Cosplitting and co-regular graphs

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Abstract

The graph $S(G)$ obtained from a graph $G(V,E)$, by adding a new vertex $w$ for every vertex $v \in V$ and joining $w$ to all neighbours of $v$ in $G$, is called the splitting graph of $G$. The cosplitting graph $CS(G)$ is obtained from $G$, by adding a new vertex $w$ for each vertex $v \in V$ and joining $w$ to those vertices of $G$ which are not adjacent to $v$ in $G$. In this paper, we introduce the concept of cosplitting graph and characterise the graphs for which splitting and cosplitting graphs are isomorphic.

Keywords: Cosplitting graph, splitting graph, degree splitting graph, co–regular graph.

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1 Introduction

Throughout this paper, we consider only finite, simple and undirected graphs. For notations and terminology, we follow [2]. A graph $G$ is said to be $r$-regular if every vertex of $G$ has degree $r$. For $r \neq k$, a graph $G$ is said to be $(r,k)$-biregular if $d(v)$ is either $r$ or $k$ for any vertex $v$ in $G$. A $1$–factor of $G$ is a $1$–regular spanning subgraph of $G$ and it is denoted by $F$. For any vertex $v \in V$ in a graph $G(V,E)$, the open neighbourhood $N(v)$ of $v$ is the set of all vertices adjacent to $v$. That is, $N(v) = \{u \in V \mid uv \in E\}$. The closed neighbourhood $N[v]$ of $v$ is defined by $N[v] = N(v) \cup \{v\}$.

A vertex of degree one is called a pendant vertex. A vertex $v$ is said to be a $k$–regular adjacency vertex (or simply a $k$–RA vertex) if $d(u) = k$ for all $u \in N(v)$. A vertex is called an RA vertex if it is a RA vertex for some $k \geq 1$. A graph $G$ in which every vertex is an RA vertex, is said to be an RA graph. A full vertex of a graph $G$ is a vertex which is adjacent to all other vertices of $G$.

Let $G_1$ and $G_2$ be any two graphs. The graph $G_1 \circ G_2$ obtained from one copy of $G_1$ and $|V(G_1)|$ copies of $G_2$ by joining each vertex in the $i$th copy of $G_2$ to the $i$th vertex of $G_1$ is called the corona of $G_1$ and $G_2$.

The cartesian product of $G_1$ and $G_2$ is denoted by $G_1 \times G_2$, whereas, the join of $G_1$ and $G_2$ is denoted by $G_1 \vee G_2$. $\gamma(G)$ denotes the domination number of a graph $G$ and $\chi(G)$ denotes its chromatic number.
The concept of splitting graph was introduced by Sampath Kumar and Walikar [4]. The graph \( S(G) \), obtained from \( G \), by adding a new vertex \( w \) for every vertex \( v \in V \) and joining \( w \) to all vertices of \( G \) adjacent to \( v \), is called the splitting graph of \( G \). For example, a graph \( G \) and its splitting graph \( S(G) \) are shown in Figure 1.

![Figure 1: A graph G and its splitting graph S(G).](image1.png)

In [4], the following result has been proved.

**Result 1.1.** [4] A graph \( G \) is a splitting graph if and only if \( V(G) \) can be partitioned into two sets \( V_1 \) and \( V_2 \) such that there exists a bijective mapping \( f \) from \( V_1 \) to \( V_2 \) and \( N(f(v)) = N(v) \cap V_i \), for any \( v \in V_1 \).

On a similar line, Ponraj and Somasundaram [3] have introduced the concept of degree splitting graph \( DS(G) \) of a graph \( G \). For a graph \( G = (V, E) \) with vertex set partition \( V_i = \{v \in V \mid d(v) = i\} \), the degree splitting graph \( DS(G) \) is obtained from \( G \), by adding a new vertex \( w_i \) for each partition \( V_i \) that contains at least two vertices and joining \( w_i \) to each vertex of \( V_i \). For example, a graph \( G \) and its degree splitting graph \( DS(G) \) are shown in Figure 2.

![Figure 2: A graph G and its degree splitting graph DS(G).](image2.png)

It is obvious that every graph is an induced subgraph of \( DS(G) \). The following results on \( DS(G) \) have been proved in [1]:
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Result 1.2. [1] The degree splitting graph $DS(G)$ is regular if and only if $G \cong K_r$, $r \geq 1$ or $(K_{2k} - F) \lor K_1$, where $F$ is a 1-factor of $K_{2k}$ and $k \geq 1$.

