ON YANG MEANS III

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Abstract. Optimal inequalities involving the p-Yang means are established. Bounding quantities are either the arithmetic or geometric or the harmonic combinations of the p-geometric and the p-quadratic means.

1. Introduction

Recently Z. -H Yang [24] introduced two bivariate means denoted in the sequel by $V$ and $U$. For the sake of presentation we include below explicit formulas for these means.

Throughout the sequel the letters $a$ and $b$ will stand for two positive and unequal numbers. The Yang means of $a$ and $b$ are defined as follows:

\begin{equation}
V(a, b) = \frac{a - b}{\sqrt{2} \sinh^{-1} \left( \frac{a - b}{\sqrt{2}ab} \right)}
\end{equation}

and

\begin{equation}
U(a, b) = \frac{a - b}{\sqrt{2} \tan^{-1} \left( \frac{a - b}{\sqrt{2}ab} \right)}.
\end{equation}

These means have been studied extensively in [24, 25] and recently in [16, 17].

This paper is a continuation of a research initiated in [16, 17] and is organized as follows. Definitions of other bivariate means utilized in this work are given in Section 2. List of those means include two Seiffert means, logarithmic mean, Neuman-Sándor mean and the Schwab-Borchardt mean $SB$. The latter plays a crucial role in our presentation. Concept of the p-mean is recalled in Section 3.

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Therein we include some facts about the p-means. Optimal inequalities involving the p-means $U_p$ and $V_p$ which are derived, respectively, from the Yang means $V$ and $U$ established in Section 4. The lower and upper bounds are either the convex arithmetic combination or the convex geometric combination or the convex harmonic combination of the p-means derived from the geometric and the quadratic means of $a$ and $b$.

2. Definitions and preliminaries

Recall that the unweighted arithmetic mean of $a$ and $b$ is defined as

\[
A = \frac{a + b}{2}.
\]

Other unweighted bivariate means used in this paper are the harmonic mean $H$, geometric mean $G$, root-square mean (quadratic mean) $Q$ and the contra-harmonic mean $C$ which are defined as follows (cf. [2])

\[
H = \frac{2ab}{a + b}, \quad G = \sqrt{ab}, \quad Q = \sqrt{\frac{a^2 + b^2}{2}}, \quad C = \frac{a^2 + b^2}{a + b}.
\]

Let

\[
v = \frac{a - b}{a + b}.
\]

Clearly $0 < |v| < 1$. One can easily verify that the means defined in (2.1) all can be expressed in terms of $A$ and $v$. We have

\[
H = A(1 - v^2), \quad G = A\sqrt{1 - v^2},
\]

\[
Q = A\sqrt{1 + v^2}, \quad C = A(1 + v^2).
\]

Other bivariate means utilized in this paper include the first and the second Seiffert means, denoted by $P$ and $T$, respectively, the Neuman-Sándor mean $M$, and the logarithmic mean $L$. Recall that

\[
P = A\frac{v}{\sinh^{-1}v}, \quad T = A\frac{v}{\tanh^{-1}v},
\]

\[
M = A\frac{v}{\sin^{-1}v}, \quad L = A\frac{v}{\tan^{-1}v}
\]

(see [21], [22], [19]). All the means mentioned above are comparable. It is known that

\[
H < G < L < P < A < M < T < Q < C
\]

(see, e.g., [19]).
The four means listed in (2.4) are special cases of the Schwab-Borchardt mean $SB$ which is defined as follows

$$SB(a,b) \equiv SB = \begin{cases} \sqrt{b^2 - a^2} \cos^{-1}(a/b) & \text{if } a < b, \\ \sqrt{a^2 - b^2} \cosh^{-1}(a/b) & \text{if } b < a. \end{cases}$$

(see, e.g., [1], [3]). This mean has been studied extensively in [11], [19], and [20]. It is well known that the mean $SB$ is strict, nonsymmetric and homogeneous of degree one in its variables.

