ containing $x$ such that $U^\gamma \cap A = \phi$ and $V^\gamma \cap B = \phi$. By [[1], Definition 2.14], there exists an $\alpha$-open set $W$ containing $x$ such that $W^\gamma \subseteq U^\gamma \cap V^\gamma$. Thus, we have that $W^\gamma \cap (A \cup B) \subseteq U^\gamma \cap V^\gamma \cap (A \cup B) \subseteq [U^\gamma \cap A] \cup [V^\gamma \cap B] = \phi$, that is, $W^\gamma \cap (A \cup B) = \phi$. Namely, we have $x \notin \alpha Cl_{\gamma}(A \cup B)$ and so $\alpha Cl_{\gamma}(A \cup B) \subseteq \alpha Cl_{\gamma}(A) \cup \alpha Cl_{\gamma}(B)$. We can obtain (6) by using (5).

**Theorem 3.33.** Let $(X, \tau)$ be a topological space and $\gamma$ an $\alpha$-monotone operation on $\alpha O(X)$. If $A$ is a subset of $X$, then

1. For every $\alpha_\gamma$-open set $G$ of $X$, we have that $\alpha Cl_{\gamma}(A) \cap G \subseteq \alpha Cl_{\gamma}(A \cap G)$. 


2. For every $\alpha_\gamma$-closed set $F$ of $X$, we have that $\alpha\text{Int}_\gamma(A \cup F) \subseteq \alpha\text{Int}_\gamma(A) \cup F$.

**Proof:** (1) Let $x \in \alpha\text{Cl}_\gamma(A) \cap G$ and let $U$ be an $\alpha$-open set containing $x$. Since $x \in \alpha\text{Cl}_\gamma(A)$, implies that $U \cap A \neq \emptyset$. Since $G$ is an $\alpha_\gamma$-open set, there exists an $\alpha$-open set $V$ of $X$ containing $x$ such that $V \cap A \subseteq G$. Thus $(U \cap V) \cap A \neq \emptyset$, this implies that $U \cap A \cap G \neq \emptyset$ by $\alpha$-monotone and hence $x \in \alpha\text{Cl}_\gamma(A \cap G)$. Therefore $\alpha\text{Cl}_\gamma(A) \cap G \subseteq \alpha\text{Cl}_\gamma(A \cap G)$.

(2) Follows from (1) and Theorem 3.3 (3).

The following example shows that the condition $\gamma$ is $\alpha$-monotone is necessary for the above theorem.

**Example 3.34.** Consider $X = \{a, b, c\}$ with the discrete topology on $X$. We define an operation $\gamma$ on $\alpha O(X)$ by

$$A^\gamma = \begin{cases} A & \text{if } A = \{a, b\} \text{ or } \{a, c\} \text{ or } \{b, c\} \\ X & \text{otherwise.} \end{cases}$$

Since $\gamma$ is not $\alpha$-monotone, so if we let $A = \{a, c\}$ and $G = \{b, c\}$, then $\alpha Cl_\gamma(A) = X$ and $\alpha Cl_\gamma(A) \cap G = \{b, c\}$, this implies that $\alpha Cl_\gamma(A) \cap G = \{b, c\} \not\subseteq \alpha Cl_\gamma(A \cap G) = \alpha Cl_\gamma(\{c\}) = \{c\}$.

**Remark 3.35.** Let $(X, \tau)$ be a topological space and $\gamma$ an $\alpha$-regular operation on $\alpha O(X)$. If $A$ is a subset of $X$, then

1. For every $\alpha_\gamma$-open set $G$ of $X$, we have $\alpha_\gamma Cl(A) \cap G \subseteq \alpha_\gamma Cl(A \cap G)$.
2. For every $\alpha_\gamma$-closed set $F$ of $X$, we have $\alpha_\gamma Int(A \cup F) \subseteq \alpha_\gamma Int(A) \cup F$.

