Fuzzy semi-connectedness and fuzzy pre-connectedness in fuzzy closure space

U.D. Tapi, Bhagyashri A. Deole
Department of Applied Mathematics and Computational Science
Shri G. S. Institute of Technology and Science
Indore (M.P.), India.
utapi@sgsits.ac.in, deolebhagyashri@gmail.com

Abstract

In this paper we introduce the concepts of fuzzy semi-connectedness and fuzzy pre-connectedness in fuzzy Čech closure space and study some of their fundamental properties.

Keywords: Fuzzy Čech closure space, fuzzy connectedness in fuzzy Čech closure space, Fσ-continuous mapping, Fp-continuous mapping, fuzzy semi-connectedness and fuzzy pre-connectedness in fuzzy Čech closure space.

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1 Introduction


The concept of fuzzy semi-open set was introduced by Azad [1] and fuzzy pre-open set was introduced by Bin Shahna [2] in fuzzy topological space. In this paper, we [10] introduce the semi-connectedness and pre-connectedness in Čech closure space. We are introducing the concepts of fuzzy semi-connectedness and fuzzy pre-connectedness in fuzzy Čech closure space and study some of their fundamental properties.

2 Preliminaries

Definition 2.1.[5] Let X be a non-empty fuzzy set. A function \( k: \mathbb{I}^X \rightarrow \mathbb{I}^X \) is called fuzzy Čech closure operator on \( X \) if it satisfies the following conditions

1. \( k(\emptyset) = \emptyset \).
2. \( A \subseteq k(A) \), for all \( A \in \mathbb{I}^X \).
3. \( k(A \cup A_2) = k(A_1 \cup k(A_2)) \) for all \( A_1, A_2 \in \mathbb{I}^X \).

The pair \((X, k)\) is called fuzzy Čech closure space.
Definition 2.2.[9] Let $X$ be a nonempty fuzzy set. A function $k: I^X \to I^X$ is called fuzzy Čech closure operator on $X$. A fuzzy Čech closure space $(X, k)$ is said to be fuzzy connected if and only if there exist any $F$-continuous map $f$ from $X$ to the fuzzy discrete space $\{0, 1\}$ is constant.

Definition 2.3.[3] A subset $A$ in a Čech closure space $(X, k)$ is called Čech semi-open in $X$ if $A \subseteq k(\text{int}(A))$. The class of all semi-open sets of Čech closure space $(X, k)$ is denoted by $SO(X, k)$.

Definition 2.4.[3] A subset $A$ in Čech closure space $X$ is called Čech pre-open if $A \subseteq \text{int}(k(\text{A}))$. The family of all pre-open sets of Čech closure space $(X, k)$ is denoted by $PO(X, k)$.

3 Fuzzy Semi-connectedness In Fuzzy Closure Space

Definition 3.1. A fuzzy set $A$ in a fuzzy Čech closure space $(X, k)$ is said to be fuzzy semi-open set if $A \subseteq k(\text{int}(A))$. The complement of fuzzy semi-open set is called a fuzzy semi-closed set. The class of all fuzzy semi-open sets of fuzzy Čech closure space $(X, k)$ is denoted by $FSO(X, k)$.

Example 3.2. Let $X = \{a, b, c\}$ be a non-empty fuzzy set. Define fuzzy Čech closure operator

$$k: I^X \to I^X \text{ such that } k(A) = \begin{cases} 0_X; & \text{if } A = 0_X \\ 1_{(b,c)}; & \text{if } 0 \prec A \leq 1_{(b,c)} \\ 1_{(b,c)}; & \text{if } 0 \prec A \leq 1_{(b)} \\ 1_{(b,c)}; & \text{if } 0 \prec A \leq 1_{(c)} \\ 1_X; & \text{otherwise.} \end{cases}$$

Then $(X, k)$ is called a fuzzy Čech closure space.

