sb*- Separation axioms

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Abstract

The aim of this paper is to introduce some new type of separation axioms and study some of their basic properties. Some implications between $T_0$, $T_1$ and $T_2$ axioms are also obtained.

Keywords: sb*-open sets, sb*-closed sets, sb*-T$_0$, sb*-T$_1$, sb*-T$_2$.

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1 Introduction

Andrijevic[1] introduced a new class of generalized open sets called b-open sets in topological spaces. This type of sets was discussed by [5] under the name of $\gamma$ - open sets. Several research papers [2,3,4,13,15] with advance results in different aspects came into existence. Further, Caldas and Jafari [4], introduced and studied b-T$_0$, b-T$_1$, b- T$_2$, b-D$_0$, b-D$_1$ and b-D$_2$ via b-open sets. After to that Keskin and Noiri [7], introduced the notion of b-T$_{1/2}$. Recently, the authors[16,17,18] introduced and studied about the sb*- closed sets, sb*-open map, sb*-continuous map, sb*- irresolute and Homeomorphisms in topological spaces. In the present paper, sb*-seperation axioms are introduced via sb*-open sets and some of its basic properties are discussed.

2 Preliminaries

Throughout this paper, X and Y denote the topological spaces $(X, \tau)$ and $(Y,\sigma)$ respectively and on which no separation axioms are assumed unless otherwise explicitly stated. Let A be a subset of the space X. The interior and closure of a set A in X are denoted by int(A) and cl(A) respectively. The complement of A is denoted by (X-A) or $A^c$. In this section, let us recall some definitions and results which are useful in the sequel.

Definition 2.1. [1] A subset A of a topological space $(X, \tau)$ is called b-open set if $A \subseteq (cl(int(A)) \cup int(cl(A)))$. The complement of a b-open set is said to be b-closed. The family of all b-open subsets of a space X is denoted by BO(X).
Definition 2.2. A subset A of a space X is called
(1) semi-open if \( A \subseteq (\text{cl}(\text{int}(A))) \);
(2) \( \alpha \)-open if \( A \subseteq \text{int}(\text{cl}(\text{int}(A))) \).

The complement of a semi-open (resp. \( \alpha \)-open) set is called semiclosed [12] (resp. \( \alpha \)-closed[19]).

Definition 2.3. [16] A subset A of a topological space \((X, \tau)\) is called a \(\text{sb}^*\)-closed set (briefly \(\text{sb}^*\)-closed) if \(\text{cl}(\text{int}(A)) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is b-open in X. The complement of \(\text{sb}^*\)-closed set is called \(\text{sb}^*\)-open. The family of all \(\text{sb}^*\)-open sets of a space X is denoted by \(\text{sb}^*\text{O}(X)\).

Definition 2.4. [4] A space X is said to be:
(1) \(b-T_0\) if for each pair of distinct points x and y in X, there exists a b-open set A containing x but not y or a b-open set B containing y but not x.
(2) \(b-T_1\) if for each pair \(x; y\) in X, \(x \neq y\), there exists a b-open set G containing x but not y and a b-open set B containing y but not x.

Definition 2.5. [15] A space X is said to be \(\text{b-T}_2\) if for any pair of distinct points x and y in X, there exist \(U \in \text{BO}(X, x)\) and \(V \in \text{BO}(X, y)\) such that \(U \cap V = \emptyset\).

Definition 2.6. A space X is said to be:
(1) \(\alpha-T_0\) if for each pair of distinct points in X, there is an \(\alpha\) - open set containing one of the points but not the other[9].
(2) \(\alpha-T_1\) if for each pair of distinct points x and y of X, there exists \(\alpha\)-open sets U and V containing x and y respectively such that \(y \notin U\) and \(x \notin V\).[9]
(3) \(\alpha-T_2\) if for each pair of distinct points x and y of X, there exist disjoint \(\alpha\)-open sets U and V containing x and y respectively[11].

Definition 2.7. [10] (i) Let X be a topological space. For each \(x \neq y \in X\), there exists a set U, such that \(x \in U, y \notin U\), and there exists a set V such that \(y \in V, x \notin V\), then X is called w-\(T_1\) space, if U is open and V is w-open sets in X.
(ii) Let X be a topological space. And for each \(x \neq y \in X\), there exist two disjoint sets U and V with \(x \in U\) and \(y \in V\), then X is called w-\(T_2\) space if U is open and V is w-open sets in X.