**Theorem 3.36.** Let $(X, \tau)$ be a topological space, $N$ a subset of $X$ and $\gamma$ an $\alpha$-open operation on $\alpha O(X)$. Then, $\alpha Int_\gamma(\alpha Cl_\gamma(N)) = \phi$ if and only if any one of the following conditions hold:

1. $\alpha Cl_\gamma(X \setminus \alpha Cl_\gamma(N)) = X$.
2. $N \subseteq \alpha Cl_\gamma(X \setminus \alpha Cl_\gamma(N))$.

**Proof:** (1) $\alpha Int_\gamma(\alpha Cl_\gamma(N)) = \phi$ if and only if $X \setminus \alpha Cl_\gamma(X \setminus \alpha Cl_\gamma(N)) = \phi$ by Theorem 3.3 (3) if and only if $X \subseteq \alpha Cl_\gamma(X \setminus \alpha Cl_\gamma(N))$ if and only if $X = \alpha Cl_\gamma(X \setminus \alpha Cl_\gamma(N))$.

(2) $N \subseteq X = \alpha Cl_\gamma(X \setminus \alpha Cl_\gamma(N))$ by (1). Conversely, $N \subseteq \alpha Cl_\gamma(X \setminus \alpha Cl_\gamma(N))$, implies that $\alpha Cl_\gamma(N) \subseteq \alpha Cl_\gamma(X \setminus \alpha Cl_\gamma(N))$ by [[1], Theorem 2.26 (2)]. Since $X = \alpha Cl_\gamma(N) \cup (X \setminus \alpha Cl_\gamma(N))$, implies that $X \subseteq \alpha Cl_\gamma(X \setminus \alpha Cl_\gamma(N)) \cup (X \setminus \alpha Cl_\gamma(N)) = \alpha Cl_\gamma(X \setminus \alpha Cl_\gamma(N))$. Hence $X = \alpha Cl_\gamma(X \setminus \alpha Cl_\gamma(N))$.

**Theorem 3.37.** Let $(X, \tau)$ be a topological space, $N$ a subset of $X$ and $\gamma$ be both $\alpha$-regular and $\alpha$-open operation on $\alpha O(X)$. If $\alpha Int_\gamma(\alpha Cl_\gamma(N)) = \phi$ then every non empty $\alpha_\gamma$-open set $U$ contains a non empty $\alpha_\gamma$-open set $A$ disjoint with $N$. 
**Proof:** Given $\alpha Int_\gamma(\alpha Cl_\gamma(N)) = \phi$. This implies that $\alpha Cl_\gamma(N)$ does not contain any non empty $\alpha_\gamma$-open set. Hence for any non empty $\alpha_\gamma$-open set $U$, $U \cap (X \setminus \alpha Cl_\gamma(N)) \neq \phi$. Thus by [[1], Theorem 2.26 (2) and Proposition 2.18] $A = U \cap (X \setminus \alpha Cl_\gamma(N)) = U \setminus \alpha Cl_\gamma(N)$ is a non empty $\alpha_\gamma$-open set contained in $U$ and disjoint with $N$. 

**Definition 3.38.** Let $A$ be a subset of a topological space $(X, \tau)$ and $\gamma$ an operation on $\alpha O(X)$. The $\alpha_\gamma$-kernel of $A$, denoted by $\alpha_\gamma Ker(A)$ is defined to be the set $\alpha_\gamma Ker(A) = \cap \{V : A \subseteq V, V \in \alpha O(X, \tau)_\gamma\}$.

**Proposition 3.39.** Let $(X, \tau)$ be a topological space with an operation $\gamma$ on $\alpha O(X)$ and $x \in X$. Then $y \in \alpha_\gamma ker(\{x\})$ if and only if $x \in \alpha_\gamma Cl(\{y\})$.

**Proof:** Suppose that $y \notin \alpha_\gamma ker(\{x\})$. Then there exists an $\alpha_\gamma$-open set $V$ containing $x$ such that $y \notin V$. Therefore, we have $x \notin \alpha_\gamma Cl(\{y\})$. The proof of the converse case can be done similarly.

**Proposition 3.40.** Let $(X, \tau)$ be a topological space with an operation $\gamma$ on $\alpha O(X)$ and $A$ be a subset of $X$. Then, $\alpha_\gamma ker(A) = \{x \in X : \alpha_\gamma Cl(\{x\}) \cap A \neq \phi\}$.

