Some square graceful graphs

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Abstract

A \((p,q)\) graph \(G(V,E)\) is said to be a square graceful graph if there exists an injection \(f:V(G)\rightarrow\{0,1,2,3,...,q^2\}\) such that the induced mapping \(f_p:E(G)\rightarrow\{1,4,9,...,q^2\}\) by \(f_p(uv)=|f(u)-f(v)|\) is a bijection. The function \(f\) is called a square graceful labeling of \(G\). In this paper, we prove the graph obtained by the subdivision of the edges of stars of bistar \(B_{n,n}\), the graph obtained by the subdivision of the edges of bistar \(B_{n,n}\), the graph obtained by the subdivision of the edges of the path \(P_n\) in a comb \(P_n\Theta K_1\), \(<C_5,K_{1,s}>\), \(<S_n:m>\) and \(<C_s,K_{1,s}>\) are square graceful graph.

Keywords: Square graceful graph, square graceful labeling.

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1 Introduction

All graphs in this paper are finite, simple and undirected graphs. Let \((p,q)\) be a graph with \(p=|V(G)|\) vertices and \(q=|E(G)|\) edges. A detailed survey of graph labeling can be found in [1]. Terms not defined here are used in the sense of Harary in [2]. There are different types of graceful labelings in the graph labeling. The concept of square graceful labeling was first introduced in [5] and some results on square graceful labeling of graphs are discussed in [5]. In this paper, we investigate some more graphs for square graceful labeling. We use the following definitions in the subsequent sections.

Definition 1.1. [5] A \((p,q)\) graph \(G(V,E)\) is said to be a square graceful graph if there exists an injection \(f:V(G)\rightarrow\{0,1,2,...,q^2\}\) such that the induced mapping \(f_p:E(G)\rightarrow\{1,4,9,...,q^2\}\) defined by \(f_p(uv)=|f(u)-f(v)|\) is a bijection. The function \(f\) is called a square graceful labeling of \(G\).

Definition 1.2. [6] The corona \(G_1\Theta G_2\) of two graphs \(G_1\) and \(G_2\) is defined as the graph \(G\) obtained by taking one copy of \(G_1\) (which has \(p\) points) and \(p\) copies of \(G_2\) and then joining the \(i^{th}\) point of \(G_1\) to every point in the \(i^{th}\) copy of \(G_2\).
Definition 1.3. [1] A complete bipartite graph $K_{1,n}$ is called a star and it has $n+1$ vertices and $n$ edges.

Definition 1.4. [1] The bistar graph $B_{m,n}$ is the graph obtained from a copy of star $K_{1,m}$ and a copy of star $K_{1,n}$ by joining the vertices of maximum degree by an edge.

Definition 1.5. [3] A subdivision of a graph $G$ is a graph that can be obtained from $G$ by a sequence of edge subdivisions.

Definition 1.6. [1] The graph $<S_n:m>$ is the graph obtained by taking $m$ disjoint copies of star $S_n$ and joining a new vertex to the centres of the $m$ copies of star $S_n$.

Definition 1.7. [4] The graph $<C_m*K_{1,n}>$ is the graph obtained from $C_m$ and $K_{1,n}$ by identifying any one of the vertices of $C_m$ with a pendent vertex of $K_{1,n}$ (that is a non-central vertex of $K_{1,n}$).

Definition 1.8. [4] The graph $<C_m,K_{1,n}>$ is the graph obtained from $C_m$ and $K_{1,n}$ by identifying any one of the vertices of $C_m$ with the central vertex of $K_{1,n}$.

2 Main Results

Theorem 2.1. The graph obtained by the subdivision of the edges of stars of the bistar $B_{m,n}$ is a square graceful graph.

Proof: Let $B_{m,n}$ be a bistar with $m+n+2$ vertices and $m+n+1$ edges. The vertex and edge sets are given by $V(B_{m,n}) =$ $\{ u_i, v_j : 1 \leq i \leq m+1, 1 \leq j \leq n+1 \}$ and $E(B_{m,n}) =$ $\{ u_i u_{m+1}, v_j v_{n+1}, u_{m+1}v_{n+1} : 1 \leq i \leq m, 1 \leq j \leq n \}$.