Definition 2.8. [10] A topological space X is (1) semi \(T_0\) if to each pair of distinct points \(x, y\) of X, there exists a semi open set A containing x but not y or a semi open set B containing y but not x.
(2) semi \(T_1\) if to each pair of distinct points \(x, y\) of X, there exists a semi open set A containing x but not y and a semi open set B containing y but not x.
(3) semi \(T_2\) if to each pair of distinct points \(x, y\) of X, there exist disjoint semi open sets A and B in X s.t. \(x \in A, y \in B\).
**Definition 2.9.** [20] A topological space $X$ is called a $T_0$ space if and only if it satisfies the following axiom of Kolmogorov. $(T_0)$ If $x$ and $y$ are distinct points of $X$, then there exists an open set which contains one of them but not the other.

**Definition 2.10.** [20] A topological space $X$ is a $T_1$-space if and only if it satisfies the following separation axiom of Frechet. $(T_1)$ If $x$ and $y$ are two distinct points of $X$, then there exists two open sets, one containing $x$ but not $y$ and the other containing $y$ but not $x$.

**Definition 2.11.** [20] A topological space $X$ is said to be a $T_2$ - space or hausdorff space if and only if for every pair of distinct points $x,y$ of $X$, there exists two disjoint open sets one containing $x$ and the other containing $y$.

**Theorem 2.12.** [16] (i)Every open set is $sb^*$-open.
(ii)Every $\alpha$ open set is $sb^*$-open.
(iii)Every $w$-open set is $sb^*$-open.
(iv)Every $sb^*$-open set is $b$ - open.

**Definition 2.13.** Let $A$ be a subset of a space $X$. Then the $sb^*$-closure of $A$ is defined as the intersection of all $sb^*$-closed sets containing $A$. i.e., $sb^*$-cl($A$) = $\cap \{F: F$ is $sb^*$-closed, $A \subseteq F\}$.

**Definition 2.14.** [17] Let $X$ and $Y$ be topological spaces. A map $f: X \to Y$ is called strongly $b^*$ - continuous ($sb^*$ - continuous) if the inverse image of every open set in $Y$ is $sb^*$ - open in $X$.

**Definition 2.15.** [17] Let $X$ and $Y$ be a topological spaces. A map $f: X \to Y$ is called strongly $b^*$ - closed ($sb^*$ - closed) map if the image of every closed set in $X$ is $sb^*$ - closed in $Y$.

**Definition 2.16.** [18] Let $X$ and $Y$ be topological spaces. A map $f: (X,\tau) \to (Y,\sigma)$ is said to be $sb^*$ - Irresolute if the inverse image of every $sb^*$ - closed set in $Y$ is $sb^*$ - closed set in $X$.

**Definition 2.17.** Let $X$ be a topological space. A subset $A \subseteq X$ is called a $sb^*$ - neighbourhood (Briefly $sb^*$ - nbd) of a point $x \in X$ if there exists a $sb^*$ - open set $G$ such that $x \in G \subseteq A$.

### 3  $sb^*$ - $T_0$ Spaces

In this section, we define $sb^*$ - $T_0$ space and study some of their properties.

**Definition 3.1.** A topological space $X$ is said to be $sb^*$-$T_0$ if for every pair of distinct points $x$ and $y$ of $X$, there exists a $sb^*$-open set $G$ such that $x \in G$ and $y \notin G$ or $y \in G$ and $x \notin G$.

**Theorem 3.2.** Every $\alpha$-$T_0$ space is $sb^*$-$T_0$.

**Proof:** Let $X$ be a $\alpha$-$T_0$ space. Let $x$ and $y$ be any two distinct points in $X$. Since $X$ is $\alpha$-$T_0$, there exists a $\alpha$ open set $U$ such that $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$. By Theorem 2.11(ii), $U$ is a $sb^*$-open set such that $x \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$. Thus $X$ is $sb^*$-$T_0$. $\blacksquare$
Theorem 3.3. Every topological space $X$ is sb*-T$_0$.

Proof: Since every topological space is $\alpha$-T$_0$ and by the above Theorem every topological space $X$ is sb*-T$_0$.