**Proof:** Let $x \in \alpha_\gamma ker(A)$ and suppose $\alpha_\gamma Cl(\{x\}) \cap A = \phi$. Hence $x \notin X \setminus \alpha_\gamma Cl(\{x\})$ which is an $\alpha_\gamma$-open set containing $A$. This is impossible, since $x \in \alpha_\gamma ker(A)$. Consequently, $\alpha_\gamma Cl(\{x\}) \cap A \neq \phi$. Let $x \in X$ such that $\alpha_\gamma Cl(\{x\}) \cap A \neq \phi$ and suppose that $x \notin \alpha_\gamma ker(A)$. Then, there exists an $\alpha_\gamma$-open set $V$ containing $A$ and $x \notin V$. Let $y \in \alpha_\gamma Cl(\{x\}) \cap A$. Hence, $V$ is an $\alpha_\gamma$-open set containing $y$ which does not contain $x$. By this contradiction $x \in \alpha_\gamma ker(A)$ and the claim.

**Proposition 3.41.** The following properties hold for the subsets $A, B$ of a topological space $(X, \tau)$ with an operation $\gamma$ on $\alpha O(X)$:

1. $A \subseteq \alpha_\gamma ker(A)$.
2. $A \subseteq B$ implies that $\alpha_\gamma ker(A) \subseteq \alpha_\gamma ker(B)$.
3. If $A$ is $\alpha_\gamma$-open in $(X, \tau)$, then $A = \alpha_\gamma ker(A)$.
4. $\alpha_\gamma ker(\alpha_\gamma ker(A)) = \alpha_\gamma ker(A)$.

**Proof:** (1), (2) and (3): Immediate consequences of $\alpha_\gamma ker(A) = \cap \{U \in \alpha O(X)_\gamma : A \subseteq U\}$.

(4): First observe that by (1) and (2), we have $\alpha_\gamma ker(A) \subseteq \alpha_\gamma ker(\alpha_\gamma ker(A))$. If $x \notin \alpha_\gamma ker(A)$, then there exists $U \in \alpha O(X, \tau)_\gamma$ such that $A \subseteq U$ and $x \notin U$. Hence $\alpha_\gamma ker(A) \subseteq U$, and so we have $x \notin \alpha_\gamma ker(\alpha_\gamma ker(A))$. Thus $\alpha_\gamma ker(\alpha_\gamma ker(A)) = \alpha_\gamma ker(A)$.
Proposition 3.42. The following statements are equivalent for any points \( x \) and \( y \) in a topological space \( (X, \tau) \) with an operation \( \gamma \) on \( \alpha O(X) \):

1. \( \alpha_\gamma \ker \{ \{ x \} \} \neq \alpha_\gamma \ker \{ \{ y \} \} \).
2. \( \alpha_\gamma \Cl \{ \{ x \} \} \neq \alpha_\gamma \Cl \{ \{ y \} \} \).

Proof: (1) \( \Rightarrow \) (2): Suppose that \( \alpha_\gamma \ker \{ \{ x \} \} \neq \alpha_\gamma \ker \{ \{ y \} \} \), then there exists a point \( z \) in \( X \) such that \( z \in \alpha_\gamma \ker \{ \{ x \} \} \) and \( z \notin \alpha_\gamma \ker \{ \{ y \} \} \). From \( z \in \alpha_\gamma \ker \{ \{ x \} \} \) it follows that \( \{ x \} \cap \alpha_\gamma \Cl \{ \{ z \} \} \neq \emptyset \) which implies \( x \in \alpha_\gamma \Cl \{ \{ z \} \} \). By \( z \notin \alpha_\gamma \ker \{ \{ y \} \} \), we have \( \{ y \} \cap \alpha_\gamma \Cl \{ \{ z \} \} = \emptyset \). Since \( x \in \alpha_\gamma \Cl \{ \{ z \} \} \), \( \alpha_\gamma \Cl \{ \{ x \} \} \subseteq \alpha_\gamma \Cl \{ \{ z \} \} \) and \( \{ y \} \cap \alpha_\gamma \Cl \{ \{ x \} \} = \emptyset \). Therefore, it follows that \( \alpha_\gamma \Cl \{ \{ x \} \} \neq \alpha_\gamma \Cl \{ \{ y \} \} \). Now \( \alpha_\gamma \ker \{ \{ x \} \} \neq \alpha_\gamma \ker \{ \{ y \} \} \) implies that \( \alpha_\gamma \Cl \{ \{ x \} \} \neq \alpha_\gamma \Cl \{ \{ y \} \} \).