Let $G$ be the graph obtained by the subdivision of the edges of stars of $B_{m,n}$. Let $w_i$ divide $u_i u_{m+1}$ for $1 \leq i \leq m$ and $z_j$ divide $v_j v_{n+1}$ for $1 \leq j \leq n$. Then the vertex and edge set of $G$ are given by

$V(G) =$ $\{ u_i, v_j : 1 \leq i \leq m+1, 1 \leq j \leq n+1 \} \cup \{ w_i, z_j : 1 \leq i \leq m, 1 \leq j \leq n \}$ and $E(G) =$ $\{ w_i u_{m+1}, u_i w_i : 1 \leq i \leq m \} \cup \{ z_j v_{n+1}, v_j z_j : 1 \leq j \leq m \} \cup \{ u_{m+1}v_{n+1} \}$

Case (i): $m < n$.

Define an injection $f : V(G) \rightarrow \{0,1,2,3,\ldots,(2m+2n+1)^2\}$ by

$f(u_{m+1}) = 1$ ; $f(v_{n+1}) = 0$ ;

For $1 \leq i \leq m$ , $f(w_i) = (2m+n+2-i)^2 + 1$ ; $f(u_i) = (3m+2n+4-2i)(m+1)$.

For $1 \leq j \leq n$ , $f(z_j) = (2m+2n+2-j)^2$ ; $f(v_j) = (2m+3n+4-2j)(2m+n)$.

Then, $f$ induces a bijection $f_p : E(G) \rightarrow \{1,4,9,\ldots,(2m+2n+1)^2\}$.

In this case the edge labels of $G$ are as follows:

$f_p(u_{m+1}v_{n+1}) = 1$ ;

$f_p(w_i u_{m+1}) = (2m+n+2-i)^2$ and $f_p(u_i w_i) = (m+n+2-i)^2$ for $1 \leq i \leq m$;

$f_p(z_j v_{n+1}) = (2m+2n+2-j)^2$ and $f_p(v_j z_j) = (n+2-j)^2$ for $1 \leq j \leq n$.
Case(ii): $m=n$.

Define an injection $f : V(G) \to \{0,1,2,3,\ldots,(4n+1)^2\}$ by

\[ f(u_{ni}) = 0 \; \; \; ; \; \; f(v_{ni}) = 1 \; ; \]

For $1 \leq i \leq n$, $f(w_i) = (4n+2-i)^2$; $f(u_i) = (5n+4-2i)(3n)$.

For $1 \leq j \leq n$, $f(z_j) = (3n+2-j)^2 + 1$; $f(v_j) = (5n+4-2j)(n)+1$.

Then, $f$ induces a bijection $f_p : E(G) \to \{1,4,9,\ldots,(4n+1)^2\}$.

In this case the edge labels of $G$ are as follows:

\[ f_p(u_{ni},v_{ni}) = 1; \]

\[ f_p(w_i,u_{ni}) = (4n+2-i)^2 \; \; \; \; \text{and} \; \; \; f_p(u_i,w_i) = (n+2-i)^2 \; \; \; \text{for} \; \; \; 1 \leq i \leq n. \]

\[ f_p(z_j,v_{ni}) = (3n+2-j)^2 \; \; \; \; \text{and} \; \; \; f_p(v_j,z_j) = (2n+2-j)^2 \; \; \; \text{for} \; \; \; 1 \leq j \leq n. \]

Case(iii): $m > n$.

Define an injection $f : V(G) \to \{0,1,2,3,\ldots,(2m+2n+1)^2\}$ by

\[ f(u_{mi}) = 0 \; \; \; ; \; \; f(v_{mi}) = 1 \; ; \]

For $1 \leq i \leq m$, $f(w_i) = (2m+2n+2-i)^2$; $f(u_i) = (3m+2n+4-2i)(m+2n)$.

For $1 \leq j \leq n$, $f(z_j) = (m+2n+2-j)^2 + 1$; $f(v_j) = (2m+3n+4-2j)(n)+1$.

Then, $f$ induces a bijection $f_p : E(G) \to \{1,4,9,\ldots,(2m+2n+1)^2\}$.