Fuzzy open sets = $\{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, I_X, 0_X\}.$

$FSO(X, k) = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, I_X, 0_X\}.$

Definition 3.3. Let $(X, k_1)$ and $(Y, k_2)$ be two fuzzy Čech closure spaces. A mapping $f: X \to Y$ is $F$-continuous if the inverse image of every fuzzy open set in $Y$ is fuzzy semi-open in $X$.

Example 3.4. Let $X = \{a, b, c\}$ be a non-empty fuzzy set. Define fuzzy Čech closure operator

$$k_i: I^X \to I^X \text{ such that } k_i(A) = \begin{cases} 0_X; & \text{if } A = 0_X \\ 1_{(b,c)}; & \text{if } 0 \prec A \leq 1_{(b,c)} \\ 1_{(b,c)}; & \text{if } 0 \prec A \leq 1_{(b)} \\ 1_{(b,c)}; & \text{if } 0 \prec A \leq 1_{(c)} \\ 1_X; & \text{otherwise.} \end{cases}$$

Then $(X, k_i)$ is called a fuzzy Čech closure space.

Fuzzy open sets = $\{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, I_X, 0_X\}.$

$FSO(X, k_i) = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, I_X, 0_X\}.$

Let $Y = \{a, b, c\}$ be a non-empty fuzzy set. Define fuzzy Čech closure operator $k_2: I^Y \to I^Y$ such that
Then \((Y, k_2)\) is called a fuzzy Čech closure space.

Fuzzy open sets = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, I_y, 0_y\},

\[FSO(Y, k_2) = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, I_y, 0_y\} .\]

Define \(F\)-mapping \(f: X \rightarrow Y\) such that

\[f^{-1}\{a\} = \{a, b\}, f^{-1}\{b\} = \{b\}, f^{-1}\{c\} = \{a, c\}, f^{-1}\{a, b\} = \{a\},\]

\[f^{-1}\{b, c\} = \{c\}, f^{-1}\{a, c\} = X, f^{-1}\{I_y\} = I_x,\]

\[f^{-1}\{0_y\} = 0_x.\]

Hence, \(f\) is an \(Fs\)-continuous mapping.

**Definition 3.5.** A fuzzy Čech closure space \((X, k)\) is said to be a fuzzy semi-connected fuzzy Čech closure space if and only if there exists a constant \(Fs\)-continuous map \(f\) from \(X\) to the fuzzy discrete space \(\{0, 1\}\). A fuzzy subset \(A\) in a fuzzy Čech closure space \((X, k)\) is said to be a fuzzy semi-connected fuzzy Čech closure space if \(A\) with the subspace topology is a fuzzy semi-connected fuzzy Čech closure space.

**Example 3.6.** Let \(X = \{a, b, c\}\) be a non-empty fuzzy set. Define fuzzy Čech closure operator

\[
k_3: I^x \rightarrow I^y \text{ such that } k_3(A) =
\begin{cases}
0_x; & A = 0_x, \\
1_{\{b, c\}}; & \text{if } 0 < A \leq 1_{\{b, c\}}, \\
1_{\{b, c\}}; & \text{if } 0 < A \leq 1_b, \\
1_{\{b, c\}}; & \text{if } 0 < A \leq 1_c, \\
1_X; & \text{otherwise.}
\end{cases}
\]

Then \((X, k_3)\) is called a fuzzy Čech closure space.

Fuzzy open sets = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, I_x, 0_x\}.

\[FSO(X, k_3) = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, I_x, 0_x\}.\]

Consider an \(Fs\)-continuous map \(f: X \rightarrow \{0, 1\}\) such that

\[f^{-1}\{1\} = \{a\} = \{b\} = \{c\} = \{a, b\} = \{b, c\} = \{a, c\} = I_x,\]

\[f^{-1}\{0\} = 0_x \text{ is constant.}\]

Hence, \((X, k_3)\) is a fuzzy semi-connected fuzzy Čech closure space.