(2) \( \Rightarrow \) (1): Suppose that \( \alpha_\gamma \Cl \{ \{ x \} \} \neq \alpha_\gamma \Cl \{ \{ y \} \} \). Then there exists a point \( z \) in \( X \) such that \( z \in \alpha_\gamma \Cl \{ \{ x \} \} \) and \( z \notin \alpha_\gamma \Cl \{ \{ y \} \} \). Then, there exists an \( \alpha_\gamma \)-open set containing \( z \) and therefore \( x \) but not \( y \), namely, \( y \notin \alpha_\gamma \ker \{ \{ x \} \} \) and thus \( \alpha_\gamma \ker \{ \{ x \} \} \neq \alpha_\gamma \ker \{ \{ y \} \} \).

\[
\square
\]

Proposition 3.43. Let \( (X, \tau) \) be a topological space and \( \gamma \) be an operation on \( \alpha O(X) \). Then, \( \cap \{ \alpha_\gamma \Cl \{ \{ x \} \} : x \in X \} = \emptyset \) if and only if \( \alpha_\gamma \ker \{ \{ x \} \} \neq X \) for every \( x \in X \).

Proof: Necessity, suppose that \( \cap \{ \alpha_\gamma \Cl \{ \{ x \} \} : x \in X \} = \emptyset \). Assume that there is a point \( y \) in \( X \) such that \( \alpha_\gamma \ker \{ \{ y \} \} = X \). Let \( x \) be any point of \( X \). Then \( x \in V \) for every \( \alpha_\gamma \)-open set \( V \) containing \( y \) and hence \( y \in \alpha_\gamma \Cl \{ \{ x \} \} \) for any \( x \in X \). This implies that \( y \in \cap \{ \alpha_\gamma \Cl \{ \{ x \} \} : x \in X \} \). But this is a contradiction.

Sufficiency, assume that \( \alpha_\gamma \ker \{ \{ x \} \} \neq X \) for every \( x \in X \). If there exists a point \( y \) in \( X \) such that \( y \in \cap \{ \alpha_\gamma \Cl \{ \{ x \} \} : x \in X \} \), then every \( \alpha_\gamma \)-open set containing \( y \) must contain every point of \( X \). This implies that the space \( X \) is the unique \( \alpha_\gamma \)-open set containing \( y \). Hence \( \alpha_\gamma \ker \{ \{ y \} \} = X \) which is a contradiction. Therefore, \( \cap \{ \alpha_\gamma \Cl \{ \{ x \} \} : x \in X \} = \emptyset \).

\[
\square
\]

Definition 3.44. [1] A subset \( A \) of a topological space \( (X, \tau) \) is called an \( \alpha_\gamma \)-D-set if there are two \( U, V \in \alpha O(X, \tau) \) such that \( U \neq X \) and \( A = U \setminus V \).

Proposition 3.45. If a singleton \( \{ x \} \) is an \( \alpha_\gamma \)-D-set of \( (X, \tau) \), then \( \alpha_\gamma \ker \{ \{ x \} \} \neq X \).

