ON EXPONENTIAL BOUNDS OF HYPERBOLIC COSINE

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Abstract. In this note, natural exponential bounds for $\cosh x$ are established. The inequalities thus obtained are interesting and sharp.

1. Introduction

The well-known Lazarević inequality [1, 2] states that

\begin{equation}
\cosh x < \left( \frac{\sinh x}{x} \right)^p ; x > 0 \text{ if and only if } p \geq 3.
\end{equation}

Chen, Zhao and Qi [3] obtained the inequality -

\begin{equation}
\cosh x \leq \left( \frac{\pi^2 + 4x^2}{\pi^2 - 4x^2} \right) ; x \in [0, \pi/2).
\end{equation}

which is Redheffer - type [4].

The inequality (1.2) later was generalised and sharpened by Zhu and Sun [5] as follows -

\begin{equation}
\left( \frac{r^2 + x^2}{r^2 - x^2} \right)^\alpha \leq \cosh x \leq \left( \frac{r^2 + x^2}{r^2 - x^2} \right)^\beta \text{ for } 0 \leq x < r
\end{equation}

if and only if $\alpha \leq 0$ and $\beta \geq \frac{r^2}{1}$.

Below are the bounds of $\cosh x$ given in [6] -

\begin{equation}
\left( \frac{1}{\cos x} \right)^{2/3} < \cosh x < \frac{1}{\cos x} ; x \in (0, \pi/4).
\end{equation}
Yupei Lv, Wang et al. [7] give the refinement of (1.4) as follows -

\[(1.5) \quad \left( \frac{1}{\cos x} \right)^a < \cosh x < \frac{1}{\cos x}; \quad x \in (0, \pi/4) \quad \text{and} \quad a \approx 0.811133.\]

For \(x \in (0, 1)\) the following inequality [6, 8] -

\[(1.6) \quad \frac{3}{3 - x^2} \leq \cosh x \leq \frac{2}{2 - x^2}\]

holds.

In this paper, we shall obtain more sharp bounds than given in the above inequalities (1.1) - (1.6) for \(\cosh x\) by using natural exponential function.

2. Main Results

We obtain our main results by using the following l’Hôpital’s Rule of Monotonicity [9, Thm. 1.25] -

**Lemma 2.1.** Let \(f, g : [a, b] \to \mathbb{R}\) be two continuous functions which are differentiable on \((a, b)\) and \(g' \neq 0\) in \((a, b)\). If \(f'/g'\) is increasing (or decreasing) on \((a, b)\), then the functions \(\frac{f(x) - f(a)}{g(x) - g(a)}\) and \(\frac{f(x) - f(b)}{g(x) - g(b)}\) are also increasing (or decreasing) on \((a, b)\). If \(f'/g'\) is strictly monotone, then the monotonicity in the conclusion is also strict.

Now we give our Main results.

**Theorem 2.1.** If \(x \in (0, 1)\) then

\[(2.1) \quad e^{ax^2} < \cosh x < e^{x^2/2}\]

with the best possible constants \(a \approx 0.433781\) and \(1/2\).

**Proof.** Let \(e^{ax^2} < \cosh x < e^{bx^2}\), which implies that, \(a < \frac{\log(\cosh x)}{x^2} < b\).

Then \(f(x) = \frac{\log(\cosh x)}{x^2} = \frac{f_1(x)}{f_2(x)}\),

where \(f_1(x) = \log(\cosh x)\) and \(f_2(x) = x^2\) with \(f_1(0) = f_2(0) = 0\). By differentiation we get

\[\frac{f'_1(x)}{f'_2(x)} = \frac{\tanh x}{2x} = \frac{f_3(x)}{f_4(x)}\]

where \(f_3(x) = \tanh x\) and \(f_4(x) = 2x\), with \(f_3(0) = f_4(0) = 0\). Again differentiation gives us -

\[\frac{f'_3(x)}{f'_4(x)} = \frac{\text{sech}^2 x}{2},\]

which is clearly strictly decreasing in \((0, 1)\). By Lemma 2.1, \(f(x)\) is strictly decreasing in \((0, 1)\). Consequently, \(a = f(1) = \log(\cosh 1) \approx 0.4333781\) and \(b = f(0+) = 1/2\) by l’Hôpital’s Rule. \(\square\)
Remark 2.1. For $-r < x < r$,
\begin{equation}
(2.2) \quad e^{Ax^2} < \cosh x < e^{x^2/2}, \text{ where } A = \frac{\log(\cosh r)}{r^2}.
\end{equation}

Proof. For any $r > 0$, clearly $\text{sech}^2 x$ is strictly increasing in $(-r, 0)$ and strictly decreasing in $(0, r)$. Applying Lemma 2.1, we get, $A \approx \log(\cosh r)/r^2$. \qed

For the application of Thm. 2.1, we give another proof of the following theorem [6, Thm.1.2]:

**Theorem 2.2.** If $x \in (0, 1)$ then
\begin{equation}
(2.3) \quad \frac{1}{\cosh x} < \frac{x^2}{\sinh^2 x} < \left(\frac{1}{\cosh x}\right)^{1/2}.
\end{equation}

Proof. As $e^{-ax^2} < e^{-x^2/3}$, for $a \approx 0.433781$ and by theorem 3 in [10] -
\[ e^{-x^2/3} < \frac{x^2}{\sinh^2 x} < e^{-bx^2} \]
where $x \in (0, 1)$ and $b \approx 0.322878$. Using these inequalities with (2.1), it is clear that -
\[ \frac{1}{\cosh x} < \frac{x^2}{\sinh^2 x} < e^{-bx^2} < e^{-x^2/4} < \left(\frac{1}{\cosh x}\right)^{1/2}. \]
This completes the proof. \qed

References


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