If $K_{n,2n+1}$ is the complete bipartite graph with bipartition $(X,Y)$ where $X = \{ v_1, v_2, \ldots, v_n \}$ and $Y = \{ w_1, w_2, \ldots, w_{2n+1} \}$, then $K_{n,2n+1}^*$ denotes the graph obtained from $K_{n,2n+1}$ by deleting the edges $v_iw_{2i-1}$ and $v_iw_{2i}$ for all $i$, $1 \leq i \leq n$.

Result 1.3. [1] Let $G$ be a connected graph. Then $DS(G)$ is a biregular RA graph if and only if $G \cong K_{1,n}$ or $K_{n,2n+1}^*$, where $n \geq 2$.

Result 1.4. [1] For any $n \geq 2$, there are $n$ non isomorphic graphs whose degree splitting graphs are all isomorphic.

We define the cosplitting graph $CS(G)$ of a graph $G$ as follows:

Let $G$ be a graph with vertex set $\{v_1, v_2, \ldots, v_n\}$. The cosplitting graph $CS(G)$ is the graph obtained from $G$, by adding a new vertex $w_i$ for each vertex $v_i$ and joining $w_i$ to all vertices which are not adjacent to $v_i$ in $G$. For example, a graph $G$ and its cosplitting graph $CS(G)$ are shown in Figure 3.

**Figure 3:** A graph $G$ and its cosplitting graph $CS(G)$.

In this paper, we characterise the graphs for which the cosplitting graph is regular, biregular or bipartite. Also we give a necessary and sufficient condition for a graph to be a cosplitting graph. And finally we characterise the graphs for which the splitting graph and the cosplitting graph are isomorphic.

2 Properties of Cosplitting Graph

Let $K(m,n)$ denote the bipartite graph with vertex set bipartition $(X,Y)$ where $X = \{ u_1, u_2, \ldots, u_{m+n} \}$ and $Y = \{ v_1, v_2, \ldots, v_{m+n} \}$ and edge set $E(K(m,n)) = \{ u_i v_j / 1 \leq i \leq m \text{ and } 1 \leq j \leq m+n \} \cup \{ u_i v_j / 1 \leq i \leq m+n \text{ and } 1 \leq j \leq n \}$. For example, the graph $K(2,3)$ is shown in Figure 4.

**Figure 4:** The graph $K(2,3)$. 
For any graph $G$ of order $n$, clearly $CS(G)$ contains $2n$ vertices. Let $v_1, v_2, \ldots, v_n$ be the vertices of $G$ and $w_1, w_2, \ldots, w_n$ be the corresponding newly added vertices in $CS(G)$. Let $d'(v)$ and $d^*(v)$ denote the degrees of a vertex $v$ in $CS(G)$ and $S(G)$ respectively.

For the cosplitting graph $CS(G)$, the following results can be easily verified:

**Result 2.1.** $d'(v_i) = n$ and $d'(w_i) + d(v_i) = n$, for all $i$, $1 \leq i \leq n$.

**Result 2.2.** If $G$ has $n$ vertices and $m$ edges, then $CS(G)$ has $2n$ vertices and $n^2 - m$ edges.

**Result 2.3.** For a connected graph $G$, $1 \leq d'(w_i) \leq n - 1$. $d'(w_i) = 1$ implies that $v_i$ is a full vertex in $G$ and $d'(w_i) = n - 1$ implies that $v_i$ is a pendant vertex in $G$.

It is important to note that Result 2.3 is also true for any disconnected graph $G$ unless $G$ contains an isolated vertex. In other words, $d'(w_i) = n$ if and only if $v_i$ is an isolated vertex. Hence $\Delta(CS(G)) = n$. Also $CS(G)$ contains $n + m$ vertices of degree $n$, if and only if $G$ contains $m$ isolated vertices. Let them be denoted by $u_1, u_2, \ldots, u_m$. Note that in such case, $CS(G)$ contains $K_{m,n}$ as an induced subgraph. The removal of the $2m$ vertices that induces $K_{m,n}$ from $CS(G)$ results in a graph which is isomorphic to $CS(G \setminus \{u_1, u_2, \ldots, u_m\})$.

