Spectral conditions for a graph to contain some subgraphs

Rao Li
Department of Mathematical Sciences
University of South Carolina Aiken
Aiken, SC 29801
USA.
raol@usca.edu

Abstract

In this paper, using the upper bound for the spectral radius for a graph obtained by Cao, we present sufficient conditions based on the spectral radius for a graph to contain some subgraphs.

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1 Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [2]. For a graph $G = (V, E)$, we use $n$ and $e$ to denote its order $|V|$ and size $|E|$, respectively. The largest and smallest degrees of a graph $G$ are denoted by $\Delta(G)$ and $\delta(G)$, respectively. The eigenvalues of a graph are defined as the eigenvalues of its adjacency matrix. The largest eigenvalue of a graph $G$, denoted $\rho(G)$, is called the spectral radius of $G$. If no confusion arises, we may drop $G$ for those invariants. We use $C_k$ to denote a cycle of length $k$. We also call $C_3$ as a triangle. The circumference of a graph is defined as the length of the longest cycle in the graph.

Cao [3] obtained the following upper bound for the spectral radius of a graph.

Theorem 1.1. [3] Let $G$ be a graph of order $n$ and size $e$ with minimum degree $\delta \geq 1$ and maximum degree $\Delta$. Then $\rho(G) \leq \sqrt{2e - \delta(n - 1) + (\delta - 1)\Delta}$ with equality if and only if $G$ is regular, a star plus copies of $K_2$, or a complete graph plus a regular graph with smaller degree of vertices.

2 Main Results

Using Theorem 1.1, Li [4] obtained sufficient conditions which are based on the spectral radius for some Hamiltonian properties of graphs. In this note, we use some of the ideas in [4] to obtain spectral conditions for a connected graph to contain some subgraphs.

Theorem 2.1. Let $G$ be a connected graph of order $n$ and size $e$. Suppose $k \geq 2$ is an integer. If $\rho > \sqrt{(1 - \frac{1}{k})n^2 - \delta(n - 1) + (\delta - 1)\Delta}$, then $G$ contains $K_{k+1}$. 

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**Proof:** Let $G$ be a connected graph satisfying the conditions in Theorem 2.1. Turán [6] proved that if a graph $G$ does not contain $K_{k+1}$ then $e \leq \left(1 - \frac{1}{k}\right) \frac{n^2}{2}$.

Suppose that $G$ does not contain $K_{k+1}$. Then, by Theorem 1.1, we have that

$$\rho \leq \sqrt{2e - \delta(n - 1) + (\delta - 1)\Delta} \leq \sqrt{\left(1 - \frac{1}{k}\right) n^2 - \delta(n - 1) + (\delta - 1)\Delta},$$

which is a contradiction. This completes the proof. 

Let $k = 2$ in Theorem 2.1. Then we have the following corollary.

**Corollary 2.2.** Let $G$ be a connected graph of order $n$ and size $e$. If $\rho > \sqrt{\frac{n^2}{2} - \delta(n - 1) + (\delta - 1)\Delta}$, then $G$ contains a triangle.

Let $H = K_{r,r}$, where $r \geq 2$. Then, for any $\epsilon > 0$,

$$r = \rho(H) > r - \epsilon = \sqrt{\frac{(n(H))^2}{2} - \delta(H)(n(H) - 1) + (\delta(H) - 1)\Delta(H) - \epsilon},$$

and $H$ does not contain a triangle. Thus Corollary 2.2 is best possible.

**Theorem 2.3.** Let $G$ be a connected graph of order $n$ and size $e$. Suppose $G$ is not bipartite. If

$$\rho > \sqrt{\frac{(n - 1)^2}{2} + 2 - \delta(n - 1) + (\delta - 1)\Delta},$$

then $G$ contains a triangle.

**Proof:** Let $G$ be a connected graph satisfying the conditions in Theorem 2.3. By Exercise 7.3.3(c) on Page 111 in [2], we have that if a non-bipartite graph $G$ does not contain a triangle then $e \leq \frac{(n-1)^2}{4} + 1$. Suppose that the non-bipartite graph $G$ does not contain a triangle. Then, by Theorem 1.1, we have

$$\rho \leq \sqrt{2e - \delta(n - 1) + (\delta - 1)\Delta} \leq \sqrt{\frac{(n - 1)^2}{2} + 2 - \delta(n - 1) + (\delta - 1)\Delta},$$

which is a contradiction. This completes the proof.

**Theorem 2.4.** Let $G$ be a connected graph of order $n$ and size $e$. If

$$\rho > \sqrt{\frac{n}{2}(1 + \sqrt{4n - 3}) - \delta(n - 1) + (\delta - 1)\Delta},$$

then $G$ contains $C_4$. 

Proof: Let \( G \) be a connected graph satisfying the given conditions. Reiman [5] proved that if a graph \( G \) does not contain \( C_4 \), then \( e \leq \frac{n}{4} (1 + \sqrt{4n - 3}) \). Suppose that \( G \) does not contain \( C_4 \). Then, by Theorem 1.1, we have that
\[
\rho \leq \sqrt{2e - \delta(n - 1) + (\delta - 1)\Delta} \leq \sqrt{\frac{n}{2} (1 + \sqrt{4n - 3}) - \delta(n - 1) + (\delta - 1)\Delta}
\]
which is a contradiction. This completes the proof.