Theorem 3.4. A space $X$ is sb*-T$_0$ space if and only if sb*-closures of distinct points are distinct.

Proof: Necessity: Let $x, y \in X$ with $x \neq y$ and $X$ be a sb*-T$_0$ space. Since $X$ is sb*-T$_0$, by Definition 3.1, there exists an sb*-open set $G$ such that $x \in G$ but $y \notin G$. Also $x \notin X-G$ and $y \in X-G$, where $X-G$ is a sb*-closed set in $X$. Since sb*cl({$y$}) is the smallest sb*-closed set containing $y$, sb*cl({$y$}) is the smallest sb*-closed set containing $y$, Hence $y \in$ sb*cl({$y$}) but $x \notin$ sb*cl({$y$}) as $x \notin X-G$. Consequently sb*cl({$x$}) $\neq$ sb*cl({$y$}).

Sufficiency: Suppose that for any pair of distinct points $x, y \in X$, sb*cl({$x$}) $\neq$ sb*cl({$y$}). Then there exists at least one point $z \in X$ such that $z \in$ sb*cl({$x$}) but $z \notin$ sb*cl({$y$}). Suppose we claim that $x \notin$ sb*cl({$y$}). For, if $x \in$ sb*cl({$y$}), then sb*cl({$x$}) $\subseteq$ sb*cl({$y$}). So $z \in$ sb*cl({$y$}), which is a contradiction. Hence $x \notin$ sb*cl({$y$}). Which implies that $x \in$ X-sb*cl({$y$}) is a sb*-open set in $X$ containing $x$ but not $y$. Hence $X$ is a sb*-T$_0$ space.

Theorem 3.5. Every subspace of a sb*-T$_0$ space is sb*-T$_0$.

Proof: Let $(Y, \tau^*)$ be a subspace of a space $X$ where $\tau^*$ is the relative topology of $\tau$ on $Y$. Let $y_1, y_2$ be two distinct points of $Y$. As $Y \subseteq X$, $y_1$ and $y_2$ are distinct points of $X$ and there exists a sb*-open set $G$ such that $y_1 \in G$ but $y_2 \notin G$ since $X$ is sb*-T$_0$. Then $G \cap Y$ is a sb*-open set in $(Y, \tau^*)$ which contains $y_1$ but does not contain $y_2$. Hence $(Y, \tau^*)$ is a sb*-T$_0$ space.

4 sb*- T$_1$ Spaces

Definition 4.1. A space $X$ is said to be sb*-T$_1$ if for every pair of distinct points $x$ and $y$ in $X$, there exist sb*- open sets $U$ and $V$ such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$.

Proposition 4.2. (i) Every w-T$_1$ space is sb*- T$_1$.

(ii) Every sb*-T$_1$ space is b-T$_1$.

Proof: (i) Suppose $X$ is a w- T$_1$ space. Let $x$ and $y$ be two distinct points in $X$. Since $X$ is w-T$_1$, there exist w- open sets $U$ and $V$ such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$. By Theorem 2.11(iii), $U$ and $V$ are sb*- open sets such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$. Hence $X$ is sb*-T$_1$.

(ii) Suppose $X$ is a sb*-T$_1$ space. Let $x$ and $y$ be two distinct points in $X$. Since $X$ is sb*-T$_1$, there exist sb*-open sets $U$ and $V$ such that $x \in U$ but $y \notin U$ and $y \in V$ and $x \notin V$. By Theorem 2.11(iv) , $U$ and $V$ are b-open sets such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$. Thus $X$ is b-T$_1$. 

**Remark 4.3.** The converse of the above proposition is not true as shown in the following examples.

**Example 4.4.** Consider the space \((X, \tau)\), where \(X = \{a, b, c\}\) and \(\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}\). Clearly \((X, \tau)\) is sb*-T\(_1\) but not w-T\(_1\). This shows that sb*-T\(_1\) does not imply w-T\(_1\).

**Example 4.5.** Consider the space \((X, \tau)\) where \(X = \{a, b, c, d\}\) and \(\tau = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}, X\}\). Then \((X, \tau)\) is b-T\(_1\) but not sb*-T\(_1\). This shows that b-T\(_1\) does not imply sb*-T\(_1\).