**Result 2.4.** $CS(K_n) \cong K_n \circ K_1$, $CS(K_n^e) \cong K_{n,n}$ and $CS(K_{m,n}) \cong K(m,n)$.

It is easy to observe that $G \circ K_1$ is a spanning subgraph of $CS(G)$ and $G \circ K_f = CS(G)$ if and only if $G \cong K_n$.

**Result 2.5.** Every graph $G$ is an induced subgraph of its cosplitting graph $CS(G)$.

**Result 2.6.** In $CS(G)$, the subgraph induced by the set of all vertices of degree $n$ is isomorphic to $G$.

**Result 2.7.** For any graph $G$, the cosplitting graph $CS(G)$ is always connected. But in case of splitting graph, $S(G)$ is connected if and only if $G$ is connected.

**Result 2.8.** The cosplitting graph $CS(G)$ is $r$-regular if and only if $G \cong K_r^c$.

**Result 2.9.** The cosplitting graph $CS(G)$ is $(r, n-r)$-biregular if and only if $G$ is an $r$-regular graph for any positive integer $r$.

**Result 2.10.** In the cosplitting graph of a connected graph, every newly added vertex that corresponds to a non-full vertex lies on at least one new cycle.

**Result 2.11.** For any graph $G$, $\chi(CS(G)) = \chi(G)$ or $\chi(G) + 1$.

The following theorem gives a characterisation of cosplitting graphs.

**Theorem 2.12.** A graph $G$ is a cosplitting graph if and only if $V(G)$ can be partitioned into two sets $V_1$ and $V_2$ such that there exists a bijection $f$ from $V_1$ to $V_2$ which satisfies the following conditions:

(i) $N(v) \cup N(f(v)) = V \setminus f(N(v))$ and

(ii) $N(v) \cap N(f(v)) = \emptyset$, for any $v \in V_1$. 
**Proof:** Let $G$ be a cosplitting graph of a graph $H$. To construct $G$ from $H$, we add a new vertex $w$ for each vertex $v$ of $H$ and join $w$ with every vertex of $H$ which is not adjacent to $v$. Let $V_1 = V(H)$ and $V_2 = V(G) \setminus V(H)$. For $v_i \in V_1$, let $w_i \in V_2$, be the corresponding newly added vertex where $1 \leq i \leq |V_1|$.

Now define a function $f : V_1 \to V_2$ by $f(v_i) = w_i$, $1 \leq i \leq |V_1|$. Then clearly $f$ is a bijection from $V_1$ onto $V_2$. Also by definition $N(f(v_i)) = V_1 \setminus N(v_i)$. Hence (ii) is proved. In $H$, each $v_i$ is adjacent not only to its neighbours in $G$, but also to all newly added vertices corresponding to its non-neighbours. Therefore we get $N(v_i) \cup N(f(v_i)) = V \setminus f(N(v_i))$.

Conversely, let the given conditions be true for a graph $G$. Let $H$ be the subgraph of $G$ induced by $V_1$. We claim that $CS(G) \cong G$. Since $f$ is bijective, it is clear that for every vertex $v_i$ in $H$, there is a unique vertex $f(v_i)$ in $G \setminus H$. Also by the assumptions (i) and (ii), $v_i$ and $f(v_i)$ are adjacent for every $i$, $1 \leq i \leq n$ and every vertex in $V_1$ is a neighbour of either $v_i$ or $f(v_i)$ but not both. Let us prove that $< G \setminus H >$ contains no edge. Suppose not, let $f(v_i)$ and $f(v_j)$ be adjacent for some $i \neq j$. Then by assumption (ii), $f(v_j) \notin N(v_i)$. In other words, $v_i \notin N(f(v_j))$ which implies that $v_i \notin N(v_j)$ which is a contradiction to (i) since $N(v_i) \cup N(f(v_j))$ does not contain any vertex of $f(N(v_j))$. Therefore $< G \setminus H >$ is a null graph. Hence if we consider $f(v_i)$ to be the corresponding newly added vertex for $v_i$, then $G$ is the cosplitting graph of $H$.

The following theorem characterises all bipartite cosplitting graphs.

**Theorem 2.13.** For any graph $G$, $CS(G)$ is bipartite if and only if $G \cong K_{m,n}$ or $K_{n}^C$.

**Proof:** Let $G$ be any graph for which $CS(G)$ is bipartite. Since $G$ is an induced subgraph of $CS(G)$, $G$ is also bipartite. Let $(X,Y)$ be the bipartition of $G$.