Theorem 2.5. Let \( G \) be a connected graph of order \( n \) and size \( e \). If
\[
\rho > \sqrt{n\sqrt{(r - 1)n + \frac{n}{2} - \delta(n - 1) + (\delta - 1)\Delta}}
\]
then \( G \) contains \( K_{2,r} \) \( (r \geq 2) \).

Proof: Let \( G \) be a connected graph satisfying the given conditions. By Exercise 7.3.4(b) on Page 111 in [2], we have that if a graph \( G \) does not contain \( K_{2,r} \) \( (r \geq 2) \) then \( e \leq \frac{n\sqrt{(r-1)n}}{2} + \frac{n}{4} \).

Suppose that \( G \) does not contain \( K_{2,r} \). Then, by Theorem 1.1, we have
\[
\rho \leq \sqrt{2e - \delta(n - 1) + (\delta - 1)\Delta} \leq \sqrt{n\sqrt{(r - 1)n + \frac{n}{2} - \delta(n - 1) + (\delta - 1)\Delta}}
\]
which is a contradiction. This completes the proof.

Theorem 2.6. Let \( G \) be a connected graph of order \( n \) and size \( e \). If
\[
\rho > \sqrt{(r - 1)\frac{1}{2} n^{2 - \frac{1}{r}} + (r - 1)n - \delta(n - 1) + (\delta - 1)\Delta}
\]
then \( G \) contains \( K_{r,r} \).

Proof: Let \( G \) be a connected graph satisfying the given conditions. By Exercise 7.3.5 on Page 112 in [2], we have that if a graph \( G \) does not contain \( K_{r,r} \) then \( e \leq \frac{(r-1)^{\frac{1}{r}} n^{2 - \frac{1}{r}}}{2} + \frac{(r-1)n}{2} \).

Suppose that \( G \) does not contain \( K_{r,r} \). Then by Theorem 1.1, we have
\[
\rho \leq \sqrt{2e - \delta(n - 1) + (\delta - 1)\Delta} \leq \sqrt{(r - 1)\frac{1}{2} n^{2 - \frac{1}{r}} + (r - 1)n - \delta(n - 1) + (\delta - 1)\Delta}
\]
which is a contradiction. This completes the proof.

Theorem 2.7. Let \( G \) be a connected graph of order \( n \) and size \( e \). Suppose \( c \) satisfies \( 3 \leq c \leq n \). If
\[
\rho \geq \sqrt{(n - 1)(c - 1 - \delta) + (\delta - 1)\Delta + 2}
\]
then the circumference of \( G \) is at least \( c \).

Proof: Let \( G \) be a connected graph satisfying the given conditions. By Theorem 4.9 on Page 137 in [1], we have that if the circumference of a graph \( G \) is less than \( c \) then \( e < \frac{(c-1)(n-1)}{2} + 1 \).
Suppose that the circumference of $G$ is less than $c$. Then, by Theorem 1.1, we have

$$\rho \leq \sqrt{2e - \delta(n - 1)} + (\delta - 1)\Delta < \sqrt{(n - 1)(c - 1) + (\delta - 1)\Delta + 2},$$

which is a contradiction. This completes the proof.

**Theorem 2.8.** Let $G$ be a connected graph of order $n$ and size $e$. Suppose $c$ is the circumference of $G$. If

$$\rho > \sqrt{\frac{c(2n - c)}{2} - \delta(n - 1) + (\delta - 1)\Delta},$$

then $G$ contains $C_r$ for each $r$ with $3 \leq r \leq c$.

**Proof:** Let $G$ be a connected graph satisfying the given conditions. By Theorem 5.2 on Page 149 in [1], we have that if $G$ does not contain $C_r$ for some $r$ with $3 \leq r \leq c$ then $e \leq \frac{c(2n-c)}{4}$. Suppose that $G$ does not contain $C_r$ for some $r$ with $3 \leq r \leq c$. Then, by Theorem 1.1, we have that

$$\rho \leq \sqrt{2e - \delta(n - 1)} + (\delta - 1)\Delta \leq \sqrt{\frac{c(2n - c)}{2} - \delta(n - 1) + (\delta - 1)\Delta},$$

which is a contradiction. This completes the proof.

**Corollary 2.9.** Let $G$ be a connected graph of order $n$ and size $e$. Suppose $G$ is Hamiltonian. If

$$\rho > \sqrt{n^2/2 - \delta(n - 1) + (\delta - 1)\Delta},$$

then $G$ contains $C_r$ for each $r$ with $3 \leq r \leq n$.

Let $H = K_{r, r}$, where $r \geq 2$. Then $H$ is Hamiltonian and, for any $\epsilon > 0$,

$$r = \rho(H) > r - \epsilon = \sqrt{\frac{(n(H))^2}{2} - \delta(H)(n(H) - 1) + (\delta(H) - 1)\Delta(H)} - \epsilon,$$

and $H$ does not contain $C_s$ when $s$ is odd such that $3 \leq s \leq n(H)$. Thus Corollary 2.9 is possible.

**References**