Proof: Since \( \{ x \} \) is an \( \alpha_\gamma \)-D-set of \( (X, \tau) \), then there exist two subsets \( U_1, U_2 \in \alpha O(X, \tau) \) such that \( \{ x \} = U_1 \setminus U_2 \), \( \{ x \} \subseteq U_1 \) and \( U_1 \neq X \). Thus, we have that \( \alpha_\gamma \ker \{ \{ x \} \} \subseteq U_1 \neq X \) and so \( \alpha_\gamma \ker \{ \{ x \} \} \neq X \).

\[
\square
\]

Definition 3.46. A subset \( A \) of a topological space \( (X, \tau) \) is said to be \( \alpha_\gamma \)-generalized closed (\( \alpha_\gamma \)-g-closed) set if \( \alpha \Cl \gamma(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is an \( \alpha_\gamma \)-open set of \( (X, \tau) \).
Definition 3.47. [1] A subset $A$ of the space $(X, \tau)$ is said to be $\alpha_\gamma$-generalized closed (briefly, $\alpha_\gamma$-g.closed) if $\alpha_\gamma \text{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is an $\alpha_\gamma$-open set in $(X, \tau)$. The complement of an $\alpha_\gamma$-g.closed set is called an $\alpha_\gamma$-g.open set.

Definition 3.48. [5] A subset $A$ of a topological space $(X, \tau)$ is called an $(\alpha, \alpha)$-generalized closed set (briefly, $(\alpha, \alpha)$-g-closed) if $\alpha \text{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\alpha$-open.

Theorem 3.49. Let $A$ be a subset of a topological space $(X, \tau)$ and $\gamma$ an operation on $\alpha O(X)$. Then, the following statements are true:

1. If $A$ is $\alpha_\gamma$-g.closed in $X$, then $A$ is $(\alpha, \alpha)$-g-closed.

2. If $A$ is $\alpha_\gamma$-g.closed in $X$, then $A$ is $\alpha_\gamma$-g.closed.

Proof: Follows from Theorem 2.24 [1].

Remark 3.50. By Theorem 3.49, every $\alpha_\gamma$-g.closed is $(\alpha, \alpha)$-g-closed.

Remark 3.51. It is clear that every $\alpha_\gamma$-closed set is $\alpha_\gamma$-g.closed, but the converse is not true in general as shown in the following example.

Example 3.52. Let $X = \{1, 2, 3\}$ with the topology $\tau = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, X\}$. We define an operation $\gamma$ on $\alpha O(X)$ by

$$A^\gamma = \begin{cases} A & \text{if } A = \{2\} \text{ or } \{1, 3\} \\ X & \text{otherwise.} \end{cases}$$

Now, if we let $A = \{1\}$, since the only $\alpha_\gamma$-open supersets of $A$ are $\{1, 3\}$ and $X$, then $A$ is $\alpha_\gamma$-g.closed. But it is easy to see that $A$ is not $\alpha_\gamma$-closed.

Theorem 3.53. If $A$ is $\alpha_\gamma$-open and $\alpha_\gamma$-g.closed, then $A$ is $\alpha_\gamma$-closed.

Proof: Suppose that $A$ is $\alpha_\gamma$-open and $\alpha_\gamma$-g.closed. Since $A \subseteq A$, we have $\alpha Cl_\gamma(A) \subseteq A$, also $A \subseteq \alpha Cl_\gamma(A)$, therefore $\alpha Cl_\gamma(A) = A$. That is, $A$ is $\alpha_\gamma$-closed.

Theorem 3.54. Let $\gamma : \alpha O(X) \to P(X)$ be an operation on $\alpha O(X)$ and $A$ a subset of a topological space $(X, \tau)$. Then the following statements are equivalent:

1. $A$ is $\alpha_\gamma$-g.closed in $(X, \tau)$.

2. $\alpha_\gamma Cl(\{x\}) \cap A \neq \emptyset$ for every $x \in \alpha Cl_\gamma(A)$.

3. $\alpha Cl_\gamma(A) \subseteq \alpha_\gamma Ker(A)$ holds.
Proof: (1) ⇒ (2): Let $A$ be an $α$-γ-g-closed set of $(X, τ)$. Suppose that there exists a point $x ∈ αClγ(A)$ such that $αγCl(\{x\}) ∩ A = \phi$. By [[1], Theorem 2.22 (2)], $αγCl(\{x\})$ is $αγ$-closed. Put $U = X \setminus αγCl(\{x\})$. Then, we have $A ⊆ U$, $x ∉ U$ and $U$ is an $αγ$-open set of $(X, τ)$. Since $A$ is an $αγ$-g-closed set, $αClγ(A) ⊆ U$. Thus, we have $x ∉ αClγ(A)$. This is a contradiction.