**Remark 4.6.** The concepts of sb*-T\(_1\) and semi-T\(_1\) are independent as shown in the following examples.

**Example 4.7.** Consider the space \((X, \tau)\), where \(X = \{a, b, c, d\}\) and \(\tau = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}, X\}\). Clearly \((X, \tau)\) is semi-T\(_1\) but not sb*-T\(_1\). This shows that semi-T\(_1\) does not imply sb*-T\(_1\).

**Example 4.8.** Consider the space \((X, \tau)\), where \(X = \{a, b, c, d\}\) and \(\tau = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}, X\}\). Then \((X, \tau)\) is sb*-T\(_1\) but not semi-T\(_1\). This shows that sb*-T\(_1\) does not imply semi-T\(_1\).

**Theorem 4.9.** Let \(f: X \rightarrow Y\) be a sb*-irresolute, injective map. If \(Y\) is sb*-T\(_1\), then \(X\) is sb*-T\(_1\).

**Proof:** Assume that \(Y\) is sb*-T\(_1\). Let \(x, y \in Y\) be such that \(x \neq y\). Then there exists a pair of sb*-open sets \(U, V\) in \(Y\) such that \(f(x) \in U, f(y) \in V\) and \(f(x) \notin V, f(y) \notin U\). Then \(x \in f^{-1}(U), y \notin f^{-1}(U)\) and \(y \in f^{-1}(V), x \notin f^{-1}(V)\). Since \(f\) is sb*-irresolute, \(X\) is sb*-T\(_1\).

**Theorem 4.10.** A space \((X, \tau)\) is sb*-T\(_1\) if and only if for every \(x \in X\), \(sb^* cl\{x\} = \{x\}\).

**Proof:** Let \((X, \tau)\) be sb*-T\(_1\) and \(x \in X\). Then for each \(y \neq x\), there exists a sb*-open set \(G\) such that \(x \in G\) but \(y \notin G\). This implies that \(y \notin sb^* cl\{x\}\), for every \(y \in X\) and \(y \neq x\). Thus \(\{x\} = sb^* cl\{x\}\).

Conversely, suppose \(sb^* cl\{x\} = \{x\}\) for every \(x \in X\). Let \(x, y\) be two distinct points in \(X\). Then \(x \notin \{y\} = sb^* cl\{y\}\) implies there exists a sb*-closed set \(B_1\) such that \(y \in B_1\) and \(x \notin B_1\). This implies \(B_1\) is a sb*-open set such that \(x \in B_1 \neq y \notin B_1\).

Also \(y \notin \{x\} = sb^* cl\{x\} \Rightarrow \) there exists a sb*-closed set \(B_2\) such that \(x \in B_2\) and \(y \notin B_2\). Which implies \(B_2\) is a sb*-open set such that \(y \in B_2 \neq x \notin B_2\). By Definition 4.1, \((X, \tau)\) is sb*-T\(_1\).

**Theorem 4.11.** Let \(f: X \rightarrow Y\) be bijective.

(i) If \(f\) is sb* continuous and \((Y, \tau_2)\) is T\(_1\), then \((X, \tau_1)\) is sb*-T\(_1\).

(ii) If \(f\) is sb*-open and \((X, \tau)\) is sb*-T\(_1\) then \((Y, \tau_2)\) is sb*-T\(_1\).
Proof: Let $f: (X, \tau_1) \to (X, \tau_2)$ be bijective.

(i) Suppose $f: (X, \tau_1) \to (Y, \tau_2)$ is sb*-continuous and $(Y, \tau_2)$ is $T_1$. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Since $f$ is bijective, $y_1 = f(x_1) \neq f(x_2) = y_2$ for some $y_1, y_2 \in Y$. Since $(Y, \tau_2)$ is $T_1$, there exist open sets $G$ and $H$ such that $y_1 \in G$ but $y_2 \notin G$ and $y_2 \in H$ but $y_1 \notin H$. Since $f$ is bijective, $x_1 = f^{-1}(y_1) \in f^{-1}(G)$ but $x_2 = f^{-1}(y_2) \notin f^{-1}(G)$ and $x_2 = f^{-1}(y_2) \in f^{-1}(H)$ but $x_1 = f^{-1}(y_1) \notin f^{-1}(H)$. Since $f$ is sb*-continuous, $f^{-1}(G)$ and $f^{-1}(H)$ are sb*-open sets in $(X, \tau_1)$. It follows that $(X, \tau_1)$ is sb*- $T_1$. This proves (i).