It has been proven in [19] that

$$(2.6) \quad P = SB(G,A), \quad T = SB(A,Q), \quad M = SB(Q,A), \quad L = SB(A,G).$$

Yang means can also be represented in terms of the Schwab-Borchardt mean. We have

$$(2.7) \quad V = SB(Q,G) \quad \text{and} \quad U = SB(G,Q)$$

(see [16]).

The following chain of inequalities

$$(2.8) \quad L < V < P < U < M < T$$

is known (see [16]).

For the sake of presentation we include new formulas for means $SB$. We have

$$(2.9) \quad SB(x,y) \equiv SB = \begin{cases} \frac{\sin r}{r} = \frac{x \tan r}{r} & \text{if } 0 \leq x < y, \\ \frac{\sinh s}{s} = \frac{x \tanh s}{s} & \text{if } y < x, \end{cases}$$

where

$$(2.10) \quad \cos r = x/y \quad \text{if } x < y \quad \text{and} \quad \cosh s = x/y \quad \text{if } x > y.$$

Clearly

$$0 < r \leq r_0, \quad \text{where} \quad r_0 = \max\{\cos^{-1}(x/y) : 0 \leq x < y\}$$

and

$$0 < s \leq s_0, \quad \text{where} \quad s_0 = \max\{\cosh^{-1}(x/y) : x > y > 0\}.$$
3. Definition and basic properties of the $p$-means

We begin this section with a simple construction of a family of bivariate means which depend on the parameter $p$ which satisfies $|p| \leq 1$. This idea has been introduced in author’s paper [13].

First two nonnegative numbers $w_1$ and $w_2$ are defined as follows:

\[(3.1) \quad w_1 = \frac{1+p}{2}, \quad w_2 = \frac{1-p}{2}.\]

Clearly $w_1 + w_2 = 1$. We associate with the pair $(a, b)$ a pair of positive numbers $(x, y)$, where

\[(3.2) \quad x = w_1 a + w_2 b, \quad y = w_1 b + w_2 a.\]

Thus $x$ and $y$ are the convex combinations of $a$ and $b$. One can easily verify that $a < x < y < b$ if $a < b$ or $b < y < x < a$ if $b < a$.

For the sake of presentation let $N$ stand for a bivariate symmetric mean. We define a mean $N_p(a, b) \equiv N_p$ as follows

\[(3.3) \quad N_p(a, b) = N(x, y).\]

We call the mean $N_p$ the $p$-mean or the $p$-mean generated by $N$.

For the reader’s convenience we present now some elementary properties of the $p$-means. It follows (3.3), (3.1), and (3.2) we see that

\[(3.4) \quad N_0 = A; \quad N_1 = N;\]

Moreover, the function $p \rightarrow N_p$ is strictly decreasing if $N < A$, i.e.,

\[(3.5) \quad N_1 \leq N_p \leq N_0\]

or is strictly increasing if $N > A$, i.e.,

\[(3.6) \quad N_0 \leq N_p \leq N_1.\]

We now present formulas for the $p$-means. Let us begin with the case when $N = A$. We have

\[A_p = A_p(a, b) = A(x, y) = A.\]

Thus we shall always write $A$ instead of $A_p$ when no confusion would arise. To obtain the $p$-versions of the four means listed in (2.3) let us introduce a quantity $u$, where

\[(3.7) \quad u = \frac{x - y}{x + y}.\]

Using (3.2) and (2.2) we obtain

\[(3.8) \quad u = pv.\]

Since $0 < |v| < 1$, $0 < |u| < p \leq 1$
Formulas for the p-means derived from means listed in (2.3) read as follows

\begin{align}
H_p &= A(1 - u^2), \quad G_p = A\sqrt{1 - u^2}, \\
Q_p &= A\sqrt{1 + u^2}, \quad C_p = A(1 + u^2).
\end{align}