(2) ⇒ (3): Follows from Proposition 3.40.

(3) ⇒ (1): Let $U$ be any $αγ$-open set such that $A ⊆ U$. Let $x$ be a point such that $x ∈ αClγ(A)$. By (3), $x ∈ αγKer(A)$ holds. Namely, we have that $x ∈ U$, because $A ⊆ U$ and $U ∈ αO(X, τ)γ$.

Theorem 3.55. Let $(X, τ)$ be a topological space and $γ$ an operation on $αO(X)$. If a subset $A$ of $X$ is $αγ$-g-closed, then $αClγ(A) \setminus A$ does not contain any non-empty $αγ$-g-closed set.

Proof: Suppose that there exists a non-empty $αγ$-g-closed set $F$ such that $F ⊆ αClγ(A) \setminus A$. Then we have $A ⊆ X \setminus F$ and $X \setminus F$ is $αγ$-open. It follows from the assumption that $αClγ(A) ⊆ X \setminus F$ and so $F ⊆ (αClγ(A) \setminus A) \cap (X \setminus αClγ(A))$. Therefore, we have $F = \phi$.

Remark 3.56. In the above theorem, if $γ$ is an $α$-open operation, then the converse of the above theorem is true.

Proof: Let $U$ be an $αγ$-open set such that $A ⊆ U$. Since $γ$ is an $α$-open operation, it follows from [[1], Theorem 2.26] that $αClγ(A)$ is $αγ$-g-closed in $(X, τ)$. Thus by [[1], Definition 2.2 and Theorem 2.11], we have $αClγ(A) ∩ (X \setminus U) = F$ is $αγ$-g-closed in $(X, τ)$. Since $X \setminus U ⊆ X \setminus A$, $F ⊆ αClγ(A) \setminus A$. Using the assumptions of the converse of Theorem 3.55 above, $F = \phi$ and hence $αClγ(A) ⊆ U$.

Theorem 3.57. Let $(X, τ)$ be a topological space and $γ$ an operation on $αO(X)$. Then for each $x ∈ X$, $\{x\}$ is $αγ$-g-closed or $X \setminus \{x\}$ is $αγ$-g-closed in $(X, τ)$.

Proof: Suppose that $\{x\}$ is not $αγ$-g-closed, then $X \setminus \{x\}$ is not $αγ$-g-open. Let $U$ be any $αγ$-g-open set such that $X \setminus \{x\} ⊆ U$. Then $U = X$. Hence, $αClγ(X \setminus \{x\}) ⊆ U$. Therefore, $X \setminus \{x\}$ is an $αγ$-g-closed set.

Proposition 3.58. A subset $A$ of $X$ is $αγ$-g-open if and only if $F ⊆ αγInt(A)$ whenever $F ⊆ A$ and $F$ is $αγ$-g-closed in $X$.

Proof: Let $A$ be $αγ$-g-open and $F ⊆ A$ where $F$ is $αγ$-g-closed. Since $X \setminus A$ is $αγ$-g-closed and $X \setminus F$ is an $αγ$-g-open set containing $X \setminus A$ implies $αγCl(X \setminus A) ⊆ X \setminus F$. By Proposition 3.21 (1), $X \setminus αγInt(A) ⊆ X \setminus F$. That is $F ⊆ αγInt(A)$.

Conversely, suppose that $F$ is $αγ$-g-closed and $F ⊆ A$ implies $F ⊆ αγInt(A)$. Let $X \setminus A ⊆ U$ where $U$ is $αγ$-open. Then $X \setminus U ⊆ A$ where $X \setminus U$ is $αγ$-g-closed. By hypothesis $X \setminus U ⊆ αγInt(A)$.

That is $X \setminus αγInt(A) ⊆ U$. By Proposition 3.21 (1), $αγCl(X \setminus A) ⊆ U$. This implies $X \setminus A$ is $αγ$-g-closed and $A$ is $αγ$-g-open.
References


