LEFT ZEROID AND RIGHT ZEROID ELEMENTS
OF SEMIRINGS

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Abstract. In this paper, we introduce the notion of a left zeroid and a right
zeroid elements of semirings. We prove that a left zeroid $\mu$ of a simple semiring
$M$ is regular if and only if $M$ is a regular semiring and studied some of their
properties.

1. Introduction and Preliminaries

The notion of a semiring is an algebraic structure with two associative binary
operations where one of them distributes over the other, was first introduced by
Vandiver [6] in 1934 but semirings had appeared in studies on the theory of ideals of
rings. A universal algebra $S = (S, +, \cdot)$ is called a semiring if and only if $(S, +), (S, \cdot)$
are semigroups which are connected by distributive laws, i.e.,

$\quad a(b + c) = ab + ac$
$\quad (a + b)c = ac + bc$

for all $a, b, c \in S$. In structure, semirings lie between
semigroups and rings. The results which hold in rings but not in semigroups hold
in semirings, since semiring is a generalization of ring. The study of rings shows
that multiplicative structure of ring is an independent of additive structure whereas
in semiring multiplicative structure of semiring is not an independent of additive
structure of semiring. The additive and the multiplicative structure of a semiring
play an important role in determining the structure of a semiring. The theory of
rings and theory of semigroups have considerable impact on the development of
theory of semirings. Semirings play an important role in studying matrices and
determinants. Semirings are useful in the areas of theoretical computer science
as well as in the solution of graph theory, optimization theory, in particular for
studying automata, coding theory and formal languages. Semiring theory has many
applications in other branches.

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Clifford and Miller [3] studied zeroid elements in semigroups. Dawson [4] studied semigroups having left or right zeroid elements. The zeroid of a semiring was introduced by Bourne and Zassenhaus. In this paper, we extend the concept of left or right zeroid elements of semigroup to semiring. We prove that, a left zeroid \( \mu \) of a simple semiring \( M \) is regular if and only if \( M \) is a regular.

An element \( u \) of a semigroup \( M \) is called a zeroid element of \( M \) if, for each element \( a \) of \( M \), there exist \( x \) and \( y \) in \( M \) such that \( ax = ya = u \).

A semiring \( (M, +, \cdot) \) is said to be division semiring if \( (M \setminus 0, \cdot) \) is a group.

2. A left zeroid and a right zeroid elements in semirings.

In this section we introduce the notion of a left zeroid and a right zeroid elements in semirings and we study their properties.

Definition 2.1. An element \( x \) of a semiring \( M \) is called a left zeroid (right zeroid) if for each \( y \in M \), there exists \( a \in M \) such that \( ay = x \) (\( ya = x \)).

Theorem 2.1. Let \( M \) be a semiring with a left zeroid element \( x \) of \( M \) and an idempotent \( e \) of \( M \). Then \( xe = x \).

Proof. Let \( x \) be a left zeroid element and \( e \) be an idempotent of \( M \). Then there exists \( a \in M \) such that \( ae = x \). Therefore \( xe = aee = ae = x \).

Corollary 2.1. Let \( M \) be a semiring with a right zeroid element \( x \) and an idempotent \( e \). Then \( ex = x \).

Theorem 2.2. Let \( M \) be a semiring and \( e \) be a left zeroid element of \( M \). Then \( xe \) is a left zeroid of \( M \) for all \( x \in M \).

Proof. Suppose \( x \in M \) is a left zeroid, then there exists \( b \in M \) such that \( bx = e \), since \( e \) is a left zeroid of \( M \). Thus \( xty = xe \). Hence \( xe \) is a left zeroid of \( M \).

Corollary 2.2. Let \( M \) be a semiring, \( e \) be a left zeroid of \( M \). Then every element of \( Me \) is a left zeroid of \( M \).

Definition 2.2. Let \( M \) be a semiring and \( a \in M \). If there exists \( b \in M \) such that \( b + a = b \) (\( a + b = b \)) then \( a \) is said to be additively left(right) zeroid of \( M \).

Theorem 2.3. Let \( M \) be a semiring with identity \( a + ab = a \) for all \( a, b \in M \). If \( x \) is a left zeroid of \( M \), then \( x \) is an additively left zeroid of \( M \).