**Case (i):** Suppose $G$ is connected. Let $x \in X$ and $y \in Y$. We claim that $x$ and $y$ are adjacent in $G$. Suppose not, then there exists an $(x,y)$ - path $P$ of odd length in $G$. Also the newly added vertex $w$ corresponding to $x$, is adjacent to both $x$ and $y$ in $CS(G)$. Therefore the path $P$ together with the edges $xw$ and $wy$ forms a cycle of odd length in $CS(G)$, which is a contradiction. Therefore every $x \in X$ is adjacent to any $y \in Y$ in $G$ and we have $G \cong K_{m,n}$.

**Case (ii):** Suppose $G$ is disconnected. If $G \not\cong K_{n}^C$, then there is a component, say $G_1$ of $G$ containing at least one edge $xy$. Let $v$ be a vertex of $G$ not in $G_1$ and let $w$ be the newly added vertex corresponding to $v$ in $CS(G)$. Clearly $w$ is adjacent to both $x$ and $y$ in $CS(G)$. Thus $wxyw$ forms a triangle in $CS(G)$. This is a contradiction to the assumption that $CS(G)$ is bipartite. Hence $G \cong K_{n}^C$.

Conversely if $G \cong K_{m,n}$ or $K_{n}^C$, then $CS(G) \cong K(m,n)$ or $K_{n,n}$ respectively and hence the result follows.

**Corollary 2.14.** $CS(G)$ is a tree if and only if $G \cong K_{1,1}$ or $K_1$.

**Proof:** Suppose $CS(G)$ is a tree. Then $CS(G)$ is bipartite and $G$ is acyclic. Therefore, by the above theorem, $G \cong K_{1,1}$ or $K_1$. And the converse is obvious.

From the above corollary, $P_2$ and $P_4$ are the only cosplitting trees.
Next we prove that $K_n \circ K_1$ and $C_4$ are the only unicyclic cosplitting graphs.

**Theorem 2.15.** The cosplitting graph $CS(G)$ of a graph $G$ is unicyclic if and only if $G \cong K_3$ or $K_3^c$.

**Proof:** Let $G$ be any graph such that $CS(G)$ is unicyclic with the cycle $C$. Let $v_1, v_2, ..., v_n$ be the vertices of $G$ and $w_1, w_2, ..., w_n$ be the corresponding newly added vertices in $CS(G)$. Since $\{w_1, w_2, ..., w_n\}$ is independent, either $V(C) \subset V(G)$ or $w_i \in V(C)$ for some $i$.

**Case (i):** Suppose $V(C) \subset V(G)$.

It is clear that the cosplitting graph of a disconnected graph other than $K_3^c$ contains more than one triangle. Hence $G$ must be connected. Also by Result 2.10, every vertex of $G$ is a full vertex and therefore the newly added vertices do not form any new cycle. Hence, $G \cong K_3$.

**Case (ii):** Suppose $w_i \in V(C)$ for some $i$.

Then $G$ is acyclic and so every component of $G$ is a tree. Since $CS(G)$ is unicyclic, by Result 2.10 every component of $G$ contains only one non full vertex. This is possible only when $G$ is empty. If $G$ contains more than two isolated vertices, then $CS(G)$ is not unicyclic. Thus $G \cong K_3^c$.

Conversely, the cosplitting graphs of $K_3$ and $K_n^c$ are $K_3 \circ K_1$ and $C_4$ respectively which are unicyclic.

**Theorem 2.16.** No two non-isomorphic graphs can have the same cosplitting graph.

**Proof:** Suppose there are two non-isomorphic graphs $G_1$ and $G_2$ such that $CS(G_1) \cong CS(G_2)$.

**Case (i):** Suppose $G_1$ has no isolated vertex. Then by Result 2.3, no newly added vertex in $CS(G_1)$ is of degree $n$. Therefore the subgraph induced by the set of all vertices of degree $n$ in $CS(G_1)$ is isomorphic to $G_1$. Since $CS(G_1) \cong CS(G_2)$, we have $CS(G_2)$ also contains exactly $n$ vertices of degree $n$, and the subgraph induced by them is isomorphic to $G_2$. This implies that $G_1 \cong G_2$, a contradiction.