In this case the edge labels are as follows:

\[ f_p(u_{mi},v_{mi}) = 1; \]

\[ f_p(w_i,u_{mi}) = (2m+2n+2-i)^2 \; \; \; \; \text{and} \; \; \; f_p(u_i,w_i) = (m+2-i)^2 \; \; \; \text{for} \; \; \; 1 \leq i \leq m. \]

\[ f_p(z_j,v_{mi}) = (m+2n+2-j)^2 \; \; \; \; \text{and} \; \; \; f_p(v_j,z_j) = (m+n+2-j)^2 \; \; \; \text{for} \; \; \; 1 \leq j \leq n. \]

Hence, the graph obtained by the subdivision of the edges of stars of the bistar $B_{m,n}$ is a square graceful graph.

Example 2.2. A square graceful labeling of the graph obtained by the subdivision of the edges of stars of bistar $B_{7,5}$ is shown in Figure 1.

![Figure 1: Square graceful labeling of the graph obtained by the subdivision of the edges of the stars of $B_{7,5}$.](image)
Theorem 2.3. The graph obtained by the subdivision of the edges of the bistar $B_{m,n}$ is a square graceful graph.

Proof: Let $B_{m,n}$ be a bistar with $m+n+2$ vertices and $m+n+1$ edges. The vertex and edge sets are given by, $V(B_{m,n}) = \{ u, v_j : 1 \leq i \leq m+1, 1 \leq j \leq n+1 \}$ and $E(B_{m,n}) = \{ u, u_{m+1}, v_j, v_{m+1}, u_{m+1}v_{m+1} : 1 \leq i \leq m, 1 \leq j \leq n \}$.

Let $G$ be the graph obtained by the subdivision of the edges of the bistar $B_{m,n}$. Let $w_i$ divide $u, u_{m+1}$ for $1 \leq i \leq m$ and $z_j$ divide $v_j, v_{m+1}$ for $1 \leq j \leq n$. Let $v$ divide $u_{m+1}, v_{m+1}$. Then the vertex and edge set of $G$ are given by

$V(G) = \{ u, v_j : 1 \leq i \leq m+1, 1 \leq j \leq n+1 \} \cup \{ w_i, z_j : 1 \leq i \leq m, 1 \leq j \leq n \} \cup \{ v \}$ and

$E(G) = \{ w_iu_{m+1}, w_iw_i, u_{m+1}v_{m+1}, v_jz_j : 1 \leq i \leq m \} \cup \{ v_jv_{m+1}, v_jz_j, v_iw_i : 1 \leq i \leq m \} \cup \{ vu_{m+1}, vv_{m+1} \}$

Case (i): $m < n$.

Define an injection $f : V(G) \rightarrow \{0,1,2,3,\ldots,(2m+2n+2)^2 \}$ by

$f(u_{m+1}) = 5$ ; $f(v_{m+1}) = 0$ ; $f(v) = 1$ ;

For $1 \leq i \leq m$, $f(w_i) = (2m+2n+3-i)^2 + 5$ ; $f(u_i) = (3m+2n+6-2i)(m)+5$ .

For $1 \leq j \leq n$, $f(z_j) = (2m+2n+3-i)^2$ ; $f(v_j) = (2m+3n+6-2i)(2m+n)$ .

Then, $f$ induces a bijection $f_p : E(G) \rightarrow \{1,4,9,\ldots,(2m+2n+2)^2 \}$.

In this case the induced edge labels of $G$ are as follows:

$f_p(wu_{m+1}) = (2m+n+3-i)^2$ and $f_p(u_iw_i) = (m+n+3-i)^2$ for $1 \leq i \leq m$.

$f_p(z_jv_{m+1}) = (2m+2n+3-j)^2$ and $f_p(v_jz_j) = (n+3-j)^2$ for $1 \leq j \leq n$.

Case (ii): $m = n$.

Define an injection $f : V(G) \rightarrow \{0,1,2,3,\ldots,(4n+2)^2 \}$ by

$f(u_{m+1}) = 5$ ; $f(v_{m+1}) = 0$ ; $f(v) = 1$ .

For $1 \leq i \leq n$, $f(w_i) = (3n+3-i)^2 + 5$ ; $f(u_i) = (5n+6-2i)(n)+5$ .