**Definition 3.7.** A fuzzy Čech closure space \((X, k)\) is called a fuzzy semi-disconnected fuzzy Čech closure space if and only if there exists a surjective \(Fs\)-continuous map \(f\) from \(X\) to the fuzzy discrete space \(\{0, 1\}\).
Theorem 3.8. A fuzzy Čech closure space \((X, k)\) is fuzzy semi-connected if and only if every \(Fs\)-continuous mapping \(f\) from \(X\) into a fuzzy discrete space \(Y = \{0, 1\}\) with at least two points is constant.

Proof: Let a fuzzy Čech closure space \((X, k)\) be fuzzy semi-connected. Then there exists an \(Fs\)-continuous mapping \(f\) from the fuzzy Čech closure space \(X\) into the fuzzy discrete space \(Y = \{0, 1\}\). For each \(y \in I, f^{1}(y) = 0_{X}\), or \(I_{X}\). If \(f^{1}(y) = 0_{X}\) for all \(y \in I\), then \(f\) ceases to be a mapping. Therefore, \(f^{1}(y_{0}) = I_{X}\) for a unique \(y_{0} \in I\). This implies that \(f\{I_{X}\} = \{y_{0}\}\) and hence \(f\) is a constant mapping.

Conversely, let \(Fs\)-continuous mapping \(f\) from \(X\) into a fuzzy discrete space \(Y = \{0, 1\}\) be constant. Suppose \(U\) is a fuzzy semi-open set in a fuzzy Čech closure space \((X, k)\). If \(U \neq 0_{X}\), we show that \(U = I_{X}\).

Otherwise, choose two fixed points \(y_{1}\) and \(y_{2}\) in \(I_{Y}\). Define \(f: X \rightarrow Y\) by

\[
f(x) = \begin{cases} y_{1} & \text{if } x \in U \\ y_{2} & \text{otherwise.} \end{cases}
\]

Then for any open set \(V\) in \(I_{Y}\),

\[f^{-1}(V) = \begin{cases} U & \text{if } V \text{ contains } y_{1} \text{ only,} \\ 1_{X}/U & \text{if } V \text{ contains } y_{2} \text{ only,} \\ 1_{X} & \text{if } V \text{ contains both } y_{1} \text{ and } y_{2}, \\ 0_{X} & \text{otherwise.} \end{cases}\]

In all the cases \(f^{-1}(V)\) is fuzzy semi-open in \(I_{X}\). Hence \(Fs\)-continuous mapping \(f\) is not constant, which is a contradiction. This proves that the only fuzzy semi-open subsets of \(X\) are \(0_{X}\) and \(I_{X}\). Hence, \((X, k)\) is a fuzzy semi-connected fuzzy Čech closure space.

Theorem 3.9. The following assertions are equivalent:

1. \((Y, k)\) is a fuzzy semi-connected fuzzy Čech closure space.
2. The only fuzzy subsets of \(Y\) both \(Fs\)-open and \(Fs\)-closed are \(0_{Y}\) and \(I_{Y}\).
3. No \(Fs\)-continuous mapping \(f: Y \rightarrow \{0, 1\}\) is surjective.

Proof: [1] \(\Rightarrow\) [2]: Let \((Y, k)\) be a fuzzy semi-connected fuzzy Čech closure space. Suppose \(G \leq Y\) is both fuzzy semi-open and fuzzy semi-closed such that \(G \neq 0_{Y}\) and \(G \neq I_{Y}\), then \(I_{Y} = G \lor G^{c}\), where \(G^{c}\) is the complement of \(G\) in \(Y\). Hence, \(Fs\)-continuous mapping \(f: Y \rightarrow \{0, 1\}\) is not constant. That is, \((Y, k)\) is not a fuzzy semi-connected fuzzy Čech closure space, which is a contradiction. Hence, the only fuzzy subsets of \(Y\) which are both fuzzy semi-open and fuzzy semi-closed are \(0_{Y}\) and \(I_{Y}\).