**Case (ii):** Let $G_1 = H_1 \cup K_n^c$, where $H_1$ contains no isolated vertex. Then $CS(G_1)$ contains $n + m$ vertices of degree $n$ and it contains $K_{n,m}$ as an induced subgraph. Since $CS(G_1) \cong CS(G_2)$, it is clear that $CS(G_2)$ also contains $n + m$ vertices of degree $n$. Therefore, $G_2 = H_2 \cup K_n^c$, for some graph $H_2$ which contains no isolated vertex. From Result 2.3, by removing $2m$ vertices that induces $K_{n,m}$ in $CS(G_1)$ and $CS(G_2)$, we get $CS(H_1)$ and $CS(H_2)$ respectively. This implies that $CS(H_1) \cong CS(H_2)$. Now using Case (i), we conclude that $H_1 \cong H_2$ and so $G_1 \cong G_2$, which is again a contradiction. Hence the result follows.

### 3 Co-regular Graphs

In this section, we define a new type of graphs called co-regular graphs and prove that co-regular graphs are the only graphs for which splitting and cosplitting graphs are isomorphic.

Let $G$ be a graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$. Then the **co-regular graph** of $G$ denoted by $CR(G)$ is the graph with vertex set $V(CR(G)) = \{u_1, u_2, ..., u_m, w_1, w_2, ..., w_n\}$ and edge set $E(CR(G)) = \{u_iw_j \mid v_iw_j \in E(G), i \neq j \text{ and } 1 \leq i, j \leq n\} \cup \{u_iw_j \mid uv_iw_j \not\in E(G) \text{ and } 1 \leq i, j \leq n\}$.

For example, a graph $G$ and its co-regular graph $CR(G)$ are shown in Figure 5.
The following results can be easily verified for a co-regular graph:

**Result 3.1.** A co-regular graph is an \( n \) - regular graph on \( 2n \) vertices.

**Result 3.2.** \( G \times P_2 \) is a spanning subgraph of \( CR(G) \). In particular, \( CR(K_n) = K_n \times P_2 \).

**Result 3.3.** \( CR(K_n^r) = K_n^r \cup K_n^c \cong K_{n,n} \).

**Result 3.4.** For any graph \( G \), \( CR(G) \) is connected.

For, if \( G \) is connected since \( G \times P_2 \) is a spanning subgraph of \( CR(G) \), then \( CR(G) \) is also connected. If \( G \) is disconnected, then every vertex in each component of one copy of \( G \) is adjacent to all vertices in the other components of another copy of \( G \) and hence \( CR(G) \) is connected.

**Result 3.5.** For any graph \( G \), \( \gamma(CR(G)) = 2 \).

For, \( CR(G) \) does not contain a full vertex and hence \( \gamma(CR(G)) \neq 1 \), and \( \{u_i, w_i\} \) is a minimum dominating set of \( CR(G) \) for any \( i, 1 \leq i \leq n \).

**Theorem 3.6.** A graph \( G \) is co-regular if and only if its vertex set can be partitioned into two element subsets \( \{u_i, w_i\}, 1 \leq i \leq n \), such that for any \( i, N(u_i) \) and \( N(w_i) \) form a partition of \( V(G) \), that is, such that \( N(u_i) \cup N(w_i) = V(G) \) and \( N(u_i) \cap N(w_i) = \emptyset \), for every \( i = 1, 2, \ldots, n \).

**Proof:** Let \( G \) be the co-regular graph of some graph \( H \). Let \( V(G) = \{u_1, u_2, \ldots, u_n, w_1, w_2, \ldots, w_n\} \) such that \( \langle\{u_i, u_2, \ldots, u_n\}\rangle \equiv \langle\{w_1, w_2, \ldots, w_n\}\rangle \equiv H \). Without loss of generality, let \( u_i \) be the isomorphic image of \( w_i \). Consider the pair \( \{u_i, w_i\} \). By the definition of co-regular graph, any vertex \( u_i, 1 \leq j \leq n, i \neq j \), is adjacent to either \( u_i \) or \( w_i \) but not both. Similar condition holds with any \( w_i \), \( 1 \leq j \leq n, i \neq j \). Since \( u_i \) and \( w_i \) are adjacent, \( u_i \in N(w_i) \) and \( w_i \in N(u_i) \). Therefore, the neighbor sets of \( u_i \) and \( w_i \) form a partition of \( V(G) \).