For $1 \leq j \leq n$, $f(z_j) = (4n+3-j)^2$ ; $f(v_j) = (5n+6-2j)(3n)$ .

Then, $f$ induces a bijection $f_p : E(G) \rightarrow \{1,4,9,\ldots,(4n+2)^2 \}$.

In this case the edge labels of $G$ are as follows:

$f_p(vu_{m+1}) = 4$ ; $f_p(vv_{m+1}) = 1$ ; $f_p(wuv_{m+1}) = (3n+3-i)^2$ and $f_p(u_iw_i) = (2n+3-i)^2$ for $1 \leq i \leq n$.

$f_p(z_jv_{m+1}) = (4n+3-j)^2$ and $f_p(v_jz_j) = (n+3-j)^2$ for $1 \leq j \leq n$.

Case (iii): $m > n$.

Define an injection $f : V(G) \rightarrow \{0,1,2,3,\ldots,(2m+2n+2)^2 \}$ by

$f(u_{m+1}) = 0$ ; $f(v_{m+1}) = 5$ ; $f(v) = 1$ .

In this case the edge labels of $G$ are as follows:

$f_p(wu_{m+1}) = (2m+n+3-i)^2$ and $f_p(u_iw_i) = (m+n+3-i)^2$ for $1 \leq i \leq n$.

$f_p(z_jv_{m+1}) = (4n+3-j)^2$ and $f_p(v_jz_j) = (n+3-j)^2$ for $1 \leq j \leq n$.
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For $1 \leq i \leq m$, $f(w_i) = (2m + 2n + 3 - i)^2$ ; $f(u_i) = (3m + 2n + 6 - 2i)(m + 2n)$ .

For $1 \leq j \leq n$, $f(z_j) = (m + 2n + 3 - j)^2 + 5$ ; $f(v_j) = (2m + 3n + 6 - 2j)(n + 5)$ .

Then, $f$ induces a bijection $f_p : E(G) \rightarrow \{1, 4, 9, \ldots, (2m + 2n + 2)^2\}$.

In this case the edge labels of $G$ are as follows: $f_p(v_{u_{w_1}}) = 1$ ; $f_p(v_{v_{w_1}}) = 4$ ; $f_p(w_{u_{w_1}}) = (2m + 2n + 3 - i)^2$ and $f_p(u_{w_i}) = (m + 3 - i)^2$ for $1 \leq i \leq m$.

$f_p(z_{v_{w_1}}) = (m + 2n + 3 - j)^2$ and $f_p(v_{z_j}) = (m + n + 3 - j)^2$ for $1 \leq j \leq n$.

Example 2.4. A square graceful labeling of the graph obtained by the subdivision of the edges of bistar $B_{3,7}$ and $B_{3,9}$ are shown in Figure 2 and Figure 3 respectively.

![Figure 2: Square graceful labeling of the graph obtained by the subdivision of the edges of $B_{3,7}$.](image)

![Figure 3: Square graceful labeling of the graph obtained by the subdivision of the edges of $B_{9,4}$.](image)

Theorem 2.5. The graph obtained by the subdivision of the edges of the path $P_n$ in comb $P_n \Theta K_1$ is a square graceful graph.

Proof: Let $G$ be the graph obtained by the subdivision of the edges of the path $P_n$ in comb $P_n \Theta K_1$.

Let $V(G) = \{u_i, v_j, w_k : 1 \leq i \leq n, 1 \leq j \leq n, 1 \leq k \leq n - 1\}$ and $E(G) = \{u_iw_k, w_ku_{i+1} : 1 \leq i \leq n - 1, 1 \leq k \leq n - 1\} \cup \{u_iv_j : 1 \leq i \leq n, 1 \leq j \leq n\}$.

Define an injection $f : V(G) \rightarrow \{0, 1, 2, 3, \ldots, (3n - 2)^2\}$ by

$f(u_i) = (3n - 2)^2$;

$f(u_{i,2}) = \frac{i(i - 1)(2i - 1)}{6}$ for $i = 2, 4, 6, \ldots, 2n - 2$. 

