A NOTE OF NEIGHBOR-TOUGHNESS OF GRAPHS

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Abstract. In this note, we point out some mistakes in Kürkçü and Aksan (2016, [2]). We also give the correct definition of neighbor-toughness. Finally, some examples, comments and generalized results related to the computation of the parameter are presented.

1. Introduction

Let \( G = (V, E) \) be a graph and \( u \in V(G) \). We call \( N(u) = \{v \in V(G) | u \neq v, u \) and \( v \) are adjacent\} the open neighborhood of \( u \), and \( N[u] = N(u) \cup \{u\} \) the closed neighborhood of \( u \). A vertex \( u \) of \( G \) is said to be subverted if its closed neighborhood \( N[u] \) is deleted from \( G \). A set of vertices \( S \subseteq V(G) \) is called a vertex subversion strategy of \( G \) if each of the vertices in \( S \) is subverted from \( G \). By \( G/S \) we denote the survival subgraph that remains after each vertex of \( S \) is subverted from \( G \). A vertex set \( S \) is called a cut strategy of \( G \) if the survival subgraph \( G/S \) is disconnected, or is a clique, or is empty.

Kürkçü and Aksan [2] claim that they introduce a new vulnerability parameter, neighbor-toughness. The parameter is defined as

\[ NT(G) = \min \{\frac{|S|}{\omega(G/S)} : \omega(G/S) \geq 1\}, \]

where \( S \) is any vertex subversion strategy of \( G \) and \( \omega(G/S) \) is the number of connected components in the graph \( G/S \). By two examples, the authors assert that the neighbor-toughness is a better parameter than the neighbor scattering.
number. This parameter, mentioned above is defined as \[ VNS(G) = \max_{S \subseteq V(G)} \{\omega(G/S) - |S|\}, \]
where the maximum is taken over all \( S \), the cut-strategy of \( G \), and \( \omega(G/S) \) is the number of components of \( G/S \).

We have sufficient reason to show that the above definition and statement in [2] are not proper. To the best of our knowledge, the concept of neighbor-toughness appeared firstly in [4]. In the next section, we will discuss and revise these items.

2. Main result

In 2013, Wei et al. [4] introduced the concept neighbor-toughness (for connected, non-complete graphs) as

\[ t_{VN}(G) = \min \left\{ \frac{|S|}{\omega(G/S)} \right\}, \]

where \( S \) is any cut strategy of \( G \) and \( \omega(G/S) \) is the number of components in \( G/S \). A set \( S^* \subseteq V(G) \) is called a \( t_{VN}\)-set of \( G \) if

\[ t_{VN}(G) = \frac{|S^*|}{\omega(G/S^*)}. \]

For the complete graph, subverting any one vertex will betray the entire graph, its neighbor-toughness is defined to be 0.

The mistake of the definition in [2] is that \( S \) should be a cut strategy instead of a vertex subversion strategy.

For example, consider the graph \( C_6 \) in Figure 1. By the definition in [2], \( \{u\} \) is a \( t_{VN}\)-set of \( C_6 \), since \( \frac{|\{u\}|}{\omega(G/\{u\})} = 1 < 2 = \frac{|\{u,v\}|}{\omega(G/\{u,v\})} \). But in [2], the authors show that \( t_{VN}(C_6) = 2 \), a contradiction. In fact, \( \{u\} \) is not a \( t_{VN}\)-set of \( C_6 \), because \( C_6/\{u\} \) is \( P_3 \), a connected graph. Obviously, \( \{u,v\} \) is a \( t_{VN}\)-set(cut strategy) of \( C_6 \) and \( t_{VN}(C_6) = 2 \).

On the other hand, consider the Petersen graph \( P(5, 2) \). Although \( \frac{|\{x\}|}{\omega(P(5, 2)/\{x\})} = 1 \), \( \{x\} \) is not a \( t_{VN}\)-set of \( P(5, 2) \), since \( P(5, 2)/\{x\} \) is \( C_6 \), a connected graph. By the definition of neighbor-toughness in [4], \( \{x, y\} \) is a real \( t_{VN}\)-set(cut strategy) of \( P(5, 2) \).

It can be concluded from the above discussion and [1, 6] that the definition of neighbor-toughness in [2] is wrong, and the definition in [4] is correct.
As two new graph parameters, neighbor-toughness and neighbor scattering number can be used to measure the invulnerability of spy networks. Undoubtedly, although formally related, they are independent. Which is a better parameter? It cannot be said simply by special examples. In fact, contrary to the author’s examples (see [2], $VNS(G_1) = VNS(G_2) = 1$, but $NT(G_1) = \frac{1}{4}$, $NT(G_2) = \frac{1}{2}$), there are more examples to show that neighbor scattering number is “better” than neighbor-toughness. Both of the following two graphs are with order 12, and they have equal connectivity and neighbor connectivity 1, as well as equal neighbor-toughness $\frac{1}{2}$, but $VNS(G_1) = 1$, $VNS(G_2) = 2$.

![Graphs G1 and G2](image)

At last, we generalize a result about the neighbor-toughness of bipartite graphs given in [2]. For a bipartite graph $K_{m,n}$, Kürkçü and Aksan prove that

$$t_{VN}(K_{m,n}) = \begin{cases} \frac{1}{m-1}, & \text{if } n < m; \\ \frac{1}{n-1}, & \text{if } n \geq m. \end{cases}$$

We show that the above formula is a corollary of the following theorem 2.1 (it is obvious, so we omit the proof).

**Theorem 2.1.** Let $K_{n_1,n_2,\ldots,n_k}$ be a complete $k$-partite graph, where $n_1 + n_2 + \cdots + n_k \geq k + 1$. Then

$$t_{VN}(K_{n_1,n_2,\ldots,n_k}) = \frac{1}{\max\{n_1-1,n_2-1,\ldots,n_k-1\}}.$$

A comet, denoted by $C_{n,k}$, is a graph by coincide an end point of path $P_{n-k}$ with the center point of a star $S_{1,k}$, where $1 \leq k \leq n - 2$ and $n \geq 4$. The order of comet $C_{n,k}$ is $n$.

**Theorem 2.2.** Let $C_{n,k}$ be a comet with order $n(\geq 5)$ and $k \leq n - 2$. Then

$$t_{VN}(C_{n,k}) = \begin{cases} \frac{1}{k+1}, & \text{if } k \leq n - 4; \\ \frac{1}{k^2}, & \text{if } k = n - 2 \text{ or } n - 3. \end{cases}$$

**Proof.** It is easy to know that the vertex in $P_{n-k}$ which is adjacent to the center of star $S_{1,k}$ is a $t_{VN}$-set of $C_{n,k}$. When $n \geq 5$ and $k \leq n - 4$, $n - k \geq 4$, the survival subgraph is a path $P_{n-k-3}$ with $k$ isolated vertex; when $k = n - 2$ or $n - 3$, the survival subgraph is $k$ isolated vertex, the conclusion holds. □

It is more meaningful to consider the neighbor-toughness computation of general graphs such as trees, Cartesian Product or composition of paths, cycles [1]. This is the work we are doing.
References


Received by editors 23.02.2018; Revised version 02.03.2018; Available online 12.03.2018.

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