[2] \(\Rightarrow\) [3]: Suppose the only fuzzy subsets of \(I_{Y}\) which are both fuzzy semi-open and fuzzy semi-closed are \(0_{Y}\) and \(I_{Y}\). Let \(f: Y \rightarrow \{0, 1\}\) be \(Fs\)-continuous and surjective. Then \(f^{-1}(0) \neq 0_{Y}\), \(f^{-1}(0) \neq I_{Y}\). But \(\{0\}\) is both fuzzy open and fuzzy closed in \(\{0, 1\}\). Hence \(f^{-1}(0)\) is fuzzy semi-open and fuzzy semi-closed in \(I_{Y}\). This is a contradiction to our assumption. Hence no \(Fs\)-continuous mapping \(f: Y \rightarrow \{0, 1\}\) is surjective.

[3] \(\Rightarrow\) [1]: Let no \(Fs\)-continuous mapping \(f: Y \rightarrow \{0, 1\}\) be surjective. If possible let the fuzzy Čech closure space \((Y, k)\) be not a fuzzy semi-connected fuzzy Čech closure space. So, \(Y = AVB\), \(A\) and \(B\) are also fuzzy semi-closed sets.
Let \( X \equiv (x) = \begin{cases} 1; & x \in A \\ 0; & x \not\in A \end{cases} \)

Then \( X \equiv (x) \) is \( Fs \)-continuous surjection which is a contradiction. Hence fuzzy Čech closure space \((Y, k)\) is a fuzzy semi-connected fuzzy Čech closure space.

**Theorem 3.10.** The \( Fs \)-continuous image of a fuzzy semi-connected fuzzy Čech closure space is a fuzzy semi-connected fuzzy Čech closure space.

**Proof:** Let \((X, k)\) be a fuzzy semi-connected fuzzy Čech closure space and consider an \( F \)-continuous mapping \( f: X \rightarrow f(X) \) which is surjective. If \( f(x) \) is not a fuzzy semi-connected fuzzy Čech closure space, then there would be an \( Fs \)-continuous surjection \( g: f(x) \rightarrow \{0, 1\} \) so that the composite mapping \( gof: X \rightarrow \{0, 1\} \) is also an \( Fs \)-continuous surjection, which is a contradiction to fuzzy semi-connectedness of fuzzy Čech closure space \((X, k)\). Hence \( f(x) \) is a fuzzy semi-connected fuzzy Čech closure space.

### 4 Fuzzy Pre-connectedness In Fuzzy Closure Space

**Definition 4.1.** Let \((X, k)\) be a fuzzy Čech closure space. A fuzzy set \( A \) in a fuzzy Čech closure space \((X, k)\) is called a fuzzy pre-open set if \( A \subseteq int(k(A)) \). The complement of a fuzzy pre-open set is called a fuzzy pre-closed set. The family of all fuzzy pre-open sets of \( X \) is denoted by \( FPO(X, k) \).

**Example 4.2.** Let \( X = \{a, b, c\} \) be a non-empty fuzzy set. Define a fuzzy Čech closure operator

\[ k_i: I^X \rightarrow I^X \text{ such that } k_i(A) = \begin{cases} 0x; & A = 0x. \\ 1_{[b,c]}; & 0 \prec A \leq 1_{[b,c]} \\ 1_{[b,c]}; & 0 \prec A \leq 1_b \\ 1_{[b,c]}; & 0 \prec A \leq 1_c \\ 1x; & \text{otherwise.} \end{cases} \]

Then \((X, k_i)\) is called a fuzzy Čech closure space.

Fuzzy open sets = \{ \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, I_X, 0_X \}.

\[ FPO(X, k_i) = \{ \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, 0_X, I_X \}. \]

**Definition 4.3.** Let \((X, k_1)\) and \((Y, k_2)\) be two fuzzy Čech closure spaces. An \( F \)-mapping \( f: X \rightarrow Y \) is pre-continuous if the inverse image of every fuzzy open set in \( Y \) is fuzzy pre-open in \( X \).

**Example 4.4.** Let \( X = \{a, b, c\} \) be a non-empty fuzzy set. Define a fuzzy Čech closure operator

\[ k_i: I^X \rightarrow I^X \text{ such that } \]

\[ k_i(A) = \begin{cases} 0x; & A = 0x. \\ 1_{[b,c]}; & 0 \prec A \leq 1_b \\ 1_{[b,c]}; & 0 \prec A \leq 1_c \\ 1_{[b,c]}; & 0 \prec A \leq 1_{[b,c]} \\ 1x; & \text{otherwise.} \end{cases} \]
Then \((X, k_1)\) is called a fuzzy Čech closure space.