Proof. Suppose \( x \in M \) is a left zeroid, then there exists \( b \in M \) such that \( bc = x \). Thus \( b + bc = b + x \) and \( b = b + x \). Therefore \( x \) is an additively left zeroid. Hence the Theorem.

Theorem 2.4. Let \( M \) be a semiring with identity \( a + ab = a \) for all \( a, b \in M \) and \( (M, +) \) be left cancellative. If \( x \) is an additively left zeroid of \( M \), then \( x \) is a left zeroid of \( M \).

Proof. Suppose \( x \) is an additively left zeroid of \( M \). Then there exists \( b \in M \) such that \( b = b + x \). Thus \( b + bc = b + x \) for all \( c \in M \) and \( bc = x \). Hence the Theorem.
Theorem 2.5. Let \( M \) be a semiring. If semiring \( M \) has both a left zeroid and a right zeroid. Then every left or right zeroid of \( M \) is a zeroid of \( M \).

Proof. Suppose \( \mu \) and \( \mu' \) are a left zeroid a right zeroid of \( M \) respectively. Then there exist \( y, z \in M \) such that \( yx = \mu \) and \( xz = \mu' \). Thus \( xzy = \mu'y \) and \( xzyx = \mu'^{\prime}y \). Finally \( xzyx = \mu'\mu \). Hence \( \mu'\mu \) is a zeroid of \( M \). Similarly we can prove \( \mu\mu' \) is a zeroid.

Let \( x \) be a left zeroid of \( M \). Then there exists \( a \in M \) such that \( ax = \mu \). Therefore \( x \) is a right zeroid. Since \( (a\mu)\mu' = x \). Hence the Theorem.

Theorem 2.6. If \( e \) is an idempotent of a semiring \( M \) then \( e \) is the left identity of \( eM \) and \( e \) is the right identity of \( Me \).

Proof. Let \( aex \in eM \). Then \( eex = ex \). Hence \( e \) is the left identity of \( eM \). Similarly we can prove \( e \) is the right identity of \( Me \).

Theorem 2.7. If \( e \) is an idempotent left zeroid of a semiring \( M \) then \( eM \) is a division semiring.

Proof. Obviously \( eM \) is a subsemiring of \( M \) and \( e \) is the left identity of \( eM \). Suppose \( eb \in eM \) there exists \( c \in M \) such that \( ec(eb) = e \). Thus \( (ec)(eb) = ee \). Therefore \( (ec)(eb) = e \). Finally, \( e \) is the left zeroid of \( eM \). Hence \( ec \) is the left inverse of \( eb \). Thus \( eM \) is a division semiring.

Theorem 2.8. Let \( U \) be a non empty set of all left zeroids of semiring \( M \). Then \( U \) is a left ideal of \( M \).

Proof. Suppose \( x_1, x_2 \in U \), \( a \in M \) and \( x \in M \). Then there exist \( y, z \in M \) such that \( yx = x_1 \) and \( zx = x_2 \). Thus \( (y+z)x = x_1 + x_2 \). Therefore \( x_1 + x_2 \) is a left zeroid of \( M \). By Theorem 2.2, \( ax_1 \) is a left zeroid of \( M \). Hence \( U \) is a left ideal of \( M \).

Corollary 2.3. Let \( M \) be a semiring. If \( M \) has a left zeroid and right zeroid. Then \( U \) is an ideal of \( M \).

Corollary 2.4. Let \( M \) be a simple semiring. If \( M \) has a left zeroid and a right zeroid then every element of \( M \) is a zeroid.

Theorem 2.9. Let \( M \) be a semiring and \( e \) be an idempotent left zeroid of \( M \). Then a mapping \( f : M \to eM \), defined by \( f(x) = ex \) is an onto homomorphism.

Proof. Let \( x_1, x_2 \in M \). Then
\[
f(x_1 + x_2) = e(x_1 + x_2) = ex_1 + ex_2 = f(x_1) + f(x_2)
\]
\[
f(x_1x_2) = e(x_1x_2) = (ex_1)x_2 = (ex_1)(ex_2) = f(x_1)f(x_2).
\]
Hence \( f \) is a homomorphism from \( M \) to \( eM \). Obviously \( f \) is onto. Hence the Theorem.