$\blacksquare$
$f(w_{k+1}) = \frac{k(k-1)(2k-1)}{6}$ for $k = 1, 3, 5, \ldots, 2n - 3$.

$f(v_{j+2}) = \frac{j(j-1)(2j-1)}{6} + \frac{(4n - 2 + j)^2}{2}$ for $j = 2, 4, 6, \ldots, 2n - 4$.

$f(v_1) = (n-1)(5n - 3)$; $f(v_n) = \frac{(n-1)(8n^2 - 10n + 3)}{3}$.

Then, $f$ induces a bijection $f_p : E(G) \rightarrow \{1, 4, 9, \ldots, (3n - 2)^2\}$.

The edge labels are as follows: $f_p(u_i w_i) = (3n - 2)^2$; $f_p(u_i v_i) = (2n - 2)^2$;

$f_p(w_i u_{i+1}) = (2i - 1)^2$ for $1 \leq i \leq n - 1$, $1 \leq k \leq n - 1$;

$f_p(u_i w_i) = 4i^2$ for $2 \leq i \leq n - 1$, $2 \leq k \leq n - 1$;

$f_p(u_i v_j) = (2n - 2 + i)^2$ for $1 \leq i \leq n - 1, 1 \leq j \leq n - 1$.

Example 2.6. A square graceful labeling of the graph obtained by the subdivision of the edges of the path $P_3$ in comb $P_3 \Theta K_1$ is shown in Figure 4.

**Theorem 2.7:** The graph $< C_3 \ast K_{1,n} >$ is a square graceful graph for $n \geq 3$.

**Proof:** Let the vertex sets of $C_3$ and $K_{1,n}$ be given by $V(C_3) = \{u_i : 1 \leq i \leq 3\}$ and $V(K_{1,n}) = \{v_j : 1 \leq j \leq n + 1\}$ where $v_{n+1}$ is the centre of the star. Identify $u_i$ of $C_3$ with $v_n$ of $K_{1,n}$ to get $< C_3 \ast K_{1,n} >$.

Then the vertex and edge sets of $< C_3 \ast K_{1,n} >$ are given by,

$V(C_3 \ast K_{1,n}) = \{u_i : 2 \leq i \leq 3; v_j : 1 \leq j \leq n + 1\}$

$u_1 = v_n$. Let $E(C_3 \ast K_{1,n}) = \{u_1 u_2, u_2 u_3, u_3 u_1\} \cup \{v_j v_{n+1} : 1 \leq j \leq n\}$

Define an injection $f : V(C_3 \ast K_{1,n}) \rightarrow \{0, 1, 2, 3, \ldots, (n + 3)^2\}$ by

$f(u_1 = v_n) = 0$; $f(u_2) = 16$; $f(u_3) = 25$; $f(v_{n+1}) = (n + 3)^2$

$f(v_{n-2}) = (n + 1)(n + 5)$; $f(v_{n-1}) = n^2 + 6n + 8$;

$f(v_j) = (2n + 6 - j)j$ for $1 \leq j \leq n - 3$.

Then, $f$ induces a bijection $f_p : E(C_3 \ast K_{1,n}) \rightarrow \{1, 4, 9, \ldots, (n + 3)^2\}$.
The induced edge labels of $< C_3^* K_{1,n} >$ are as follows:

\[ f_p(u_1 u_2) = 16 ; \quad f_p(u_2 u_3) = 9 ; \quad f_p(u_1 u_3) = 25 ; \quad f_p(v_{n-1} v_{n+1}) = 1 ; \]

\[ f_p(u_1 v_{n+1}) = (n+3)^2 ; \quad f_p(v_j v_{n+1}) = (n+3-j)^2 \quad \text{for } 1 \leq j \leq n-3 ; \quad f_p(v_{n-2} v_{n+1}) = 4 . \]

Hence, the graph $< C_3^* K_{1,n} >$ is a square graceful graph for $n \geq 3$. ■

**Example 2.8.** A square graceful labeling of $< C_3^* K_{1,8} >$ is shown in Figure 5.

![Square graceful labeling of $< C_3^* K_{1,8} >$](image)

**Figure 5:** Square graceful labeling of $< C_3^* K_{1,8} >$.

**Theorem 2.9.** The graph $< S_n : m \rangle$ is a square graceful graph.

**Proof:** Let $v_0, v_1, v_2, \cdots v_n$ be the vertices of the $j^{th}$ copy of the star $S_n$ in $< S_n : m \rangle$ where $v_0$ is the centre of the star where $1 \leq j \leq m$.