(ii) Suppose $f$ is sb*-open and $(X, \tau_1)$ is sb*- $T_1$. Let $y_1 \neq y_2 \in Y$. Since $f$ is bijective, there exist $x_1, x_2 \in X$, such that $f(x_1) = y_1$ and $f(x_2) = y_2$ with $x_1 \neq x_2$. Since $(X, \tau_1)$ is sb*- $T_1$, there exist sb*-open sets $G$ and $H$ in $X$ such that $x_1 \in G$ but $x_2 \notin G$ and $x_2 \in H$ but $x_1 \notin H$. Since $f$ is sb*-open, $f(G)$ and $f(H)$ are sb*-open in $Y$ such that $y_1 = f(x_1) \in f(G)$ and $y_2 = f(x_2) \in f(H)$. Again since $f$ is bijective, $y_2 = f(x_2) \notin f(G)$ and $y_1 = f(x_1) \notin f(H)$. Thus $(Y, \tau_2)$ is sb*- $T_1$. This proves (iii).

5 sb*- $T_2$ Spaces

In this section we introduce sb*- $T_2$ space and investigate some of their basic properties.

Definition 5.1. A space $X$ is said to be sb*- $T_2$ if for every pair of distinct points $x$ and $y$ in $X$, there are disjoint sb*-open sets $U$ and $V$ in $X$ containing $x$ and $y$ respectively.

Theorem 5.2. (i) Every $w$-$T_2$ space is sb*- $T_2$.

(ii) Every $\alpha$-$T_2$ space is sb*- $T_2$.

Proof: (i) Let $X$ be a $w$-$T_2$ space. Let $x$ and $y$ be two distinct points in $X$. Since $X$ is $w$-$T_2$, there exist disjoint w-open sets $U$ and $V$ such that $x \in U$ and $y \in V$. By Theorem 2.11(ii), $U$ and $V$ are disjoint sb*-open sets such that $x \in U$ and $y \in V$. Hence $X$ is sb*- $T_2$.

(ii) Suppose $X$ is $\alpha$-$T_2$ space. Let $x$ and $y$ be two disjoint $\alpha$ open sets $U$ and $V$ such that $x \in U$ and $y \in V$. By Theorem 2.11(i), $U$ and $V$ are disjoint sb*-open sets such that $x \in U$ and $y \in V$. Hence $X$ is sb*- $T_2$.

Remark 5.3. The converse of the statements (i) and (ii) of the above Theorem is not true as shown in the following examples.

Example 5.4. Consider the space $(X, \tau)$, where $X = \{a,b,c\}$ and $\tau = \{\emptyset, \{a,b\}, X\}$. Then $(X, \tau)$ is sb*- $T_2$ but not $w$-$T_2$. This shows that sb*- $T_2$ does not imply $w$-$T_2$.

Example 5.5. Consider the space $(X, \tau)$, where $X = \{a,b,c\}$ and $\tau = \{\emptyset, \{b\}, \{a,c\}, X\}$. It can be verified that $(X, \tau)$ is sb*- $T_2$ but not $\alpha$-$T_2$. This shows that sb*- $T_2$ does not imply $\alpha$-$T_2$.

Remark 5.6. The concepts of semi-$T_2$ and sb*- $T_2$ are independent as shown in the following examples.
Example 5.7. Consider the space \((X, \tau)\), where \(X = \{a, b, c, d\}\) and \(\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}\). It can be verified that \((X, \tau)\) is semi-\(T_2\) but not \(sb^*-T_2\). This shows that semi-\(T_2\) does not imply \(sb^*-T_2\).

Example 5.8. Consider the space \((X, \tau)\), where \(X = \{a, b, c, d\}\) and \(\tau = \{\phi, \{a, b\}, X\}\). Then \((X, \tau)\) is \(sb^*-T_2\) but not semi-\(T_2\). This shows that \(sb^*-T_2\) does not imply semi-\(T_2\).

Remark 5.9. Every \(sb^*-T_2\) space is \(b-T_2\). But the converse is not true as shown in the following example.