Theorem 2.10. If \( e \) is an idempotent left zeroid of a semiring \( M \) then \( Me \) is a regular semiring.
Proof. Obviously $e$ is a right identity of $Me$. Suppose $xe \in Me$. There exists $g \in M$ such that $gxe = e$ and $e = ee = e(ge) = (eg)(ze)$. Therefore $e$ is a left zeroid of $Me$. Suppose $x \in Me$ then there exists $y \in Me$ such that $yx = e$. Then $xxy = xx = x$. Thus $Me$ is a regular semiring. \[ \square \]

Theorem 2.11. Let $M$ be a semiring. If $e$ is the only idempotent of $M$, which is a left zeroid of $M$ then $e$ is a zeroid of $M$.

Proof. Let $e$ be the only idempotent of a semiring $M$, which is a left zeroid of $M$. Then by Theorem 2.10, $Me$ is regular. Suppose $b \in Me$. Then there exists $x \in Me$ such that $b = bx$. Therefore $bx$ is an idempotent of $M$. Hence $bx = e$. Each element of $Me$ has right inverse and $e$ is a right identity of $Me$. Therefore $Me$ is a division semiring.

Let $c \in M$, then $ce \in Me$. There exists $de \in Me$, such that $(ce)(de) = e$. Then $c(ede) = e$. Therefore $e$ is a right zero ideal of $M$. Thus $e$ is a zeroid of $M$. \[ \square \]

We define a relation $\leq$ on the non-empty set of idempotents of a semiring $M$ as follows: $e \leq f \iff ef = e$.

Theorem 2.12. Let $M$ be a semiring. If $e$ is a unique least idempotent and the left (right) zeroid of $M$ then $e$ is a zeroid of $M$.

Proof. Suppose $e$ is the least idempotent and the left zeroid of $M$. Let $M$ contains an idempotent $f$, which is a left zeroid of $M$. By Theorem 2.1, $fe = f$. Then $f \leq e$. Since $e$ is the unique least idempotent, we have $f = e$. Therefore by Theorem 2.11, $e$ is a zeroid of $M$.

Suppose that $e$ is a right zeroid of $M$. Let $M$ contains an idempotent $f$ which is a right zeroid of $M$. By Corollary 2.1, we have $fe = f$. Therefore $f \leq e$. Hence $e = f$. Thus $e$ is the only idempotent of $M$ which is a right zeroid of $M$. By Theorem 2.11, $e$ is a zeroid of $M$. Hence the Theorem. \[ \square \]

Theorem 2.13. A semiring $M$ with a left zeroid $\mu$ contains a left zeroid idempotent if and only if $\mu$ is a regular of $M$.

Proof. Suppose left zeroid $\mu$ is regular element of $M$. Then there exists $x \in M$ such that $\mu = \mu x$. Then $x \mu = \mu x \mu$. Hence $x \mu$ is a left zeroid idempotent. Conversely suppose that $e$ is a left zeroid idempotent of $M$. We can prove $e$ is a left zeroid of $M \mu$. By Theorem 2.10 $M \mu e$ is regular. Therefore $M \mu e = M(\mu e) = M \mu$. Hence $M \mu$ is regular. Thus $\mu$ is regular. \[ \square \]

Theorem 2.14. Let $M$ be a simple semiring. Then a left zeroid $\mu$ of a simple semiring $M$ is regular if and only if $M$ is a regular semiring.

Proof. Suppose $M$ is a simple semiring with a regular left zeroid $\mu$ of $M$. Since $\mu$ is regular, there exists $x \in M$ such that $\mu = \mu x$. Then $x \mu$ is an idempotent of $M$. Suppose $b \in M$. Then there exists $c \in M$ such that $cb = \mu$ and there exists $d \in M$ such that $dc = \mu$. Then $\mu b = deb = d \mu$. Therefore $M \mu b = M(d \mu) = (Md) \mu \subseteq M \mu$. Thus $M \mu$ is a right ideal of $M$. Obviously $M \mu$ is a left ideal of $M$. Hence $M \mu = M$, since $M$ is simple. Every element of $M$ is a left zeroid of $M$. Thus
$x\mu$ is a left zeroid idempotent of $M$. by Theorem 2.10 $Mx\mu$ is regular. We have $Mx\mu = M\mu x\mu = M\mu = M$.

Converse is obvious. \hfill \Box

References


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