Similarly, using (2.4) we obtain

\begin{align}
P_p &= A\frac{u}{\sin^{-1}u}, \quad T_p = A\frac{u}{\tan^{-1}u}, \\
M_p &= A\frac{u}{\sinh^{-1}u}, \quad L_p = A\frac{u}{\tanh^{-1}u}.
\end{align}

It is worth mentioning that the means \(P_p, T_p, M_p\), and \(L_p\) can be represented as the Schwab-Borchardt means. Making use of (2.6) and (2.7) we obtain

\begin{align}
P_p &= SB(G_p, A), \quad T_p = SB(A, Q_p), \\
M_p &= SB(Q_p, A), \quad L_p = SB(A, G_p), \\
V_p &= SB(Q_p, G_p), \quad U_p = SB(G_p, Q_p).
\end{align}

For this reason we call \((G_p, A), (A, Q_p), (Q_p, A), (A, G_p), (Q_p, G_p)\) and \((G_p, Q_p)\) the pairs of generating means.

We close this section with the following remarks. The idea of using the p-means was motivated by a recent development in theory of means. Let \(R\) and \(S\) be bivariate symmetric means and let \(0 \leq \lambda \leq 1\). Many researchers (see, e.g., [4], [5], [6], [7], [8], [9], [10], [23]) have studied problems of finding all values of \(\lambda\) for which inequality \(R(\lambda a + (1 - \lambda)b) < S(r, s)\) is satisfied for all positive numbers \(r\) and \(s\). Let us note that with \(\lambda = (1 + p)/2 = w_1\) we have \(1 - \lambda = (1 - p)/2 = w_2\). Thus the inequality in question can be written as \(R_p(r, s) < S(r, s)\). With the parameter \(\lambda\) used instead of \(p\) formula (3.8) should be changed \(u = (2\lambda - 1)v\), which is a little bit more cumbersome in analytic computations than (3.8) is.

4. Main results

The goal of this section is to determine coefficients of six optimal convex combinations which form both lower and upper bounds for the p-means \(U_p(a, b) \equiv U_p\) and \(V_p(a, b) \equiv V_p\). Convex combinations employed here involve the p-means \(Q_p\) and \(G_p\) of positive and unequal numbers \(a\) and \(b\). Our first result reads as follows:

**Theorem 4.1.** The two-sided inequality

\[
\alpha_1 Q_p + (1 - \alpha_1)G_p < U_p < \beta_1 Q_p + (1 - \beta_1)G_p
\]

holds true provided

\[
\alpha_1 \leq \frac{2}{\pi} \quad \text{and} \quad \beta_1 \geq \frac{2}{3}.
\]

**Proof.** Taking into account that \(G_p < Q_p\) one can rewrite (4.1) as

\[
\alpha_1 < \frac{U_p/Q_p - G_p/Q_p}{1 - G_p/Q_p} < \beta_1.
\]
Since $U_p = SB(G_p, Q_p)$ (see (3.11)) we get using (2.9) and (2.10)

$$\frac{U_p}{Q_p} = \frac{\sin r}{r} \quad \text{and} \quad \frac{G_p}{Q_p} = \cos r,$$

where $0 \leq r \leq \pi/2$. This in conjunction with (4.3) yields

$$\alpha_1 < \Phi_1(r) < \beta_1,$$

where

$$\Phi_1(r) = \frac{\sin r - r \cos r}{r(1 - \cos r)}.$$

It follows from Theorem 3 in [14] that the function $\Phi_1(r)$ is strictly decreasing on its domain. Moreover,

$$\Phi_1(0^+) = \frac{2}{3} \quad \text{and} \quad \Phi_1\left(\frac{\pi}{2}\right).$$

Hence the assertion follows.