Example 5.10. Consider the space \((X, \tau)\), where \(X = \{a, b, c\}\) and \(\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}\). Clearly \((X, \tau)\) is \(b-T_2\) but not \(sb^*-T_2\). This shows that \(b-T_2\) does not imply \(sb^*-T_2\).

Theorem 5.11. Every \(sb^*-T_2\) space is \(sb^*-T_1\).

Proof: Let \(X\) be a \(sb^*-T_2\) space. Let \(x\) and \(y\) be two distinct points in \(X\). Since \(X\) is \(sb^*-T_2\), there exist disjoint \(sb^*\)-open sets \(U\) and \(V\) such that \(x \in U\) and \(y \in V\). Since \(U\) and \(V\) are disjoint, \(x \in U\) but \(y \notin U\) and \(y \in V\) but \(x \notin V\). Hence \(X\) is \(sb^*-T_1\).

However the converse is not true as shown in the following example.

Example 5.12. Consider the space \((X, \tau)\), where \(X = \{a, b, c, d\}\) and \(\tau = \{\phi, \{a, b\}, X\}\). Then \((X, \tau)\) is \(sb^*-T_1\) but not \(sb^*-T_2\). This shows that \(sb^*-T_1\) does not imply \(sb^*-T_2\).

Theorem 5.13. For a topological space \(X\), the following are equivalent:

(i) \(X\) is a \(sb^*-T_2\) space.

(ii) For each \(y \neq x\) there exists a \(sb^*\)-open set \(U\) such that \(x \in U\) and \(y \notin \text{sb}^*\text{cl}(U)\).

(iii) For each \(x \in X\), \(\bigcap\{\text{sb}^*\text{cl}(U): U \in \text{sb}^*\text{O}(X) \text{ and } x \in U\} = \{x\}\).

Proof: (i)⇒ (ii): Suppose \(X\) is a \(sb^*-T_2\) space. Then for each \(y \neq x\) there exist disjoint \(sb^*\)-open sets \(U\) and \(V\) such that \(x \in U\) and \(y \in V\). Since \(V\) is \(sb^*\)-open, \(V^c\) is \(sb^*\) - closed and \(U \subseteq V^c\). This implies that \(sb^*\text{cl}(U) \subseteq V^c\). Since \(y \notin V^c\), \(y \notin sb^*\text{cl}(U)\).

(ii)⇒ (iii): If \(y \neq x\), then there exists a \(sb^*\)-open set \(U\) such that \(x \in U\) and \(y \notin \text{sb}^*\text{cl}(U)\). Therefore \(y \notin \bigcap\{\text{sb}^*\text{cl}(U): U \in \text{sb}^*\text{O}(X) \text{ and } x \in U\}\). Therefore \(\bigcap\{\text{sb}^*\text{cl}(U): U \in \text{sb}^*\text{O}(X) \text{ and } x \in U\} = \{x\}\). This proves (iii).

(iii)⇒ (i): Let \(y \neq x\) in \(X\). Then \(y \notin \{x\} = \bigcap\{\text{sb}^*\text{cl}(U): U \in \text{sb}^*\text{O}(X) \text{ and } x \in U\}\). This implies that there exists a \(sb^*\)-open set \(U\) such that \(x \in U\) and \(y \notin \text{sb}^*\text{cl}(U)\). Let \(V = (\text{sb}^*\text{cl}(U))^c\). Then \(V\) is \(sb^*\)-open and \(y \in V\). Now \(U \cap V = V \cap \text{sb}^*\text{cl}(U))^c \subseteq U \cap (U)^c = \phi\). Therefore \(X\) is \(sb^*-T_2\) space.

Theorem 5.14. Let \(f:X \rightarrow Y\) be a bijection.

(i) If \(f\) is \(sb^*\)-open and \(X\) is \(T_2\), then \(Y\) is \(sb^*-T_2\).

(ii) If \(f\) is \(sb^*\)-continuous and \(Y\) is \(T_2\), then \(X\) is \(sb^*-T_2\).
Proof: Let \( f: X \to Y \) be a bijection.