V($< S_n : m \rangle$) = \{v; v_j : 0 \leq i \leq n, 1 \leq j \leq m\}.

Let $E(< S_n : m \rangle) = \{v v_0 : 1 \leq j \leq m\}$

\[ v_0 v_i : 1 \leq i \leq n, 1 \leq j \leq m \]

Define an injection $f : V(< S_n : m \rangle) \rightarrow \{0, 1, 2, \ldots, (mn+m)^2\}$ by

\[ f(v) = 1 ; \quad f(v_0) = 0 ; \quad f(v_j) = j^2 + 1 \quad \text{if } 2 \leq j \leq m ; \]

\[ f(v_0) = (mn+m+1-i)^2 \quad \text{if } 1 \leq i \leq n ; \]

\[ f(v_i) = [mn+m+n+1-nj-i]^2 + j^2 + 1 \quad \text{if } 1 \leq i \leq n \text{ and } 2 \leq j \leq m . \]

Then, $f$ induces a bijection $f_p : E(< S_n : m \rangle) \rightarrow \{1, 4, 9, \ldots, (mn+m)^2\}$.

The induced edge labels of $< S_n : m \rangle$ are as follows:

\[ f_p(v v_0) = j^2 \quad \text{if } 1 \leq j \leq m ; \]

\[ f_p(v_0 v_i) = [mn+m+n+1-nj-i]^2 \quad \text{if } 1 \leq i \leq n, 1 \leq j \leq m . \]

Hence the graph $< S_n : m \rangle$ is a square graceful graph. ■
Example 2.10: A square graceful labeling of \([S_4:4]\) is shown in Figure 6.

![Figure 6: Square graceful labeling of \([S_4:4]\).](image)

Theorem 2.11: The graph \(\langle C_3, K_{1,n} \rangle\) is a square graceful graph.

Proof: Let \(V(\langle C_3, K_{1,n} \rangle) = \{u_i \mid 1 \leq i \leq 3\} \cup \{v_j \mid 1 \leq j \leq n+1\}\).

Take \(u_1 = v_{n+1}\). Let \(E(\langle C_3, K_{1,n} \rangle) = \{u_iu_2; u_iu_3; u_iu_j; u_jv_j \mid 1 \leq j \leq n\}\).

Define an injection \(f: V(\langle C_3, K_{1,n} \rangle) \to \{0,1,2,3,\ldots,(n+3)^2\}\) by:

\[
\begin{align*}
  f(u_1) &= 0; \\
  f(u_2) &= 16; \\
  f(u_3) &= 25; \\
  f(v_{n+1}) &= 4; \\
  f(v_n) &= 1; \\
  f(v_j) &= (n+3-j)^2 & \text{if } 1 \leq j \leq n-2.
\end{align*}
\]

Then, \(f\) induces a bijection \(f_p: E(\langle C_3, K_{1,n} \rangle) \to \{1,4,9,\ldots,(n+3)^2\}\).

The edge labels of \(\langle C_3, K_{1,n} \rangle\) are as follows:

\[
\begin{align*}
  f_p(u_1u_2) &= 16; \\
  f_p(u_1u_3) &= 25; \\
  f_p(u_1u_j) &= 9; \\
  f_p(u_jv_j) &= 4; \\
  f_p(u_1v_n) &= 1; \\
  f_p(u_jv_j) &= (n+4-j)^2 & \text{if } 1 \leq j \leq n-2.
\end{align*}
\]

Example 2.12: A square graceful labeling of \(\langle C_3, K_{1,8} \rangle\) and \(\langle C_3, K_{1,5} \rangle\) are shown in Figure 7(a) and Figure 7(b) respectively.

![Figure 7: Square graceful labeling of \(\langle C_3, K_{1,8} \rangle\) and \(\langle C_3, K_{1,5} \rangle\).](image)
References