A counterpart of the last theorem for the mean $V_p$ reads as follows:

**Theorem 4.2.** The two-sided inequality

$$\gamma_1 Q_p + (1 - \gamma_1)G_p < V_p < \delta_1 Q_p + (1 - \delta_1)G_p$$

holds true provided

$$\gamma_1 \leq 0 \quad \text{and} \quad \delta_1 \geq \frac{1}{3}.$$

**Proof.** We follow the lines used in the proof of the last theorem. Firstly, we rewrite (4.4) as follows

$$\gamma_1 \frac{V_p}{Q_p} + (1 - \gamma_1) \frac{G_p}{Q_p} < V_p < \delta_1 \frac{V_p}{Q_p} + (1 - \delta_1) \frac{G_p}{Q_p}$$

Since $V_p = SB(Q_p, G_p)$ (see (3.11)) we get using (2.9) and (2.10)

$$\frac{V_p}{G_p} = \frac{\sinh s}{s} \quad \text{and} \quad \frac{Q_p}{G_p} = \cosh s,$$

where $0 \leq s < \infty$. This in conjunction with (4.6) yields

$$\gamma_1 < \Psi_1(s) < \delta_1,$$

where

$$\Psi_1(s) = \frac{\sinh s - s}{s(\cosh s - 1)}.$$

It follows from [12] that the function $\Psi_1(s)$ is strictly decreasing on its domain. Moreover,

$$\Psi_1(0^+) = \frac{1}{3} \quad \text{and} \quad \Psi_1(\infty^-) = 0.$$

Hence the assertion follows.
In the next two theorems we will deal with optimal bounds for $U_p$ and $V_p$ where now bounding quantities are the geometric convex combinations of $Q_p$ and $G_p$.

**Theorem 4.3.** The following inequality

\[ G_p^{\alpha_2} Q_p^{1-\alpha_2} < U_p < G_p^{\beta_2} Q_p^{1-\beta_2} \]

is valid if

\[ \alpha_2 \geq \frac{1}{3} \quad \text{and} \quad \beta_2 \leq 0. \]

**Proof.** First we rewrite (4.7) as follows

\[ (G_p/Q_p)^{\alpha_2} < U_p/Q_p < (G_p/Q_p)^{\beta_2}. \]

Since $U_p = SB(G_p, Q_p)$ (2.9) and (2.10) yield

\[ G_p/Q_p = \cos \varphi \quad \text{and} \quad U_p/Q_p = \frac{\sin \varphi}{r}, \]

where $0 \leq r \leq \pi/2$. This in conjunction with (4.9) yields

\[ (\cos r)^{\alpha_2} < \frac{\sin r}{r} < (\cos r)^{\beta_2}. \]

Taking logarithms we can write the last two-sided inequality as

\[ \beta_2 < \Phi_2(r) < \alpha_2, \]

where

\[ \Phi_2(r) = \frac{\ln(\sin r)}{\ln(\cos r)}. \]

It follows from Lemma 2 in [15] that the function $\Phi_2(r)$ is strictly decreasing on its domain. This in conjunction with

\[ \Phi_2(0^+) = \frac{1}{3} \quad \text{and} \quad \Phi_2(\frac{\pi}{2}) = 0 \]

yields the asserted result. The proof is complete. \qed

A result for $V_p$, which is similar to that in Theorem 4.3, reads as follows:

**Theorem 4.4.** The following inequality

\[ Q_p^{\gamma_2} G_p^{1-\gamma_2} < V_p < Q_p^{\delta_2} G_p^{1-\delta_2} \]

is valid if

\[ \gamma_2 \leq \frac{1}{3} \quad \text{and} \quad \delta_2 \geq 1. \]