(i) Suppose \( f \) is sb*-open and \( X \) is T\(_2\). Let \( y_1 \neq y_2 \in Y \). Since \( f \) is a bijection, there exist \( x_1, x_2 \) in \( X \) such that \( f(x_1) = y_1 \) and \( f(x_2) = y_2 \) with \( x_1 \neq x_2 \). Since \( X \) is T\(_2\), there exist disjoint open sets \( U \) and \( V \) in \( X \) such that \( x_1 \in U \) and \( x_2 \in V \). Since \( f \) is sb*-open, \( f(U) \) and \( f(V) \) are sb*-open in \( Y \) such that \( y_1 = f(x_1) \in f(U) \) and \( y_2 = f(x_2) \in f(V) \). Again since \( f \) is a bijection, \( f(U) \) and \( f(V) \) are disjoint in \( Y \). Thus \( Y \) is sb*-T\(_2\).

(ii) Suppose \( f: X \to Y \) is sb*-continuous and \( Y \) is T\(_2\). Let \( x_1, x_2 \in X \) with \( x_1 \neq x_2 \). Let \( y_1 = f(x_1) \) and \( y_2 = f(x_2) \). Since \( f \) is one-one, \( y_1 \neq y_2 \). Since \( Y \) is T\(_2\), there exist disjoint open sets \( U \) and \( V \) containing \( y_1 \) and \( y_2 \) respectively. Since \( f \) is sb*-continuous bijective, \( f^{-1}(U) \) and \( f^{-1}(V) \) are disjoint sb*-open sets containing \( x_1 \) and \( x_2 \) respectively. Thus \( X \) is sb*-T\(_2\).

Theorem 5.15. A topological space \((X, \tau)\) is sb*-T\(_2\) if and only if the intersection of all sb*-closed, sb*-neighbourhoods of each point of the space is reduced to that point.

Proof: Let \((X, \tau)\) be sb*-T\(_2\) and \( x \in X \). Then for each \( y \neq x \) in \( X \), there exist disjoint sb*-open sets \( U \) and \( V \) such that \( x \in U \) , \( y \in V \). Now \( U \cap V = \emptyset \) implies \( x \in U \subseteq V^c \). Therefore \( V^c \) is a sb*-neighbourhood of \( x \). Since \( V \) is sb*-open, \( V^c \) is sb* closed and sb*-neighbourhood of \( x \) to which \( y \) does not belong. That is there is a sb*-closed, sb*-neighbourhoods of \( x \) which does not contain \( y \). so we get the intersection of all sb* - closed, sb*-neighbourhood of \( x \) is \( \{x\} \).

Conversely, let \( x, y \in X \) such that \( x \neq y \) in \( X \). Then by assumption, there exist a sb*-closed , sb*-neighbourhood \( V \) of \( x \) such that \( y \notin V \). Now there exists a sb*-open set \( U \) such that \( x \in U \subseteq V \). Thus \( U \) and \( V^c \) are disjoint sb*-open sets containing \( x \) and \( y \) respectively. Thus \((X, \tau)\) is sb*-T\(_2\).

Theorem 5.16. If \( f: X \to Y \) be bijective, sb*-irresolute map and \( X \) is sb*-T\(_2\), then \((X, \tau_2)\) is sb*-T\(_2\).

Proof: Suppose \( f: (X, \tau) \to (Y, \tau_2) \) is bijective. And \( f \) is sb*-irresolute, and \( (Y, \tau_2) \) is sb*-T\(_2\). Let \( x_1, x_2 \in X \) with \( x_1 \neq x_2 \). Since \( f \) is bijective, \( y_1 = f(x_1) \neq f(x_2) = y_2 \) for some \( y_1, y_2 \in Y \). Since \( (Y, \tau_2) \) is sb*-T\(_2\), there exist disjoint sb*-open sets \( G \) and \( H \) such that \( y_1 \in G \) and \( y_2 \in H \). Again since \( f \) is bijective, \( x_1 \in f^{-1}(y_1) \in f^{-1}(G) \) and \( x_2 \in f^{-1}(y_2) \in f^{-1}(H) \). Since \( f \) is sb*-irresolute, \( f^{-1}(G) \) and \( f^{-1}(H) \) are sb*-open sets in \((X, \tau_1)\). Also \( f \) is bijective, \( G \cap H = \emptyset \) implies that \( f^{-1}(G) \cap f^{-1}(H) = f^{-1}(G \cap H) = f^{-1}(\emptyset) = \emptyset \). It follows that \((X, \tau_2)\) is sb*-T\(_2\).

References