Fuzzy open sets = \(\{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, I_X, 0_X\}\).

\(\text{FPO}(X, k_1) = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, 0_X, I_X\}\).

Let \(Y = \{a, b, c\}\) be a non-empty fuzzy set. Define a fuzzy Čech closure operator \(k_2: I^Y \to I^Y\) such that

\[
k_2(A) = \begin{cases} 
0_Y & \text{if } A = 0_Y \\
1_{[a,b]} & \text{if } 0 < A \leq 1_{[a]} \\
1_{[b,c]} & \text{if } 0 < A \leq 1_{[b]} \\
1_{[a,c]} & \text{if } 0A \leq 1_{[c]} \\
1_Y & \text{otherwise.}
\end{cases}
\]

Then \((Y, k_2)\) is called a fuzzy Čech closure space.

Fuzzy open sets = \(\{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, 0_Y, I_Y\}\).

\(\text{FPO}(Y, k_2) = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, I_Y, 0_Y\}\).

There exists an \(F\)-mapping \(f: X \to Y\) such that

\[
f^I(a) = \{a, b\}, f^I(b) = \{b\}, f^I(c) = \{a, c\}, f^I(a, b) = \{a\},
\]

\[
f^I(b, c) = \{c\}, f^I(a, c) = X, f^I(1_Y) = I_X, f^I(0_Y) = 0_X.
\]

Hence \(f\) is an \(F\)-continuous mapping.

**Definition 4.5.** A fuzzy Čech closure space \((X, k)\) is called a fuzzy pre-connected fuzzy Čech closure space if and only if there exists an \(F\)-continuous map \(f\) from \(X\) to the fuzzy discrete space \(\{0, 1\}\) which is constant. A fuzzy subset \(A\) of fuzzy pre-connected fuzzy Čech closure space \((X, k)\) is said to be a fuzzy pre-connected fuzzy Čech closure space if \(A\) with the subspace topology is a fuzzy pre- connected fuzzy Čech closure space.

**Example 4.6.** Let \(X = \{a, b, c\}\) be a non-empty fuzzy set. Define a fuzzy Čech closure operator

\[
k_3: I^X \to I^X\text{ such that } k_3(A) = \begin{cases} 
0_X & \text{if } A = 0_X \\
1_{[b,c]} & \text{if } 0 < A \leq 1_{[b,c]} \\
1_{[b,c]} & \text{if } 0 < A \leq 1_{[b,c]} \\
1_{[a,c]} & \text{if } 0 < A \leq 1_{[a,c]} \\
1_X & \text{otherwise.}
\end{cases}
\]

Then \((X, k_3)\) is called a fuzzy Čech closure space.

Fuzzy open sets = \(\{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, I_X, 0_X\}\).

\(\text{FPO}(X, k_3) = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, 0_X, I_X\}\).

Consider a fuzzy pre-continuous mapping \(f: X \to \{0, 1\}\) such that

\[
f^I(1) = \{a\} = \{a, b\} = \{b, c\} = \{a, c\} = \{I_X\} = \{b\} = \{c\}, f^I(0) = 0_X\text{ is constant.}
\]

Hence, \((X, k_3)\) is a fuzzy pre-connected fuzzy Čech closure space.

**Definition 4.7.** A fuzzy Čech closure space \((X, k)\) is called a fuzzy pre-disconnected fuzzy Čech closure space if and only if every \(Fp\)-continuous map \(f\) from \(X\) to the fuzzy discrete space \(\{0, 1\}\) is surjective.
Theorem 4.8. If \( \{ A_i : i \in \Lambda \} \) is a family of fuzzy pre-connected fuzzy Čech closure subsets of a fuzzy pre-connected fuzzy Čech closure space \((X, k)\) then \( \forall A_i \) is also a fuzzy pre-connected fuzzy Čech closure subset of \( X \), where \( \Lambda \) is any index set.