**Proof.** Dividing each member of (4.11) by $G_p$ and next taking logarithms and using formulas (3.11), (2.9) and (2.10) we obtain, after a little algebra

\[ \gamma_2 < \Psi_2(s) < \delta_2, \]
where
\[ \Psi_2(s) = \frac{\ln(\sinh s)}{\ln(\cosh s)} \]
\((0 < s < \infty)\). It is known (cf. [26] and [15]) that the function \(\Psi_2(s)\) is strictly increasing on its domain and also that
\[ \Psi_2(0^+) = \frac{1}{3} \quad \text{and} \quad \Psi_2(\infty^-) = 1. \]
This in conjunction with (4.13) gives the desired result. \(\square\)

The remaining two results deal with the optimal bounds for the reciprocals of two means \(U_p\) and \(V_p\). Bounding expressions have a structure of the reciprocals of the harmonic means of \(Q_p\) and \(G_p\). We shall establish now the following

**Theorem 4.5.** The two-sided inequality
\[(4.14) \quad \frac{\alpha_3}{G_p} + \frac{1 - \alpha_3}{Q_p} < \frac{1}{U_p} < \frac{\beta_3}{G_p} + \frac{1 - \beta_3}{Q_p}\]
holds true provided
\[(4.15) \quad \alpha_3 \leq 0 \quad \text{and} \quad \beta_3 \geq \frac{1}{3}.\]

**Proof.** First we rewrite (4.14) as follows
\[(4.16) \quad \alpha_3 < \left( \frac{G_p}{U_p} \right) \frac{1 - U_p/Q_p}{1 - G_p/Q_p} < \beta_3.\]
Making use of (3.11), (2.9) and (2.10) we obtain
\[ \frac{G_p}{U_p} = \frac{r}{\tan r}, \quad \frac{U_p}{Q_p} = \frac{\sin r}{r} \quad \text{and} \quad \frac{G_p}{Q_p} = \cos r. \]
Applying these formulas to (4.16) we obtain
\[(4.17) \quad \alpha_3 < \Phi_3(r) < \beta_3,\]
where
\[ \Phi_3(r) = \frac{r - \sin r}{\tan r - \sin r} \]
\((0 < r < \pi/2)\). It follows from Lemma 2 in [15] that the function \(\Phi_3(r)\) is strictly decreasing on its domain and also that
\[ \Phi_3(0^+) = \frac{1}{3} \quad \text{and} \quad \Phi_3\left(\frac{\pi}{2}\right) = 0. \]
This in conjunction with (4.17) gives the desired result. \(\square\)

We close this section with the following

**Theorem 4.6.** The following double inequality
\[(4.18) \quad \frac{\gamma_3}{G_p} + \frac{1 - \gamma_3}{Q_p} < \frac{1}{V_p} < \frac{\delta_3}{G_p} + \frac{1 - \delta_3}{Q_p}\]
holds true provided

\[(4.19) \quad \gamma_3 \leq 0 \quad \text{and} \quad \delta_3 \geq \frac{2}{3}.
\]

**Proof.** It's easy to see that the two-sided inequality (4.19) is equivalent to the following one

\[(4.20) \quad \gamma_3 < \left(\frac{G_p}{V_p}\right) 1 - \frac{V_p/Q_p}{1 - G_p/Q_p} < \delta_3.
\]

Making use of (3.11), (2.9) and (2.10) we obtain

\[G_p/V_p = \frac{s}{\sinh s}, \quad V_p/Q_p = \frac{\tanh s}{s} \quad \text{and} \quad G_p/Q_p = \frac{1}{\cosh s}.
\]

Applying these formulas to (4.20) we obtain

\[(4.21) \quad \gamma_3 < \Psi_3(s) < \delta_3,
\]

where

\[\Psi_3(s) = \frac{s - \tanh s}{\sinh s - \tanh s}
\]

\((0 < s < \infty)\). It follows from Lemma 3 in [15] that the function \(\Psi_3(s)\) is strictly decreasing on its domain and also that

\[\Psi_3(0^+) = \frac{2}{3} \quad \text{and} \quad \Phi_3(\infty^-) = 0.
\]

This in conjunction with (4.21) yields the asserted result. \(\square\)

**References**


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