Conversely, suppose the vertex set of any graph \( G \) can be partitioned into two element subsets such that any vertex in \( G \) is a neighbour of any one vertex but not to both in each subset. Therefore \( G \) contains even number of vertices. Let \( V(G) = \{u_1, u_2, \ldots, u_n, w_1, w_2, \ldots, w_n\} \) such that \( \{u_i, w_i\}, \{u_2, w_2\}, \ldots, \{u_n, w_n\} \) be the partition of \( V(G) \).

First we claim that \( \langle\{u_1, u_2, \ldots, u_n\}\rangle \equiv \langle\{w_1, w_2, \ldots, w_n\}\rangle \). Suppose \( u_i \) is adjacent to \( u_s \). Then \( u_s \notin N(w_i) \) and hence \( w_i \in N(u_s) \). In a similar way, we prove that if \( u_i \) and \( u_s \) are non adjacent, then \( w_i \) and \( w_s \) are non adjacent. Since \( r \) and \( s \) are arbitrary, \( \langle\{u_1, u_2, \ldots, u_n\}\rangle \equiv \langle\{w_1, w_2, \ldots, w_n\}\rangle \equiv H \), say.
For $1 \leq i \leq n$, since $N(u_i) \cup N(w_i) = V(G)$, we have $u_i \in N(w_i)$. Hence, $u_i$ is adjacent to $w_i$. Also since $N(u_i) \cap N(w_i) = \emptyset$, both $u_i$ and $w_i$ have no common neighbours. Combining the two conditions we get $|N(u_i)| = |N(w_i)|$. Thus we conclude that $G = CR(H)$.

**Theorem 3.7.** Let $G$ be any graph of order $n$. Then $S(G) \cong CS(G)$ if and only if $G \cong CR(H)$ for some graph $H$.

**Proof:** Let $G$ be any graph of order $n$ such that its splitting graph $S(G)$ is isomorphic to its cosplitting graph $CS(G)$. Hence by Result 2.7, $G$ is connected. For any vertex $u$ in $G$, $d^*(u) = 2d(u)$ and $d'(u) = n$. Since $S(G) \cong CS(G)$, we have $d(u) = n/2$ for all $u \in V(G)$. That is, $G$ is an $n/2$ – regular graph on $n$ vertices.

Let $V(G) = \{u_1, u_2, ..., u_n\}$ and let $v_1, v_2, ..., v_n$ be the newly added vertices in $S(G)$. From the definition of splitting graph, for every vertex $v_i$, there exists a unique vertex $u_k \not\in N(v_i)$ in $G$ such that $N(u_k) \cap V(G) = N(v_i)$ by Result 1.1. Since $S(G) \cong CS(G)$, there will be a one to one correspondence between the newly added vertices in $S(G)$ and $CS(G)$. Therefore from the definition of cosplitting graph, corresponding to every $v_i$, there exists a unique vertex $u_m \in N(v_i)$ in $G$ such that $N(v_i) = V(G) \setminus N(u_m)$ by Theorem 2.12.

Combining the above two conditions we get $N(u_m) \cup N(u_k) = V(G), N(u_m) \cap N(u_k) = \emptyset$. Then clearly $u_k$ and $u_m$ are adjacent. Thus $u_k$ and $u_m$ are two adjacent vertices in $G$, whose neighbour sets form a partition of $V(G)$. In a similar manner, we can pair off vertices of $G$ such that each pair has distinct neighbour set whose union is $V(G)$ itself. Thus by the above theorem, $G$ is isomorphic to $CR(H)$ for some $H$.

Conversely, assume that $G$ is a co – regular graph of a graph $H$. Let $V(G) = \{u_1, u_2, ..., u_n, w_1, w_2, ..., w_n\}$ such that $\langle \{u_1, u_2, ..., u_n\} \rangle \cong \langle \{w_1, w_2, ..., w_n\} \rangle \cong H$. Without loss of generality, let $u_i$ be the isomorphic image of $w_i$. Let $a_1, a_2, ..., a_n, b_1, b_2, ..., b_n$ and $c_1, c_2, ..., c_n, d_1, d_2, ..., d_n$ be the newly added vertices in $S(G)$ and $CS(G)$ respectively corresponding to the vertices $u_1, u_2, ..., u_n, w_1, w_2, ..., w_n$. Then a function $f: S(G) \rightarrow CS(G)$ defined by $f(u_i) = u_i, f(w_i) = w_i, f(a_i) = c_i, f(b_i) = d_i, c_i = c_i$, where $1 \leq i \leq n$, can be easily verified to be an isomorphism. Hence the theorem is proved.

**References**