**Proof:** Each \( A_i, i \in \Lambda \) is a fuzzy pre-connected fuzzy Čech closure subset of \( X \). So there exists an \( Fp\)-continuous mapping \( f_i: A_i \to [0, 1] \) which is constant. Let an \( Fp\)-continuous mapping \( f: \bigvee A_i \to [0, 1] \) be not constant. Then \( f^{-1}\{1\} \neq A_i \), which is a contradiction to each \( A_i \) is fuzzy pre-connected subsets of \( X \). That is, every \( Fp\)-continuous mapping \( f \) is constant. Hence \( \forall A_i \) is a fuzzy pre-connected fuzzy Čech closure space.

**Theorem 4.9.** Let \((X, k_1)\) and \((Y, k_2)\) be two fuzzy Čech closure spaces and \( F\)-mapping \( f: (X, k_1) \to (Y, k_2) \) be bijective. Then,

(i) if \( f \) is an \( Fp\)-continuous mapping and \( X \) is a fuzzy pre-connected fuzzy Čech closure space then \( Y \) is a fuzzy connected fuzzy Čech closure space.

(ii) if \( f \) is an \( F\)-continuous mapping and \( X \) is fuzzy pre-connected fuzzy Čech closure space then \( Y \) is fuzzy connected fuzzy Čech closure space.

(iii) if \( f \) is an \( Fp\)-open mapping and \( Y \) is a fuzzy pre-connected fuzzy Čech closure space then \( X \) is a fuzzy connected fuzzy Čech closure space.

(iv) if \( f \) is an \( F\)-open mapping and \( X \) is a fuzzy connected fuzzy Čech closure space then \( Y \) is a fuzzy pre-connected fuzzy Čech closure space.

**Proof:** (i) Let \((Y, k_2)\) be a fuzzy Čech closure space and \( X \) be a fuzzy pre-connected fuzzy Čech closure space. Then there exists an \( Fp\)-continuous mapping \( fog: X \to [0, 1] \) which is constant. Consider an \( Fp\)-continuous mapping \( g: Y \to [0, 1] \), given that \( f: X \to Y \) is \( Fp\)-continuous and bijective so that \( g \) is also a constant mapping. Hence, \( Y \) is a fuzzy connected fuzzy Čech closure space.

(ii) Given that \( X \) is a fuzzy pre-connected fuzzy Čech closure space, that is, every \( Fp\)-continuous mapping \( g: X \to [0, 1] \) is constant. Since \( f^{-1}: Y \to X \) is \( F\)-continuous bijection, so that \( F\)-continuous mapping \( f^{-1}og: Y \to [0, 1] \) is constant. Hence \( Y \) is a fuzzy connected fuzzy Čech closure space.

(iii) Given that \( Y \) is a fuzzy pre-connected fuzzy Čech closure space, that is, every \( Fp\)-continuous mapping \( g: Y \to [0, 1] \) is constant. Since \( f: X \to Y \) is \( Fp\)-open and bijective, we have \( F\)-continuous mapping \( fog: X \to [0, 1] \) is constant. Hence \( X \) is fuzzy connected fuzzy Čech closure space.

(iv) Given that \( X \) is a fuzzy connected fuzzy Čech closure space, that is, an \( F\)-continuous mapping \( g: X \to [0, 1] \) is constant and \( f^{-1}: Y \to X \) is \( F\)-open mapping so that it is an \( Fp\)-open mapping then \( Fp\)-continuous mapping \( f^{-1}og: Y \to [0, 1] \) is constant. Hence \( Y \) is a fuzzy pre-connected fuzzy Čech closure space.

**Theorem 4.10.** A fuzzy Čech closure space \((X, k)\) is fuzzy pre-disconnected if and only if there exists a surjective \( Fp\)-continuous map \( f \) from \( X \) to a fuzzy discrete space \( Y = \{0, 1\} \).

**References**


